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On Algorithm Constructing Baer Subplanes of Hall plane

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Abstract

In this study, we construct an algorithm (implemented in *C#*) to determine Baer subplanes of the projective plane of order 9 coordinatized by elements of a left nearfield of order 9 and classify these subplanes. Also, affine planes embedded in Baer subplanes are determined.

Keywords: *Affine plane; Baer subplane; Near field; Projective Plane*
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1. Introduction

A projective plane π consist of a set $\mathscr{P}$ of points and a set $\mathscr{L}$ of subsets of $\mathscr{P}$, called lines, such that every pair of points is contained in exactly one line, every two distinct lines intersect in exactly one point, and there exist four points in such a position that they pairwise define six distinct lines. A subplane of a projective plane π is a set $\mathscr{B}$ of points and lines which is itself a projective plane, relative to the incidence relation given in π.
It is well known that there exist at least four non-isomorphic projective planes of order 9. The known four distinct projective planes of order 9 are extensively studied by Room Kirkpatrick [12]. These are Desarguesian plane, the left nearfield plane, the right nearfield plane and Hughes plane. These are Desarguesian plane, the left nearfield plane, the right nearfield plane and Hughes plane. The last three planes of order 9 are called "miniquaternion planes" because they can be coordinatized by the miniquaternion near field. O. Veblen and J. M. Wedderburn [13] discovered these miniquaternion planes in 1907.

In a finite projective plane π (not necessarily Desarguesian) a set K of k $(k \geq 3)$ points such that no three points of K are collinear (on a line) is called a k-arc. If the plane π has order p then $k \leq p+2$, however the maximum value of k can only be achieved if p is even. In a plane of order p , a $(p+1)$-arc is called an oval and, if p is even, a $(p+2)$-arc is called a hyperoval. A general reference for ovals is Hirschfeld [10, 11]. There are known plenty of examples of arcs in projective planes; see [5, 7].

In 2011, Akca constructed an algorithm to classify $(k,3)-$ arcs in the projective plane of order 9, coordinatized by elements of a left semifield, denoted by $SFPG(2,9)$ [1]. And also, Akca et. al. determined the points and lines of the projective plane over a left nearfield of order 9 [2]. In 2016, Akca et. al. examined the projective plane of order 9, coordinatized by elements of a left semifield and found Fano subplanes of this projective plane at least 155760 [4]. In [3], an algorithm (implemented in *C#*) to determine and classify Fano subplanes of the projective plane of order 9 coordinatized by elements of a left nearfield of order 9 is given. In [5, 7], all complete a $(k,2)$-arcs containing complete quadrangles which generate the Fano planes in the projective plane whose algebraic structure is the left nearfield of order 9 are examined. Our aim is to determine Baer subplanes of the projective plane of order 9 coordinatized by elements of a left nearfield of order 9. We give an algorithm for checking arcs and apply the algorithm classify these subplanes.

2. The Hall Plane of order 9

The original construction of Hall planes was based on a Hall quasifield (also called a Hall system) [8]. To build a Hall quasifield, start with a Galois field $F = GF(p^n)$, for p a prime and a quadratic irreducible polynomial $f(t) = t^2 - rt - s$ over F. We consider an algebraic system $(S, \oplus, \odot)$ over the Galois field $(F_3, +, \cdot)$ of order 3. The nine elements of S are $a + \lambda b,\ a,\ b \in F_3,\ \lambda \notin F_3$. Addition in S is defined by the rule

$$(a+\lambda b) \oplus (c+\lambda d) = (a+c) + \lambda\,(b+d)$$

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and multiplication by

$$(a+\lambda b)\odot(c+\lambda d)=\begin{cases} ca+\lambda\,(da) & \text{if } b=0 \\ ca-db(a^2+1)+\lambda\,(cb-da) & \text{if } b\neq 0 \end{cases}$$

where $a,\ b,\ c,\ d\in F_3,\ \lambda\notin F_3$ and $f(t)=t^2+1$ is an irreducible polynomial on F_3. For the sake of the shortness if we use ab instead of $a+\lambda b$ in addition and multiplication equations, then addition and multiplication tables are obtained as follows:

$\oplus$	00	01	02	10	11	12	20	21	22
00	00	01	02	10	11	12	20	21	22
01	01	02	00	11	12	10	21	22	20
02	02	00	01	12	10	11	22	20	21
10	10	11	12	20	21	22	00	01	02
11	11	12	10	21	22	20	01	02	00
12	12	10	11	22	20	21	02	00	01
20	20	21	22	00	01	02	10	11	12
21	21	22	20	01	02	00	11	12	10
22	22	20	21	02	00	01	12	10	11

$\odot$	00	01	02	10	11	12	20	21	22
00	00	00	00	00	00	00	00	00	00
01	00	20	10	01	21	11	02	22	12
02	00	10	20	02	12	22	01	11	21
10	00	01	02	10	11	12	20	21	22
11	00	12	21	11	20	02	22	01	10
12	00	22	11	12	01	20	21	10	02
20	00	02	01	20	22	21	10	12	11
21	00	11	22	21	02	10	12	20	01
22	00	21	12	22	10	01	11	02	20

If we use the following equalities

$$\begin{array}{lcl} 00 & \to & 0 \\ 10 & \to & 1 \\ 20 & \to & 2 \\ 01 & \to & 3 \\ 11 & \to & 4 \\ 21 & \to & 5 \\ 02 & \to & 6 \\ 12 & \to & 7 \\ 22 & \to & 8 \end{array}$$

the addition and multiplication tables in $(S,\oplus,\odot)$ can be arranged as follows:

$\oplus$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	0	4	5	3	7	8	6
2	2	0	1	5	3	4	8	6	7
3	3	4	5	6	7	8	0	1	2
4	4	5	3	7	8	6	1	2	0
5	5	3	4	8	6	7	2	0	1
6	6	7	8	0	1	2	3	4	5
7	7	8	6	1	2	0	4	5	3
8	8	6	7	2	0	1	5	3	4

$\odot$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	1	6	8	7	3	5	4
3	0	3	6	2	5	8	1	4	7
4	0	4	8	7	2	3	5	6	1
5	0	5	7	4	6	2	8	1	3
6	0	6	3	1	7	4	2	8	5
7	0	7	5	8	3	1	4	2	6
8	0	8	4	5	1	6	7	3	2

The projective plane whose point and the lines are coordinated by elements of $(S,\oplus,\odot)$.
The 91 points of P_2S are the elements of the set $N=\{(x,y):x,y\in S\}\cup\{(m):m\in S\}\cup\{(\infty):\infty\notin S\}$.
The points of the form (x,y) are called *proper* points, and the unique point (∞) and the points of the form (m) are called *ideal* points. The 91 lines of P_2S are defined as follows:

$$\mathscr{D}=\{[m,k]:m,k\in S\}\cup\{[a]:a\in S\}\cup\{[\infty]:\infty\notin S\}$$

We have 81 lines of the form $[m,k]$, 9 lines of the form $[a]$ are called *proper* lines and the unique line $[\infty]$ is called *ideal* line. And incidence relation

$$\begin{array}{l} (x,y)\circ[m,k]\Leftrightarrow y=m\odot x\oplus k,\forall m,k\in S \\ (x,y)\circ[a]\Leftrightarrow x=a \\ (m)\circ[\infty] \\ (\infty)\circ[\infty] \end{array}$$

We convert a point expressed in Cartesian coordinates to homogeneous coordinates in left nearfield plane of order 9. We have seen a point (x,y) in the P_2S has homogeneous coordinates $\lambda\odot(x,y,1)=(\lambda\odot x,\lambda\odot y,\lambda\odot 1)$, $\lambda\in S$, $\lambda\neq 0$. Homogeneous coordinates of the form $\lambda\odot(m,1,0)$ corresponds to the unique point at infinity in the P_2S.
We have seen that a line $[m,k]$ in the P_2S has homogeneous coordinates $\mu\odot[m,-1,k]=[\mu\odot m,\mu\odot(-1),\mu\odot k]$, $\mu\in S$, $\mu\neq 0$. Homogeneous coordinates of all lines $[a]$, $a\neq 0$, $a\in S$ in the P_2S is the form $\mu\odot[x,0,1]$. Homogeneous coordinates of the form $[0,0,\mu]$ corresponds to the unique line $[\infty]$ at infinity in the P_2S.
A line in the P_2S has general equation $y=m\odot x\oplus k$, $m,k\in S$. Let (x_1,x_2,x_3) , $x_3\neq 0$ be the homogeneous coordinates of a point (x,y) on the line; hence $x_3^{-1}\odot x_1=x$ and $x_3^{-1}\odot x_2=y$. Substituting for x and y in the line equation and multiplying through by x_3 , yields the condition

$$m\odot x_1\oplus(-1)\odot x_2\oplus k\odot x_3=0$$

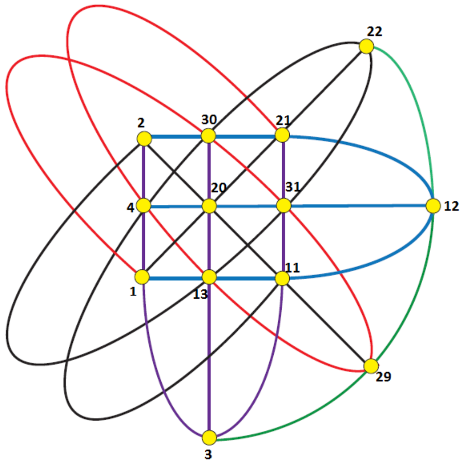
for (x_1,x_2,x_3) to be the homogeneous coordinates of a point on the line. The table of all homogeneous coordinates of the 91 points and lines in the projective plane P_2S defined in terms of a coordinate system over the Hall system $(S,\oplus,\odot)$ is given, see [5].

3. Baer Subplanes

A subplane of a projective plane is a subset of the points of the plane which themselves form a projective plane with the same incidence relations. Let $\mathbb{P}$ be a finite plane of order n with a proper subplane $\mathbb{B}$ of order m. Then either $n = m^2$ or $n \geq m^2 + m$, [4].
A proper subplane $\mathbb{B}$ of a (not necessarily Desarguesian) projective plane $\mathbb{P}$ is called a *Baer subplane* if each line of $\mathbb{P}$ contains a point in $\mathbb{B}$ and, dually, each point of $\mathbb{P}$ is incident with a line in $\mathbb{B}$.
A projective plane $\mathbb{B} = (N', D')$ is a subplane of $\mathbb{P}$, in symbols $\mathbb{B} \leq \mathbb{P}$, if $N' \subseteq N$ and $D' = \{d \cap N' \mid d \in D, 1 < \left|d \cap N'\right|\}$. With respect to $\mathbb{B}$, a line $d \in D$ is said to be an *interior line* if $D \cap D' \in \mathbb{B}$; otherwise d is called an *exterior line*. Analogously, the points in N' are called *interior points*, and the others *exterior points*.
A proper subplane $\mathbb{B}$ of $\mathbb{P}$ is a *Baer subplane* ,if $d \cap N' \neq \emptyset$ for each line $d \in D$ and, dually, each pencil D_p contains an interior line.
In [1], the list of the all quadrangles that generate Fano plane in P_2S is given. Now we take the set $A = \{O, I, X, P\}$ such that $I = (1,1,1)$, $X = (1,0,0)$, $O = (0,0,1)$, $P = (a,b,1)$ with $a, b \in F_3$ in Hall Plane. We constructed the algorithms used in the classification of $(k,2)-$arcs containing Fano planes of P_2S and containing the quadrangles not constructing the Fano planes in the projective planeP_2S over a Hall system S of order 9 by using this completion procedure in [5, 7], respectively. Now we give an algorithm for finding Baer subplanes of this Hall plane of order 9 and apply the algorithm (implemented *C#*) to determine.

Step 1
Input the points $A_i, i = \{1,2,3,4\}$ and and A_i (element of $\{1,2,3,...,92\}$ and $A_1 \neq A_2 \neq A3 \neq A_4$
Begin
$p = A_4$
$u_1 \leftarrow$ the row on $p,\ A_1$
$u_2 \leftarrow$ the row on $A_2,\ A_3$
$u \leftarrow$ the same points on u_1 and u_2
$v_1 \leftarrow$ the row on p, A_2
$v_2 \leftarrow$ the row on $A_2,\ A_3$
$v \leftarrow$ the same points on v_1 and v_2
$w_1 \leftarrow$ the row on p,A_3
$w_2 \leftarrow$ the row on $A_1,\ A_2$
$w \leftarrow$ the same points on w_1 and w_2
$a_1 \leftarrow$ the row on $u,\ v$
$a_2 \leftarrow$ the row on $w,\ p$
$a \leftarrow$ the same points on a_1 and a_2
$b_1 \leftarrow$ the row on $u,\ v$
$b_2 \leftarrow$ the row on $A_1,\ w$
$b \leftarrow$ the same points on b_1 and b_2
$c_1 \leftarrow$ the row on $p,\ A_1$
$c_2 \leftarrow$ the row on $v,\ w$
$c \leftarrow$ the same points on c_1 and c_2
$d_1 \leftarrow$ the row on $u,\ A_2$
$d_2 \leftarrow$ the row on $v,\ w$
$d \leftarrow$ the same points on d_1 and d_2
$e_1 \leftarrow$ the row on $A_1,\ v$
$e_2 \leftarrow$ the row on u,v
$e \leftarrow$ the same points on e_1 and e_2
$f_1 \leftarrow$ the row on $A_2,\ p$
$f_2 \leftarrow$ the row on u,w
$f \leftarrow$the same points on f_1 and f_2
if $(a = b)$ then print "NOT COLLINEAR , THEY CONSTRUCT FANO PLANE" and go to step1
if $\{p,b,d,e\}$ is not on same row then
go to step1
else
    if $\{c,a,e,A_2\}$ is not on same row then
    go to step1
    else
    if $\{c,b,f,A_3\}$ is not on same row
        then go to step 1
        else
        if $\{a_1,\ a,\ d,\ f\}$ is not on same row then
        print "NOT COLLINEAR"
            go to step 1
        else
        print "$\{A_1,\ A_2,\ A_3,\ p,\ u,\ v,\ w\}$ COLLINEAR, THE RESULT IS TRUE"
        end if
        end if
    end if
end if

As a result of, there are four Baer subplanes in this Hall plane not containing Fano subplanes:
$\{1,2,3,4,11,12,13,20,21,22,29,30,31\}$, $\{1,3,5,10,11,16,18,21,23,28,31,33,35\}$,
$\{1,3,6,9,11,14,17,21,24,27,31,34,37\}$, $\{1,3,7,8,11,15,19,21,25,26,31,32,36\}$. If one of these Baer Plane $\{1,2,3,4,11,12,13,20,21,22,29,30,31\}$ is choosed to be shown as follows:



There are 13 lines and 13 points. Every line has four points, three of which are finite points and one of which is on l_∞. Also, it is determined that these four Baer subplanes have 5 common points $\{1,3,11,21,31\}$. Four of these common points $\{3,11,21,31\}$ are on the same line and there is one point not on this line.

4. Affine Planes Embedded in Baer Subplanes

As known that any affine plane embeds into a projective plane. If we want to reserve to this procedure, we want to start with a projective plane, declare one line, and delete it and all its points. In section 3, four Baer subplanes are obtained. These Baer subplanes have common line including $\{3,11,21,31\}$ points. If we delete this line, the remaining structure is the affine plane. In this way, four discrete Affine planes which have one common point $X=(1,0,0)$ are obtained.

5. Conclusion

We constructed an algorithm for searching Baer subplanes containing the quadrangles $A=\{O,\ I,\ X,\ P\}$ such that $I=(1,1,1)$, $X=(1,0,0)$, $O=(0,0,1)$, $P=(a,b,1)$ with $a,\ b\in F_3$ in Hall Plane not containing Fano plane and found four different Baer subplanes which are $\{1,2,3,4,11,12,13,20,21,22,29,30,31\}$, $\{1,3,5,10,11,16,18,21,23,28,31,33,35\}$, $\{1,3,6,9,11,14,17,21,24,27,31,34,37\}$, $\{1,3,7,8,11,15,19,21,25,26,31,32,36\}$. In addition, these Baer subplanes have common five points and four discrete affine planes which embedded in Baer subplanes containing $X=(1,0,0)$ are determined.

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