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Characterizations of a Bertrand Curve According to Darboux Vector

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Abstract

In this paper, we first take a Bertrand curve pair and then we use Darboux vector instead of mean curvature vector to give characterizations of Bertrand partner curve by means of the Bertrand curve. By making use of the relations between the Frenet frames of the Bertrand curve pair we give the differential equations and sufficient conditions of harmonicity(biharmonic curve or 1-type of harmonic curve) for the Bertrand partner curve in terms of the Darboux vector of the Bertrand curve. We get some new results and finally we write an example to demonstrate how our assumptions come true.

Keywords: *Bertrand curve pair, differential equation, Darboux vector, biharmonic curve, Laplace operator.*
2010 Mathematics Subject Classification: *14H45, 53A04.*

1. Introduction and Preliminaries

In geometry one of the most commonly used fact is that we can constitute a relations between the invariants and general features of a curve. As one of the widely known exemplification revealing this relation is the Bertrand curve pair. We show that all characterizations of the Bertrand partner curve can be given in terms of the Darboux vector of Bertrand curve. In this way we give the harmonicity conditions of Bertrand partner curve by means of the Darboux vector of Bertrand curve. Referring this formula we also give differential equations representing the Bertrand partner curve through the main curve. By this method, we obtain ordinary differential equations. Also this method made it easier for us to interpret the harmonicity of the Bertrand partner curve. Now we may cite some remarkable works drawing our attention. We may make classification of biharmonic curves [1]. By this paper we recognize that some curves may be called as biharmonic curve while some of them are 1-type of harmonic. Among so many works we apply as a tool only some of them: Senyurt and Cakir [2] point out a method to classify a given curve by means of an another curve.Kocayigit et al.[3] study 1-type of harmonic curves by using the Darboux vector of the curve itself. Senyurt and Cakir [4] study biharmonic curves whose mean curvature vector field is the kernel of Laplacian. Also they give the differential equations of a curve according to unit Darboux vector of the given curve [5]. Now we may review some basic concepts of differential geometry. We can give the Frenet formulas as, [7]

$$T' = \vartheta\kappa N, \qquad N' = -\vartheta\kappa T + \vartheta\tau B, \qquad B' = -\vartheta\tau N. \tag{1.1}$$

Every Frenet frame moves along an axis which is called a Darboux vector and it is given by, [6]

$$W = \tau T + \kappa B. \tag{1.2}$$

Given that α is a differentiable curve with the principal normal N and γ is another differentiable curve. If α and γ have the common principal normal at their corresponding points then α is called a Bertrand curve and γ is called the Bertrand partner of α. In this way (α,γ) is called the Bertrand curve pair. It is obvious from this statement, [7]

$$\gamma(t) = \alpha(t) + \lambda(t)N(t), \quad \lambda(t) \in \mathbb{R}. \tag{1.3}$$

The ordered pair (α,γ) forms a Bertrand couple if and only if $\lambda\kappa + \mu\tau = 1$ where $\lambda, \mu \in \mathbb{R}$. The relation between the Frenet frames of α and γ is

$$T_\gamma = cos\theta T + sin\theta B, \quad N_\gamma = N, \quad B_\gamma = -sin\theta T + cos\theta B \tag{1.4}$$

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provided $cos\theta = < T_\gamma, T >$. The relationship between the curvatures of α and γ is

$$\kappa_\gamma(t) = \frac{\lambda\kappa - sin^2\theta}{\lambda(1-\lambda\kappa)}, \quad \tau_\gamma(t) = \frac{1}{\lambda^2\tau} sin^2\theta \cdot \tag{1.5}$$

When we take the eq.(1.2) and eq.(1.4) into consideration, we have another relation between the Darboux vectors of α and γ as follows, [8]

$$W_\gamma = \frac{1}{\tau\sqrt{\lambda^2+\mu^2}} W \tag{1.6}$$

provided that $\theta_\gamma = arctan(\frac{1}{\mu\kappa - \lambda\tau})$, shown in Fig.1.

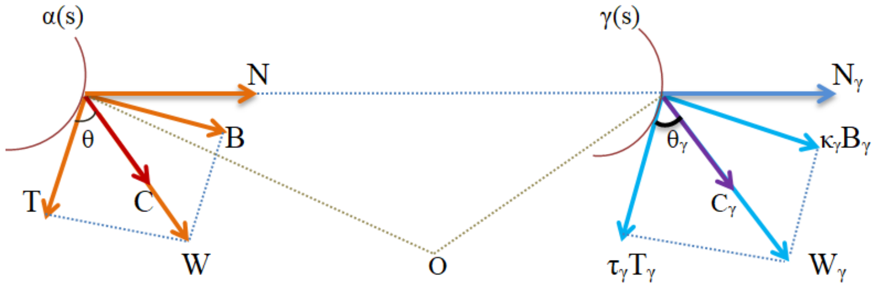

Figure 1.1: (α, γ) Bertrand curve pairs with Darboux vectors

Laplace operator can be defined as the following mapping, [3]

$$\Delta : \chi^{\perp}(\alpha(I)) \to \chi(\alpha(I)) \ \text{ such that } \ \Delta H = -D_T^2 H \tag{1.7}$$

where H is the mean curvature vector and D is Levi-Civita connection along a curve.

Theorem 1.1. [3] *Let α be a regular curve with the Darboux vector W. We have the following propositions.*

i)If $\Delta W = 0$ then α is called a biharmonic curve.

ii)If $\Delta W = \lambda W$ then α is called a 1-type of harmonic curve, $\lambda \in \mathbb{R}$.

Theorem 1.2. [9] *Let (α, β) be a Bertrand curve pair. Then the covariants derivatives of α with respect to B is given as in*

$$\begin{aligned} D_B T &= (\frac{1-cos\theta}{sin\theta})\kappa N, \\ D_B N &= -(\frac{1-cos\theta}{sin\theta})\kappa T + (\frac{1-cos\theta}{sin\theta})\tau B, \\ D_B B &= -(\frac{1-cos\theta}{sin\theta})\tau N. \end{aligned} \tag{1.8}$$

2. Discussions and Result

Throughout the present paper we use the set $\{T, N, B, \kappa, \tau, W\}$ to express the Frenet apparatus of the Bertrand curve α and also the set $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma, \tau_\gamma, W_\gamma\}$ for the Frenet elements of the Bertrand partner curve γ with the norm $\vartheta = \parallel \gamma'(s) \parallel$.

Theorem 2.1. *Let (α, γ) be a Bertrand curve pair and γ be the partner curve with Darboux vector W_γ. Differential equation characterizing the curve γ with respect to connection is given by*

$$c_{\gamma 1} D^3_{T_\gamma} W_\gamma + c_{\gamma 2} D^2_{T_\gamma} W_\gamma + c_{\gamma 3} D_{T_\gamma} W_\gamma + c_{\gamma 4} W_\gamma = 0$$

with the coefficients $c_{\gamma 1}$, $c_{\gamma 2}$, $c_{\gamma 3}$ and $c_{\gamma 4}$ as

$$\begin{aligned} c_{\gamma 1} &= \vartheta\left(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\right)^2, \\ c_{\gamma 2} &= \left(\vartheta\kappa''_\gamma\tau_\gamma - \vartheta\kappa_\gamma\tau''_\gamma - (\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma)'\right)\left(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\right), \\ c_{\gamma 3} &= \left(\kappa'''_\gamma\tau_\gamma - \kappa_\gamma\tau'''_\gamma + \vartheta^2(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma)(\kappa^2_\gamma + \tau^2_\gamma)\right)\left(\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma\right) + \left(\vartheta\kappa''_\gamma\tau_\gamma - \vartheta\kappa_\gamma\tau''_\gamma - (\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma)'\right)\left(\kappa''_\gamma\tau_\gamma - \kappa_\gamma\tau''_\gamma\right), \\ c_{\gamma 4} &= \left(\kappa'_\gamma\tau'''_\gamma - \kappa'''_\gamma\tau'_\gamma - \vartheta^2(\kappa_\gamma\kappa'_\gamma + \tau_\gamma\tau'_\gamma)(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma)\right)\left(\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma\right) + \left(\vartheta\kappa''_\gamma\tau_\gamma - \vartheta\kappa_\gamma\tau''_\gamma - (\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma)'(\kappa'_\gamma\tau''_\gamma - \kappa''_\gamma\tau'_\gamma)\right). \end{aligned}$$

Proof. From the definition of Darboux vector we have

$$W_\gamma = \tau_\gamma T_\gamma + \kappa_\gamma B_\gamma. \tag{2.1}$$

Covariant derivatives of this vector with respect to T_γ are

$$D_{T_\gamma} W_\gamma = \tau'_\gamma T_\gamma + \kappa'_\gamma B_\gamma, \tag{2.2}$$

$$D^2_{T_\gamma} W_\gamma = \tau''_\gamma T_\gamma + \left(\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma\right)N_\gamma + \kappa''_\gamma B_\gamma, \tag{2.3}$$

$$D^3_{T_\gamma} W_\gamma = \left(\tau'''_\gamma - \vartheta^2\kappa_\gamma\left(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\right)\right)T_\gamma + \left(\vartheta\kappa_\gamma\tau''_\gamma - \vartheta\kappa''_\gamma\tau_\gamma + \left(\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma\right)'\right)N_\gamma + \left(\vartheta^2\tau_\gamma\left(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\right) + \kappa'''_\gamma\right)B_\gamma. \tag{2.4}$$

From eq.(2.1) and eq.(2.2) we write T_γ and B_γ as follows

$$T_\gamma = \frac{\kappa_\gamma}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma} D_{T_\gamma}W_\gamma - \frac{\kappa'_\gamma}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma} W_\gamma \quad \text{and} \quad B_\gamma = \frac{-\tau_\gamma}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma} D_{T_\gamma}W_\gamma + \frac{\tau'_\gamma}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma} W_\gamma.$$

Setting these vectors into eq.(2.3) we get N_γ as

$$N_\gamma = \frac{1}{\vartheta\left(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\right)} D^2_{T_\gamma}W_\gamma + \frac{\kappa''_\gamma\tau_\gamma - \kappa_\gamma\tau''_\gamma}{\vartheta\left(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\right)^2} D_{T_\gamma}W_\gamma + \frac{\kappa'_\gamma\tau''_\gamma - \kappa''_\gamma\tau'_\gamma}{\vartheta\left(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma\right)^2} W_\gamma.$$

Putting the equivalents of T_γ, N_γ and B_γ into eq.(2.4) we obtain

$$\begin{aligned} D^3_{T_\gamma}W_\gamma = &\left(\left(\vartheta\kappa_\gamma\tau''_\gamma - \vartheta\kappa''_\gamma\tau_\gamma + (\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma)'\right)\frac{1}{\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma}\right)D^2_{T_\gamma}W_\gamma + \left(\left(\kappa_\gamma\tau'''_\gamma - \kappa'''_\gamma\tau_\gamma - \vartheta^2(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma)(\kappa^2_\gamma + \tau^2_\gamma)\right)\frac{1}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma}\right. \\ &+ \left.\left(\vartheta\kappa_\gamma\tau''_\gamma - \vartheta\kappa''_\gamma\tau_\gamma + (\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma)'\right)\frac{\kappa''_\gamma\tau_\gamma - \kappa_\gamma\tau''_\gamma}{\vartheta(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma)^2}\right)D_{T_\gamma}W_\gamma + \left(\left(\kappa'''_\gamma\tau'_\gamma - \kappa'_\gamma\tau'''_\gamma + \vartheta^2(\kappa_\gamma\kappa'_\gamma + \tau_\gamma\tau'_\gamma)(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma)\right)\frac{1}{\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma}\right. \\ &+ \left.\left(\vartheta\kappa_\gamma\tau''_\gamma - \vartheta\kappa''_\gamma\tau_\gamma + (\vartheta\kappa_\gamma\tau'_\gamma - \vartheta\kappa'_\gamma\tau_\gamma)'\right)\frac{\kappa'_\gamma\tau''_\gamma - \kappa''_\gamma\tau'_\gamma}{\vartheta(\kappa_\gamma\tau'_\gamma - \kappa'_\gamma\tau_\gamma)^2}\right)W_\gamma. \end{aligned}$$

Finally we arrange the linear union of $D^3_{T_\gamma}W_\gamma$, $D^2_{T_\gamma}W_\gamma$, $D_{T_\gamma}W_\gamma$, W_γ with the coefficients $c_{\gamma 1}$, $c_{\gamma 2}$, $c_{\gamma 3}$, $c_{\gamma 4}$ and this completes the proof. □

Theorem 2.2. *Let (α, γ) be a Bertrand curve pair. Then the differential equation characterizing the curve γ in terms of α with respect to connection can be given*

$$\omega_1 D^3_B W + \omega_2 D^2_B W + \omega_3 D_B W + \omega_4 W = 0$$

with the coefficients $\omega_1, \omega_2, \omega_3, \omega_4$ *as*

$$\omega_1 = c_1\frac{\sin^2\theta\left(\cos\theta + \sin\theta\right)}{\tau\sqrt{\lambda^2 + \mu^2}},$$

$$\omega_2 = c_1\frac{3\sin^2\theta\left(\cos\theta + \sin\theta\right)}{\sqrt{\lambda^2 + \mu^2}}\left(\frac{1}{\tau}\right)' + c_2\frac{\sin\theta(\cos\theta + \sin\theta)}{\tau\sqrt{\lambda^2 + \mu^2}} + \frac{\rho_3\cos\theta}{(1 - \sin\theta)(\kappa\tau' - \kappa'\tau)},$$

$$\omega_3 = \rho_1 + \frac{\rho_2\kappa - \rho_4\tau}{\kappa\tau' - \kappa'\tau} + \frac{\rho_3(\kappa\tau'' - \kappa''\tau)\cos\theta}{(\sin\theta - 1)(\kappa\tau' - \kappa'\tau)^2},$$

$$\omega_4 = \frac{\rho_4\tau' - \rho_2\kappa'}{\kappa\tau' - \kappa'\tau} + \frac{\rho_3(\kappa''\tau' - \kappa'\tau'')\cos\theta}{(\sin\theta - 1)(\kappa\tau' - \kappa'\tau)^2},$$

$$\rho_1 = c_1\frac{\sin\theta\left(2\sin 2\theta + 2\sin^2\theta + 1\right)}{\sqrt{\lambda^2 + \mu^2}}\left(\frac{1}{\tau}\right)'' + c_2\frac{2\sin\theta(\cos\theta + \sin\theta)}{\sqrt{\lambda^2 + \mu^2}}\left(\frac{1}{\tau}\right)' + c_3\frac{\cos\theta + \sin\theta}{\tau\sqrt{\lambda^2 + \mu^2}},$$

$$\rho_2 = c_1\frac{sin\theta\big(1+sin2\theta\big)\tau}{\sqrt{\lambda^2+\mu^2}}\big(\frac{1}{\tau}\big)''' + c_2\frac{(cos\theta+sin\theta)^2\tau}{\sqrt{\lambda^2+\mu^2}}\big(\frac{1}{\tau}\big)'' - c_3\frac{cos\theta+sin\theta}{\sqrt{\lambda^2+\mu^2}}\big(\frac{\tau'}{\tau}\big)$$

$$+c_1\big(\frac{cos\theta+sin\theta}{\sqrt{\lambda^2+\mu^2}}\big)\Big(\big(\frac{\tau''}{\tau}\big)'sin\theta cos\theta + cos\theta\big(cos\theta-1\big)\big(\kappa\tau'-\kappa'\tau\big)\big(\frac{\kappa\tau+\kappa}{\tau}\big)$$

$$-\big(\big(\frac{\tau'}{\tau}\big)^2\big)'sin2\theta + cos\theta\big(cos\theta+sin\theta\big)\big(\big(\frac{1}{\tau}\big)''\tau\big)' + cos\theta\big(cos\theta+sin\theta\big)\big(\frac{\tau''\tau-2(\tau')^2}{\tau^2}\big)'$$

$$-\big(\frac{\kappa}{\tau}\big)cos^2\theta\big(\kappa\tau'-\kappa'\tau\big)\Big) + c_2\frac{cos\theta(cos\theta+sin\theta)}{\tau^2\sqrt{\lambda^2+\mu^2}}\big(\tau''\tau-2(\tau')^2\big) + \frac{c_4}{\sqrt{\lambda^2+\mu^2}},$$

$$\rho_3 = c_1\big(\frac{cos\theta+sin\theta}{\sqrt{\lambda^2+\mu^2}}\big)\Big(\big(\frac{\kappa\tau''}{\tau}\big)cos\theta sin\theta + cos\theta\big(1-cos\theta\big)\Big(\big(\kappa\tau'-\kappa'\tau\big)'$$

$$+\big(\frac{\kappa}{\tau^2}\big)\big(\tau''\tau-2(\tau')^2\big) + \frac{2\kappa'\tau'-\kappa''\tau}{\tau}\Big) + \big(\frac{\kappa'\tau'}{\tau}\big)sin2\theta - \big(\kappa''\tau\big)cos\theta sin\theta$$

$$-\kappa\big(\frac{\tau'}{\tau}\big)^2 sin2\theta + \big(\frac{\kappa}{\tau^2}\big)\big(\tau''\tau-2(\tau')^2\big)cos^2\theta$$

$$+cos\theta\big(cos\theta+sin\theta\big)\big(\frac{\kappa\tau'-\kappa'\tau}{\tau}\big)' - cos^2\theta\big(\frac{\kappa''\tau-2\kappa'\tau'}{\tau}\big)\Big) + c_2\frac{cos\theta(cos\theta+sin\theta)}{\tau\sqrt{\lambda^2+\mu^2}}\big(\kappa\tau'-\kappa'\tau\big),$$

$$\rho_4 = c_1\frac{sin\theta\big(1+sin2\theta\big)\kappa}{\sqrt{\lambda^2+\mu^2}}\big(\frac{1}{\tau}\big)''' + c_2\frac{(cos\theta+sin\theta)^2\kappa}{\sqrt{\lambda^2+\mu^2}}\big(\frac{1}{\tau}\big)'' - c_3\frac{cos\theta+sin\theta}{\sqrt{\lambda^2+\mu^2}}\big(\frac{\kappa\tau'}{\tau^2}\big)$$

$$+\frac{\kappa c_4}{\tau\sqrt{\lambda^2+\mu^2}} + c_1\big(\frac{cos\theta+sin\theta}{\sqrt{\lambda^2+\mu^2}}\big)\Big(\big(\kappa\tau'-\kappa'\tau\big)\big(cos\theta+\tau(cos\theta-cos^2\theta)\big)$$

$$+\kappa'''sin\theta cos\theta - \big(\frac{\kappa'\tau'}{\tau^2}\big)'sin2\theta + cos\theta\big(cos\theta+sin\theta\big)\Big(\big(\big(\frac{1}{\tau}\big)''\kappa\big)' + \big(\frac{\kappa''\tau-2\kappa'\tau'}{\tau^2}\big)'\Big)\Big)$$

$$+c_2\frac{cos\theta(cos\theta+sin\theta)}{\tau^2\sqrt{\lambda^2+\mu^2}}\big(\kappa''\tau-2\kappa'\tau'\big)$$

and c_1, c_2, c_3, c_4

$$c_1 = \tau\sqrt{\lambda^2+\mu^2}\Big(\frac{sin^2\theta}{\lambda^2\tau}\Big)^2\Big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\Big)^2,$$

$$c_2 = \Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'' - \big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)''\Big) - \Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\Big)'\Big)$$

$$\Big(\big(\frac{sin^2\theta}{\lambda^2\tau}\big)\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\Big)\Big),$$

$$c_3 = \Big(\big(\frac{sin^2\theta}{\lambda^2\tau}\big)\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)''' - \big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)'''\Big) + \tau\big(\lambda^2+\mu^2\big)\big(\frac{sin^2\theta}{\lambda^2}\big)$$

$$\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\Big)\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)^2 + \big(\frac{sin^2\theta}{\lambda^2\tau}\big)^2\Big)\Big)$$

$$\Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\Big)\Big) + \Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'' - \big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)''\Big)$$

$$-\Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2}sin^2\theta\big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)' - \big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'\big)\Big)'\Big)\Big(\big(\frac{sin^2\theta}{\lambda^2\tau}\big)\Big(\big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu\tau}\big)'' - \big(\frac{\lambda\kappa-sin^2\theta}{\lambda\mu}\big)\big(\frac{1}{\tau}\big)''\Big)\Big),$$

$$
\begin{aligned}
c_4 \quad = \quad & \Big((\frac{sin^2\theta}{\lambda^2}) \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'(\frac{1}{\tau})''' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'''(\frac{1}{\tau})' \Big) - (\lambda^2+\mu^2)\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' + (\frac{sin^2\theta}{\lambda^2})^2(\frac{1}{\tau})' \Big) \\
& \Big(\frac{sin^2\theta}{\lambda^2}\Big)\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' \Big)\Big) \Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2} sin^2\theta \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' \Big)\Big) \\
& + \Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2} sin^2\theta \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})'' \Big) - \Big(\frac{\sqrt{\lambda^2+\mu^2}}{\lambda^2} sin^2\theta \Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu})(\frac{1}{\tau})' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})' \Big)\Big)' \\
& \frac{sin^2\theta}{\lambda^2}\Big((\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})'(\frac{1}{\tau})'' - (\frac{\lambda\kappa - sin^2\theta}{\lambda\mu\tau})''(\frac{1}{\tau})' \Big)\Big).
\end{aligned}
$$

Proof. By making use of eq.(1.6), we can write the vector W_γ of eq.(2.1) in terms of the Bertrand curve α as

$$W_\gamma = \frac{1}{\tau\sqrt{\lambda^2+\mu^2}} W.$$

Applying the eq.(1.8) we evaluate the first derivative of this vector with respect to T_γ

$$
\begin{aligned}
D_{T_\gamma} W_\gamma &= D_{\left(cos\theta T + sin\theta B\right)} \Big(\frac{1}{\tau\sqrt{\lambda^2+\mu^2}} W \Big) \\
&= cos\theta D_T \big(\frac{1}{\tau\sqrt{\lambda^2+\mu^2}} W \big) + sin\theta D_B \big(\frac{1}{\tau\sqrt{\lambda^2+\mu^2}} W \big) \\
&= \Big(\frac{cos\theta + sin\theta}{\tau\sqrt{\lambda^2+\mu^2}} \Big) D_B W + \Big(\frac{cos\theta + sin\theta}{\sqrt{\lambda^2+\mu^2}} (\frac{1}{\tau})' \Big) W
\end{aligned}
$$

and the second derivative of W_γ as

$$
\begin{aligned}
D^2_{T_\gamma} W_\gamma \quad = \quad & \frac{sin\theta(cos\theta + sin\theta)}{\tau\sqrt{\lambda^2+\mu^2}} D^2_B W + \frac{2sin\theta(cos\theta + sin\theta)}{\sqrt{\lambda^2+\mu^2}} (\frac{1}{\tau})' D_B W + \frac{(cos\theta + sin\theta)^2}{\sqrt{\lambda^2+\mu^2}} (\frac{1}{\tau})'' W + \frac{cos\theta(cos\theta + sin\theta)}{\tau^2\sqrt{\lambda^2+\mu^2}} \big(\tau''\tau - 2(\tau')^2 \big) T \\
& + \frac{cos\theta(cos\theta + sin\theta)}{\tau\sqrt{\lambda^2+\mu^2}} \big(\kappa\tau' - \kappa'\tau \big) N + \frac{cos\theta(cos\theta + sin\theta)}{\tau^2\sqrt{\lambda^2+\mu^2}} \big(\kappa''\tau - 2\kappa'\tau' \big) B.
\end{aligned}
\tag{2.5}
$$

By the similar method we obtain the third derivative of W_γ as follows

$$
\begin{aligned}
D^3_{T_\gamma} W_\gamma \quad = \quad & \frac{sin^2\theta\big(cos\theta + sin\theta\big)}{\tau\sqrt{\lambda^2+\mu^2}} D^3_B W + \frac{3sin^2\theta\big(cos\theta + sin\theta\big)}{\sqrt{\lambda^2+\mu^2}} \big(\frac{1}{\tau}\big)' D^2_B W + \frac{sin\theta\big(2sin2\theta + 2sin^2\theta + 1\big)}{\sqrt{\lambda^2+\mu^2}} \big(\frac{1}{\tau}\big)'' D_B W + \frac{sin\theta\big(1 + sin2\theta\big)}{\sqrt{\lambda^2+\mu^2}} \big(\frac{1}{\tau}\big)''' W \\
& + \big(\frac{cos\theta + sin\theta}{\sqrt{\lambda^2+\mu^2}}\big) \Big(\big(\frac{\tau''}{\tau}\big)' sin\theta cos\theta + cos\theta\big(cos\theta - 1\big)\big(\kappa\tau' - \kappa'\tau\big)\big(\frac{\kappa\tau + \kappa}{\tau}\big) - \big(\big(\frac{\tau'}{\tau}\big)^2\big)' sin2\theta + cos\theta\big(cos\theta + sin\theta\big)\big(\big(\frac{1}{\tau}\big)''\tau\big)' \\
& + cos\theta\big(cos\theta + sin\theta\big)\big(\frac{\tau''\tau - 2(\tau')^2}{\tau^2}\big)' - \big(\frac{\kappa}{\tau}\big) cos^2\theta\big(\kappa\tau' - \kappa'\tau\big)\Big) T + \big(\frac{cos\theta + sin\theta}{\sqrt{\lambda^2+\mu^2}}\big)\Big(\big(\frac{\kappa\tau''}{\tau}\big) cos\theta sin\theta + cos\theta\big(1 - cos\theta\big)\Big(\big(\kappa\tau' - \kappa'\tau\big)' \\
& + \big(\frac{\kappa}{\tau^2}\big)\big(\tau''\tau - 2(\tau')^2\big) + \frac{2\kappa'\tau' - \kappa''\tau}{\tau}\Big) + \big(\frac{\kappa'\tau'}{\tau}\big) sin2\theta - \big(\kappa''\tau\big) cos\theta sin\theta - \kappa\big(\frac{\tau'}{\tau}\big)^2 sin2\theta + \big(\frac{\kappa}{\tau^2}\big)\big(\tau''\tau - 2(\tau')^2\big) cos^2\theta \\
& + cos\theta\big(cos\theta + sin\theta\big)\big(\frac{\kappa\tau' - \kappa'\tau}{\tau}\big)' - cos^2\theta\big(\frac{\kappa''\tau - 2\kappa'\tau'}{\tau}\big)\Big) N + \big(\frac{cos\theta + sin\theta}{\sqrt{\lambda^2+\mu^2}}\big)\Big(\big(\kappa\tau' - \kappa'\tau\big)\big(cos\theta + \tau(cos\theta - cos^2\theta)\big) + \kappa''' sin\theta cos\theta \\
& - \big(\frac{\kappa'\tau'}{\tau^2}\big)' sin2\theta + cos\theta\big(cos\theta + sin\theta\big)\Big(\big(\big(\frac{1}{\tau}\big)''\kappa\big)' + \big(\frac{\kappa''\tau - 2\kappa'\tau'}{\tau^2}\big)'\Big)\Big) B.
\end{aligned}
$$

Now we may express the Frenet vectors T, N, B of the covariant derivatives $D_{T_\gamma}W_\gamma$ $D^2_{T_\gamma}W_\gamma$ and $D^3_{T_\gamma}W_\gamma$ in terms of W. In order to do this we use Frenet formulae given in eq.(1.8). From the equalities

$$W = \tau T + \kappa B \quad \text{and} \quad D_B W = \tau' T + \kappa' B$$

we can write the vectors T and B as

$$T = \Big(\frac{\kappa}{\kappa\tau' - \kappa'\tau}\Big) D_B W - \Big(\frac{\kappa'}{\kappa\tau' - \kappa'\tau}\Big) W \quad \text{and} \quad B = \Big(\frac{-\tau}{\kappa\tau' - \kappa'\tau}\Big) D_B W + \Big(\frac{\tau'}{\kappa\tau' - \kappa'\tau}\Big) W.$$

Putting these vectors into the second derivative of W given above, we write the vector N as follows

$$N = \Big(\frac{cos\theta}{(1-sin\theta)(\kappa\tau'-\kappa'\tau)}\Big)D_B^2W + \Big(\frac{cos\theta(\kappa\tau''-\kappa''\tau)}{(sin\theta-1)(\kappa\tau'-\kappa'\tau)^2}\Big)D_BW + \Big(\frac{cos\theta(\kappa''\tau'-\kappa'\tau'')}{(sin\theta-1)(\kappa\tau'-\kappa'\tau)^2}\Big)W.$$

It remains only to find out the counterparts of the coefficients $c_{\gamma 1}$, $c_{\gamma 2}$, $c_{\gamma 3}$, $c_{\gamma 4}$ of the eq.(2.1). Applying the eq.(1.5) we can write the equivalent of these coefficients c_1, c_2, c_3, c_4 as given above.
Finally we rearrange the linear combination of derivatives $D^3_{T_\gamma}W_\gamma$, $D^2_{T_\gamma}W_\gamma$, $D_{T_\gamma}W_\gamma$ and then put their coefficients computed above gives us the desired differential equation. □

Corollary 2.3. *Suppose that (α,γ) be a Bertrand curve pair with the angle θ between the vectors T_γ and T. According to Levi-Civita connection, Bertrand partner curve γ is biharmonic curve if and only if $tan\theta = -1$.*

Proof. From eq.(1.7), Laplace image of the vector W_γ is $\Delta W_\gamma = -D^2_{T_\gamma}W_\gamma$ and from eq.(2.5) we get

$$\Delta W_\gamma = -\frac{sin\theta(cos\theta+sin\theta)}{\tau\sqrt{\lambda^2+\mu^2}}D_B^2W - \frac{2sin\theta(cos\theta+sin\theta)}{\sqrt{\lambda^2+\mu^2}}(\frac{1}{\tau})'D_BW - \frac{(cos\theta+sin\theta)^2}{\sqrt{\lambda^2+\mu^2}}(\frac{1}{\tau})''W - \frac{cos\theta(cos\theta+sin\theta)}{\tau^2\sqrt{\lambda^2+\mu^2}}(\tau''\tau-2(\tau')^2)T$$

$$-\frac{cos\theta(cos\theta+sin\theta)}{\tau\sqrt{\lambda^2+\mu^2}}(\kappa\tau'-\kappa'\tau)N - \frac{cos\theta(cos\theta+sin\theta)}{\tau^2\sqrt{\lambda^2+\mu^2}}(\kappa''\tau-2\kappa'\tau')B.$$

Considering the case $\Delta W_\gamma = 0$ of the Theorem 1.1, we obtain that $cos\theta + sin\theta = 0$, that is, $tan\theta = -1$.

□

Example 2.4. *Given that (α,γ) be a Bertrand curve pair and suppose that the Bertrand curve α with the curvatures κ and τ is satisfying the condition*

$$\lambda = \frac{(\frac{\tau}{\kappa})'}{(\frac{\kappa^2+\tau^2}{\tau})'} \quad and \quad \mu = \frac{(\frac{\kappa^2+\tau^2}{\tau})' - (\frac{\kappa}{\tau})'\kappa}{(\frac{\kappa^2+\tau^2}{\tau})'\tau}.$$

Then the differential equations of γ can be evaluated as

1. According to Theorem 2.1:

$$\vartheta\Big((\kappa_\gamma)''\tau_\gamma - \kappa_\gamma(\tau_\gamma)''\Big)^2 D_{T_\gamma}W_\gamma + \vartheta\Big((\kappa_\gamma)''\tau_\gamma - \kappa_\gamma(\tau_\gamma)''\Big)W_\gamma = 0.$$

2. According to Theorem 2.2:

$$\Big((\frac{\mu\kappa-\lambda\tau}{\tau\sqrt{\lambda^2+\mu^2}})''\frac{1}{\tau\sqrt{\lambda^2+\mu^2}} - \frac{\mu\kappa-\lambda\tau}{\tau\sqrt{\lambda^2+\mu^2}}(\frac{1}{\tau\sqrt{\lambda^2+\mu^2}})''\Big)^2\Big(\frac{cos\theta+sin\theta}{\sqrt{\lambda^2+\mu^2}}\Big)D_BW$$
$$+\Big(\Big((\frac{\mu\kappa-\lambda\tau}{\tau\sqrt{\lambda^2+\mu^2}})''\frac{1}{\tau\sqrt{\lambda^2+\mu^2}} - \frac{\mu\kappa-\lambda\tau}{\tau\sqrt{\lambda^2+\mu^2}}(\frac{1}{\tau\sqrt{\lambda^2+\mu^2}})''\Big)^2\Big(\frac{cos\theta+sin\theta}{\sqrt{\lambda^2+\mu^2}}\Big)(\frac{-\tau'}{\tau})$$
$$+\Big((\frac{\mu\kappa-\lambda\tau}{\tau\sqrt{\lambda^2+\mu^2}})''\frac{1}{\tau(\lambda^2+\mu^2)^{3/2}} - \frac{\mu\kappa-\lambda\tau}{\tau(\lambda^2+\mu^2)^{3/2}}(\frac{1}{\tau\sqrt{\lambda^2+\mu^2}})''\Big)\Big)W = 0.$$

Conclusion: By making use of Darboux vector instead of mean curvature vector we give all characterizations of Bertrand partner curve in terms of the Bertrand curve. Thanks to this method we get elementary differential equations and also this method made it easier for us to comment the harmonicity of the Bertrand partner curve. We hope that this paper inspire the geometers to make similar scientific studies in non-Euclidean spaces.

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