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Some Characterizations of Spherical Indicatrix Curves Generated by Sannia Frame

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Abstract

In this study, we have first provided the relations between the Frenet frame and Sannia frame on the striction points of four ruled surfaces of each formed by taking the basis as the tangent, normal, binormal and Darboux vector. Second, we have defined the relations between the Sannia vectors and their derivatives. For each Sannia frame, we have calculated the Darboux frame and expressed those in terms of Frenet frame. Last, we have obtained the arc lengths and the geodesic curvatures according to both Euclidean space E^3 and unit sphere S^2 of Sannia vectors for each four of ruled surfaces.

Keywords: *Ruled surface; Sannia frame; Striction curve; Spherical indicatrix; Geodesic curvature.*
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1. Introduction

We engage the curves in almost many areas of our daily life. For example, we see helix curves on DNA sequencing. The cornu spiral is used in construction of highways, railways, metro or rail systems. The Catanery curve is used as a design of bridge or train tracks. We can give so many examples like this. The frames on the other hand is an essential subject in the curve theory. The most used is the Frenet frame. Researchers define some associated curves by using the vectors of Frenet frames and characterize them as a special curve. Some of those are known as the involute-evolute curves, Bertrand curves, Mannheim curves. If a curve lacks of the second derivative, then the Frenet frame cannot be established. Therefore Bishop (1975) defined an another frame and provided the corresponding relations between his frame and Frenet frame in [2]. In Euclidean space, E^3, a spherical indicatrix curve is defined to be the locus of the end points of a unit vector settled at the center of a sphere. The arc lengths and the geodesic curvatures of these curves were studied in [4]. The idea of spherical indicatrix was extended to the Minkowski space in [3]. By using Bishop frame instead Frenet, the spherical indicatrices were given in [11]. As an extension of this to the dual space, the spherical indicatrix curves were defined according to the Dual Bishop frame in [5]. There are other studies that the spherical curves were considered in different spaces and related with some associated curves [1, 9, 10].
Motivated by these, in this study we have first established the relations between the Sannia and Frenet frame by using the Darboux vector defined by Frenet vectors. Next we have defined the derivative relations of the vectors of Sannia frame. And last, we have calculated the arclengths and the geodesic curvatures of spherical indicatrices of Sannia vectors.

2. Preliminaries

In this section, we recall some basic concepts that will be used throughout the paper. Let $\alpha = \alpha(s)$ be any differentiable curve in three dimensional Euclidean space E^3. The curvatures and the Frenet vectors of α together with the corresponding Frenet formulae are given as

$$T(s) = \frac{\alpha'(s)}{\|\alpha'(s)\|}, \quad B(s) = \frac{\alpha'(s)\wedge\alpha''(s)}{\|\alpha'(s)\wedge\alpha''(s)\|}, \quad N(s) = B(s)\wedge T(s), \tag{2.1}$$

$$\kappa = \frac{\|\alpha'\wedge\alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\langle\alpha'\wedge\alpha'', \alpha'''\rangle}{\|\alpha'\wedge\alpha''\|^2}, \tag{2.2}$$

$$T' = \kappa\nu N, \quad N' = -\kappa\nu T + \tau\nu B, \quad B' = -\tau\nu N, \quad \|\alpha'\| = \nu, \tag{2.3}$$

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where $\nu = \|\alpha'\|$, κ is the curvature and τ is the torsion of the curve [7]. It is known that the Frenet vectors rotate instantaneously along the curve and this instantaneous rotation happens around an axis spanned by a vector. This vector is called as Darboux vector and according to the definition, it has the following form:

$$W = \tau T + \kappa B \tag{2.4}$$

However, if θ is taken to be the angle between the vectors B and W, then we may write,

$$\kappa = \|W\|\cos\theta, \quad \tau = \|W\|\sin\theta, \tag{2.5}$$

and may correspondingly derive the unit Darboux vector as

$$C = sin\theta T + cos\theta B. \tag{2.6}$$

Now let us consider M as a surface in E^3, and denote ξ, S and D as the normal of surface, the shape operator and Riemann connection, respectively. For $X,Y \in \chi(M)$, the following operation,

$$\bar{D}_X Y = D_X Y + \langle S(X), Y\rangle \xi \tag{2.7}$$

is called as the Gauss equation where the operand $\bar{D}$ is the derivative operator in Gauss sense. The geodesic curvature according to the E^3 is defined as

$$k_g = \|D_T T\| \tag{2.8}$$

while it is expressed as

$$\zeta_g = \|\bar{D}_T T\| \tag{2.9}$$

according to S^2, where T is the tangent vector at the point s of $\alpha(s)$.
On the other hand, if specifically the given surface is taken to be as a ruled surface then a parametrization to this is given by

$$X(s,v) = \alpha(s) + vr(s), \tag{2.10}$$

where α is called as the base curve and $r(s)$ is the director curve. Moreover, the foot of the common perpendicular to two neighbor rulings on main ruling is known as the striction (or central) point. Therefore, the locus of these points are called as the striction curve. The equation of the striction curve on a given ruled surface, $X(s,v)$ is given by [7]

$$\beta(s) = \alpha(s) - \frac{\langle \alpha', r'\rangle}{\|r'\|^2} r. \tag{2.11}$$

If the base curve is chosen to be the predefined striction curve, then we may write the following ruled surface as

$$X(s,v) = \beta(s) + vr(s). \tag{2.12}$$

It is known that there exists an orthonormal system denoted by $\{e_1, e_2, e_3\}$ on the striction curve where the unit vectors e_i, $(i = 1,2,3)$ are defined as

$$e_1 = r, \qquad e_2 = \frac{e'_1}{\|e'_1\|}, \qquad e_3 = e_1 \wedge e_2. \tag{2.13}$$

Such an orthonormal system is known as Sannia Frame [8]. (Gustavo Sannia was an Italian mathematician lived in 1875-1930.) If k_1 and k_2 are taken to be the curvatures of the striction curve, then the Frenet formulae wise derivative changes are given by

$$e'_1 = k_1 e_2, \qquad e'_2 = -k_1 e_1 + k_2 e_3, \qquad e'_3 = -k_2 e_2. \tag{2.14}$$

3. Spherical indicatrix curves generated by Sannia frame

In this section, we first form a set of ruled surfaces by choosing the directors as the each element of Frenet frame and the Darboux vector of a given curve. Then, we calculate the corresponding striction curves of every ruled surface and construct the Sannia frame on these curves. By considering the unit vectors of Sannia frame for each case, we define the spherical indicatrix curves and calculate their arc lengths and the geodesic curvatures.

Proposition 3.1. *Let $\{e_1, e_2, e_3\}$ denote the Sannia frame on the striction curve of the ruled surface that is swept out by the tangent vector of α. Then the corresponding relationships between the elements of Sannia and Frenet frame are given as follows:*

$$e_1 = T, \qquad e_2 = N, \qquad e_3 = B. \tag{3.1}$$

Proof. By the definition given in (2.13), the proof is trivial. □

Remark 3.2. *Note that the spherical indicatrix curves generated by $\{e_1, e_2, e_3\}$ Sannia frame is same as the spherical indicatrices of Frenet frame. The characterizations of the spherical indicatrices of Frenet vectors can be found in [6].*

Proposition 3.3. *Let $\{f_1, f_2, f_3\}$ denote the Sannia frame on the striction curve of the ruled surface that is swept out by the normal vector of α. Then the corresponding relationships between the elements of Sannia and Frenet frame are given as follows:*

$$f_1 = N, \qquad f_2 = -\cos\theta T + \sin\theta B, \qquad f_3 = \sin\theta T + \cos\theta B \tag{3.2}$$

where θ is the angle between B and W.

Proof. By the equation given in (2.11), we may define the striction curve of the surface ruled by principal normal as

$$\beta(s)=\alpha(s)+\frac{\kappa}{\kappa^2+\tau^2}N.$$

Now by considering the definition of the ruled surface, X_N, it is clear that $f_1=N$. When referred the equations (2.5) and (2.13), the proof is trivial for the vectors f_2 and f_3. □

Theorem 3.4. *The relationship between the vector elements, $\{f_1,f_2,f_3\}$ of Sannia frame and their derivatives are given by the following:*

$$f_1{}'=\kappa_1 f_2,\qquad f_2{}'=-\kappa_1 f_1+\tau_1 f_3,\qquad f_3{}'=-\tau_1 f_2. \tag{3.3}$$

where $\kappa_1=\|W\|$, and $\tau_1=\theta'$

Proof. If we take the derivatives of each vector $f_1,\ f_2,\ f_3$ by considering the equations at (3.2) and referring the relation (2.5), then we complete the proof by the following calculations:

$$\begin{aligned} f'_1 &= -\kappa T+\tau B \\ &= (\kappa\cos\theta+\tau\sin\theta)\,f_2+(\tau\cos\theta-\kappa\sin\theta)\,f_3 \\ &= \sqrt{\kappa^2+\tau^2}f_2 \\ &= \|W\|\,f_2 \\ &= \kappa_1 f_2, \end{aligned}$$

$$\begin{aligned} f'_2 &= \theta'\sin\theta T-(\kappa\cos\theta+\tau\sin\theta)N+\theta'\cos\theta B \\ &= \theta'\sin\theta\,(-\cos\theta f_2+\sin\theta f_3)-(\kappa\cos\theta+\tau\sin\theta)f_1 \\ &\quad +\theta'\cos\theta\,(\sin\theta f_2+\cos\theta f_3) \\ &= -\|W\|\,f_1+\theta' f_3 \\ &= -\kappa_1 f_1+\tau_1 f_3, \end{aligned}$$

$$\begin{aligned} f_3{}' &= \theta'\cos\theta T+\kappa N\sin\theta-\theta'\sin\theta B-\tau N\cos\theta \\ &= \theta'\cos\theta\,(-\cos\theta f_2+\sin\theta f_3)-\theta'\sin\theta\,(\sin\theta f_2+\cos\theta f_3) \\ &= -\theta' f_2 \\ &= -\tau_1 f_2. \end{aligned}$$

□

Theorem 3.5. *The Darboux vector denoted by W_1 corresponding to the Sannia frame, $\{f_1,f_2,f_3\}$ is given by the following:*

$$W_1=\theta' f_1+\|W\|\,f_3. \tag{3.4}$$

Proof. Let us express the Darboux vector W_1 with the linear combinations of the vectors $f_1,\ f_2,\ f_3$ as

$$W_1=x_1f_1+y_1f_2+z_1f_3,$$

where $x_1,y_1,z_1\in R$. By taking into account the relations given in (3.3), the vector product of W_1 expressed by above with each $f_1,\ f_2,\ f_3$ results the following:

$$\begin{aligned} W_1\wedge f_1=f_1{}' &\Rightarrow -y_1f_3+z_1f_2=\|W\|\,f_2 \\ &\Rightarrow y_1=0,\quad z_1=\|W\|, \end{aligned}$$

$$\begin{aligned} W_1\wedge f_2=f_2{}' &\Rightarrow x_1f_3-z_1f_1=\|W\|\,f_1+\theta' f_3 \\ &\Rightarrow z_1=\|W\|,\quad x_1=\theta'. \end{aligned}$$

When substituted the coefficients x_1,y_1,z_1, the proof is complete. □

Corollary 3.6. *If C_1 is taken to be the unit Darboux vector, then it is stated by means of Frenet vectors as following*

$$C_1=\sin\theta_1 N+\cos\theta_1 C, \tag{3.5}$$

where θ_1 is the angle between C_1 and f_3.

Proof. By referring the relation (3.4), we may write C_1 as

$$C_1 = \frac{\theta'}{\sqrt{(\theta')^2+\|W\|^2}} f_1 + \frac{\|W\|}{\sqrt{(\theta')^2+\|W\|^2}} f_3.$$

Since θ_1 is the angle between C_1 and f_3, we write

$$\sin\theta_1 = \frac{\theta'}{\sqrt{(\theta')^2+\|W\|^2}}, \quad \cos\theta_1 = \frac{\|W\|}{\sqrt{(\theta')^2+\|W\|^2}}, \quad \theta_1 = \arctan\left(\frac{\theta'}{\|W\|}\right), \tag{3.6}$$

and therefore

$$C_1 = \sin\theta_1 f_1 + \cos\theta_1 f_3,$$

which completes the proof. □

Corollary 3.7. *The spherical indiatrix curves of f_1 and f_3 are two separate spherical involutes of the C_1 spherical indicatrix curve.*

Proof. The tangent vector of moving curve C_1 and the spherical indicatrix of C_1 is common. Since $C_1 = C_1(s)$ is defined to be the unit vector in the direction of W_1, the tangent of C_1 can be calculated by following:

$$\begin{aligned}\frac{dC_1}{ds} &= (\sin\theta_1)' f_1 + (\cos\theta_1)' f_3 + \sin\theta_1(\kappa_1 f_2) + \cos\theta_1(-\tau_1 f_2)\\ &= (\sin\theta_1)' f_1 + (\cos\theta_1)' f_3 + \underbrace{(\kappa_1\sin\theta_1 - \tau_1\cos\theta_1)}_{=0} f_2\\ &= (\sin\theta_1)' f_1 + (\cos\theta_1)' f_3.\end{aligned}$$

On the other hand, as the tangents of the spherical indicatrix of (f_1) and (f_3) are given by

$$\frac{df_1}{ds} = f_1{}' = \kappa_1 f_2,$$
$$\frac{df_3}{ds} = f_3{}' = -\tau_1 f_2,$$

we write

$$\left\langle \frac{dC_1}{ds}, \frac{df_1}{ds}\right\rangle = 0 \quad \text{and} \quad \left\langle \frac{dC_1}{ds}, \frac{df_3}{ds}\right\rangle = 0.$$

The latter expression clearly shows that (f_1) and (f_3) are the spherical involutes of C_1. □

Definition 3.8. *In Euclidean space, E^3, the curves traced out on the unit sphere by a radius of each unit vectors f_1, f_2, f_3 on the $\beta(s)$ striction curve are called as f_1- indicatrix, f_2- indicatrix and f_3- indicatrix curve and these are denoted by*

$$\beta_{f_1}(s) = f_1(s), \qquad \beta_{f_2}(s) = f_2(s), \qquad \beta_{f_3}(s) = f_3(s). \tag{3.7}$$

The corresponding arc lengths of these curves are given as follows:

$$\begin{aligned}\frac{d\beta_{f_1}}{ds_{f_1}}\frac{ds_{f_1}}{ds} = f_1{}' &\Rightarrow T_{f_1}\frac{ds_{f_1}}{ds} = \|W\| f_2\\ &\Rightarrow \frac{ds_{f_1}}{ds} = \|W\|\\ &\Rightarrow s_{f_1} = \int \|W\| ds,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\frac{d\beta_{f_2}}{ds_{f_2}}\frac{ds_{f_2}}{ds} = f_2{}' &\Rightarrow T_{f_2}\frac{ds_{f_2}}{ds} = -\|W\| f_1 + \theta' f_3\\ &\Rightarrow \frac{ds_{f_2}}{ds} = \sqrt{\|W\|^2+\theta'}\\ &\Rightarrow s_{f_2} = \int \sqrt{\|W\|^2+\theta'} ds,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\frac{d\beta_{f_3}}{ds_{f_3}}\frac{ds_{f_3}}{ds} = f_3{}' &\Rightarrow T_{f_3}\frac{ds_{f_3}}{ds} = -\theta' f_2\\ &\Rightarrow \frac{ds_{f_3}}{ds} = \theta'\\ &\Rightarrow s_{f_3} = \int \theta' ds.\end{aligned} \tag{3.10}$$

Theorem 3.9. *Let k_{f_1} denote the geodesic curvature of the f_1- indicatrix, then it is defined by*

$k_{f_1} = \sec\theta_1.$

Proof. It is clear by the relation (3.8) that $T_{f_1} = f_2$. By taking the derivative of this and considering the relations (3.3), we write

$$\frac{dT_{f_1}}{ds_{f_1}}\frac{ds_{f_1}}{ds} = f_2{}'$$

$$D_{T_{f_1}}T_{f_1} = -f_1 + \frac{\theta'}{\|W\|}f_3. \tag{3.11}$$

By taking the norm of the latter and referring the ralations given in (3.6), we complete the proof by following:

$$\begin{aligned} k_{f_1} &= \sqrt{1+\left(\frac{\theta'}{\|W\|}\right)^2} \\ &= \sqrt{1+\tan^2\theta_1} \\ &= \sec\theta_1. \end{aligned}$$

□

Theorem 3.10. *Let k_{f_2} denote the geodesic curvature of the f_2- indicatrix, then it is defined by*

$k_{f_2} = \sec\theta_1.$

Proof. By using the relation (3.9), the tangent vector of f_2- indicatrix curve can be given as

$$T_{f_2} = -\frac{\|W\|}{\sqrt{\|W\|^2+\theta'}}f_1 + \frac{\theta'}{\sqrt{\|W\|^2+\theta'}}f_3.$$

We simplify this by referring the relations given in (3.6) as

$$T_{f_2} = -\cos\theta_1 f_1 + \sin\theta_1 f_3. \tag{3.12}$$

If we take the derivative of this last expression and consider the relations (3.3) and (3.4), then we get

$$\frac{dT_{f_2}}{ds_{f_2}}\frac{ds_{f_2}}{ds} = \theta_1{}'\sin\theta_1 f_1 - \|W_1\| f_2 + \theta_1{}'\cos\theta_1 f_3,$$

$$\begin{aligned} D_{T_{f_2}}T_{f_2} &= \frac{\theta_1{}'\sin\theta_1 f_1 - \|W_1\| f_2 + \theta_1{}'\cos\theta_1 f_3}{\|W_1\|}, \\ &= \frac{\theta_1{}'}{\|W_1\|}C_1 - f_2. \end{aligned} \tag{3.13}$$

By taking the norm of the last expression, we complete the proof as like below

$$\begin{aligned} k_{f_2} &= \sqrt{1+\left(\frac{\theta_1{}'}{\|W_1\|}\right)^2} \\ &= \sqrt{1+\tan^2\theta_1} \\ &= sec\theta_1. \end{aligned}$$

□

Theorem 3.11. *Let k_{f_3} denote the geodesic curvature of the f_3- indicatrix, then it is defined by*

$k_{f_3} = \csc\theta_1.$

Proof. It is clear by the relation (3.10) that $T_{f_3} = -f_2$ By taking the derivative of this and considering the relations given in (3.3), we get

$$\frac{dT_{f_3}}{ds_{f_3}}\frac{ds_{f_3}}{ds} = -f_2{}'$$

$$D_{T_{f_3}}T_{f_3} = \frac{\|W\|}{\theta'}f_1 - f_3. \tag{3.14}$$

Similarly, by taking the norm and using the relations in (3.6), we obtain that

$$\begin{aligned} k_{f_3} &= \sqrt{1+\left(\frac{\|W\|}{\theta'}\right)^2} \\ &= \sqrt{1+\cot^2\theta_1} \\ &= \csc\theta_1, \end{aligned}$$

which completes the proof. □

Theorem 3.12. *The geodesic curvatures of f_1, f_2 and f_3 indicatrices according to S^2 are given by*

$$\zeta_{f_1} = \tan\theta_1, \quad \zeta_{f_2} = \frac{\theta_1{}'}{\|W_1\|}, \quad \zeta_{f_3} = \cot\theta_1, \tag{3.15}$$

respectively.

Proof. By using the relations (2.7), (3.11), (3.13) and (3.14), we can write

$$\begin{aligned}\bar{D}_{T_{f_1}} T_{f_1} &= D_{T_{f_1}} T_{f_1} + \left\langle S\left(T_{f_1}\right), T_{f_1}\right\rangle f_1 \\ &= \frac{\theta'}{\|W\|} f_3\end{aligned}$$

$$\begin{aligned}\bar{D}_{T_{f_2}} T_{f_2} &= D_{T_{f_2}} T_{f_2} + \left\langle S\left(T_{f_2}\right), T_{f_2}\right\rangle f_2 \\ &= \frac{\theta_1{}'}{\|W_1\|} C_1\end{aligned}$$

$$\begin{aligned}\bar{D}_{T_{f_3}} T_{f_3} &= D_{T_{f_3}} T_{f_3} + \left\langle S\left(T_{f_3}\right), T_{f_3}\right\rangle f_3 \\ &= \frac{\|W\|}{\theta'} f_1.\end{aligned}$$

Now, by referring the relations in (3.6), the proof is straightforward. □

Proposition 3.13. *Let $\{g_1, g_2, g_3\}$ denote the Sannia frame along the striction curve δ of the ruled surface, $X_B(s,v) = \alpha(s) + vB(s)$. Then the corresponding relationships between the elements of Sannia and Frenet frame are given as follows:*

$$g_1 = B, \quad g_2 = -N, \quad g_3 = T. \tag{3.16}$$

Proof. The proof is straightforward, when considered the definition of Sannia frame given in (2.13). □

Theorem 3.14. *The relationship between the Sannia vectors, $\{g_1, g_2, g_3\}$ and their derivatives are given by the following:*

$$g_1{}' = \kappa_2 g_2, \quad g_2{}' = -\kappa_2 g_1 + \tau_2 g_3, \quad g_3{}' = -\tau_2 g_2 \tag{3.17}$$

where $\kappa_2 = \tau$, and $\tau_2 = \kappa$

Proof. By considering (3.16) and taking the derivatives of each $\{g_1, g_2, g_3\}$, the proof is complete by following:

$$\begin{aligned}g_1{}' &= -\tau N = \tau g_2 = \kappa_2 g_2, \\ g_2{}' &= -N' = \kappa T - \tau B = -\kappa_2 g_1 + \tau_2 g_3, \\ g_3{}' &= T' = \kappa N = -\kappa g_2 = -\tau_2 g_2.\end{aligned}$$

□

Corollary 3.15. *The Darboux vector of the Frenet frame of α is same as of the $\{g_1, g_2, g_3\}$ Sannia Frame.*

Corollary 3.16. *The arc length and the geodesic curvatures according to both E^3 and S^2 of each spherical indicatrices of tangent, normal and binormal vectors of α are the same as of g_3-, g_2- and g_1- indicatrices, respectively.*

Proposition 3.17. *Let $\{p_1, p_2, p_3\}$ denote the Sannia frame along the striction curve γ of the ruled surface, $X_C(s,v) = \alpha(s) + vC(s)$. Then the corresponding relationships between the elements of Sannia and Frenet frame are given as follows:*

$$p_1 = \sin\theta T + \cos\theta B, \quad \mathrm{p}_2 = \cos\theta T - \sin\theta B, \quad \mathrm{p}_3 = N. \tag{3.18}$$

Proof. By using the definition of striction curve given in (2.11) we write

$$\gamma(s) = \alpha(s) - \frac{1}{\varphi'}\cos\theta C.$$

It is clear from the definition of the ruled surface $X_C(s,v)$ that $p_1 = C(s)$. By referring both (2.5) and (2.13), one can easily calculate p_2 and p_3. □

Theorem 3.18. *The relationship between the Sannia vectors, $\{p_1, p_2, p_3\}$ and their derivatives are given by the following:*

$$p_1{}' = \kappa_3 p_2, \quad p_2{}' = -\kappa_3 p_1 + \tau_3 p_3, \quad p_3{}' = -\tau_3 p_2 \tag{3.19}$$

where $\kappa_3 = \theta'$ and $\tau_3 = \|W\|$.

Proof. When taken the derivatives of each vector $p_1,\ p_2,\ p_3$ and considered the equations given in (2.5) and (3.18), the proof is done with the following:

$$\begin{aligned} {p_1}' &= \theta' \cos\theta T + \underbrace{(\kappa \sin\theta - \tau \cos\theta)}_{=0} N - \theta' \sin\theta B \\ &= \theta' \cos\theta\,(\sin\theta p_1 + \cos\theta p_2) - \theta' \sin\theta\,(\cos\theta p_1 - \sin\theta p_2) \\ &= \theta' p_2 \\ &= \kappa_3 p_2, \end{aligned}$$

$$\begin{aligned} {p_2}' &= -\theta' \sin\theta T + (\kappa \cos\theta + \tau \sin\theta) N - \theta' \cos\theta B \\ &= -\theta' \sin\theta\,(\sin\theta p_1 + \cos\theta p_2) + (\kappa \cos\theta + \tau \sin\theta)\, p_3 \\ &\quad - \theta' \cos\theta\,(\cos\theta p_1 - \sin\theta p_2) \\ &= -\theta' p_1 + \|W\|\, p_3 \\ &= -\kappa_3 p_1 + \tau_3 p_3, \end{aligned}$$

$$\begin{aligned} {p_3}' &= -\kappa T + \tau B \\ &= -\kappa\,(\sin\theta p_1 + \cos\theta p_2) + \tau\,(\cos\theta p_1 - \sin\theta p_2) \\ &= -\|W\|\, p_2 \\ &= -\tau_3 p_2. \end{aligned}$$

□

Theorem 3.19. *The unit Darboux vector W_2 of corresponding Sannia frame, $\{p_1, p_2, p_3\}$ is given by*

$$W_2 = \|W\|\, p_1 + \theta' p_3. \tag{3.20}$$

Proof. The Darboux vector, W_2 can be expressed as the linear combination of $\{p_1, p_2, p_3\}$ as

$$W_2 = x_2 p_1 + y_2 p_2 + z_2 p_3,$$

where $x_2, y_2, z_2 \in R$. When considered the relation (3.19) and applied the vector production of W_2 with each $p_1,\ p_2,\ p_3$ the corresponding coefficients can be found as

$$\begin{aligned} W_2 \wedge p_1 = {p_1}' &\Rightarrow -y_2 p_3 + z_2 p_2 = \theta' p_2 \\ &\Rightarrow y_2 = 0, \quad z_2 = \theta', \end{aligned}$$

$$\begin{aligned} W_2 \wedge p_2 = {p_2}' &\Rightarrow x_2 p_3 - z_2 p_1 = -\theta' p_1 + \|W\|\, p_3 \\ &\Rightarrow x_2 = \|W\|, \end{aligned}$$

which completes the proof. □

Corollary 3.20. *If C_2 is considered to be the unit Darboux vector, then by means of Frenet vectors, it has the following equation:*

$$C_2 = \sin\theta_2 C + \cos\theta_2 N, \tag{3.21}$$

where θ_2 is the angle between C_2 and p_3.

Proof. By referring the relation (3.20), it is easy to write C_2 as

$$C_2 = \frac{W_2}{\|W_2\|} = \frac{\|W\|}{\sqrt{(\theta')^2 + \|W\|^2}} p_1 + \frac{\theta'}{\sqrt{(\theta')^2 + \|W\|^2}} p_3.$$

Now since θ_2 is the angle between C_2 and p_3, we may write

$$\sin\theta_2 = \frac{\|W\|}{\sqrt{(\theta')^2 + \|W\|^2}}, \quad \cos\theta_2 = \frac{\theta'}{\sqrt{(\theta')^2 + \|W\|^2}}, \quad \theta_2 = \arctan\left(\frac{\|W\|}{\theta'}\right). \tag{3.22}$$

Hence

$$C_2 = \sin\theta_2 p_1 + \cos\theta_2 p_3,$$

which completes the proof. □

Corollary 3.21. *The spherical indiatrix curves of p_1 and p_3 are two separate spherical involutes of the C_2 spherical indicatrix curve.*

Proof. The tangent vectors of moving curve C_2 and the spherical indicatrix of C_2 are common. Since the curve (C_2) with $C_2 = C_2(s)$ is defined to be the unit vector in the direction of W_2, the tangent vector of (C_2) can be calculated by following:

$$\begin{aligned} C_2 &= \sin\theta_2 p_1 + \cos\theta_2 p_3, \\ \frac{dC_2}{ds} &= (\sin\theta_2)' p_1 + (\cos\theta_2)' p_3 + \sin\theta_2 (\kappa_2 p_2) + \cos\theta_2 (-\tau_2 p_2) \\ &= (\sin\theta_2)' p_1 + (\cos\theta_2)' p_3 + \underbrace{(\kappa_2 \sin\theta_2 - \tau_2 \cos\theta_2)}_{=0} p_2 \\ &= (\sin\theta_2)' p_1 + (\cos\theta_2)' p_3. \end{aligned}$$

On the other hand, since the tangents of the spherical indicatrix of (p_1) and (p_3) are

$$\begin{aligned} \frac{dp_1}{ds} &= p_1{}' = \kappa_2 p_2, \\ \frac{dp_3}{ds} &= p_3{}' = -\tau_2 p_2. \end{aligned}$$

Thus, we write

$$\left\langle \frac{dC_2}{ds}, \frac{dp_1}{ds} \right\rangle = 0, \quad \text{and} \quad \left\langle \frac{dC_2}{ds}, \frac{dp_3}{ds} \right\rangle = 0.$$

This clearly means that (p_1) and (p_3) are two spherical involutes of C_2. □

Definition 3.22. *In E^3, the curves traced out on the unit sphere by a radius of each unit vectors p_1, p_2, p_3 of the striction curve $\gamma(s)$ are called as p_1- indicatrix, p_2- indicatrix and p_3- indicatrix curve and we denote them as*

$$\gamma_{p_1}(s) = p_1(s), \quad \gamma_{p_2}(s) = p_2(s), \quad \gamma_{p_3}(s) = p_3(s). \tag{3.23}$$

The arc lengths of these curves are calculated as like below:

$$\begin{aligned} \frac{d\gamma_{p_1}}{ds_{p_1}} \frac{ds_{p_1}}{ds} = p_1{}' &\Rightarrow T_{p_1} \frac{ds_{p_1}}{ds} = \theta' p_2 \\ &\Rightarrow \frac{ds_{p_1}}{ds} = \theta' \\ &\Rightarrow s_{p_1} = \int \theta' ds, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \frac{d\gamma_{p_2}}{ds_{p_2}} \frac{ds_{p_2}}{ds} = p_2{}' &\Rightarrow T_{p_2} \frac{ds_{p_2}}{ds} = -\theta' p_1 + \|W\| p_3 \\ &\Rightarrow \frac{ds_{p_2}}{ds} = \sqrt{\|W\|^2 + \theta'} \\ &\Rightarrow s_{p_2} = \int \sqrt{\|W\|^2 + \theta'} ds, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \frac{d\gamma_{p_3}}{ds_{p_3}} \frac{ds_{p_3}}{ds} = p_3{}' &\Rightarrow T_{p_3} \frac{ds_{p_3}}{ds} = -\|W\| p_2 \\ &\Rightarrow \frac{ds_{p_3}}{ds} = \|W\| \\ &\Rightarrow s_{p_3} = \int \|W\| ds. \end{aligned} \tag{3.26}$$

Theorem 3.23. *Let k_{p_1} denote the geodesic curvature of the p_1- indicatrix, then it is defined by*

$$k_{p_1} = \sec\theta_2.$$

Proof. By the relation (3.24) it is clearly seen that $T_{p_1} = p_2$. By taking the derivative of this and considering the relations (3.19), we write

$$\begin{aligned} \frac{dT_{p_1}}{ds_{p_1}} \frac{ds_{p_1}}{ds} &= p_2{}' \\ D_{T_{p_1}} T_{p_1} &= -p_1 + \frac{\|W\|}{\theta'} p_3. \end{aligned} \tag{3.27}$$

Next taking the norm of the latter and referring the ralation in (3.22), complete the proof as following:

$$\begin{aligned} k_{p_1} &= \sqrt{1 + \left(\frac{\theta'}{\|W\|} \right)^2} \\ &= \sqrt{1 + \tan^2\theta_2} \\ &= \sec\theta_2. \end{aligned}$$

□

Theorem 3.24. *Let k_{p_2} denote the geodesic curvature of the p_2- indicatrix, then it is given by*

$$k_{p_2} = \sqrt{1+\left(\frac{\theta_2{}'}{\|W_2\|}\right)^2}.$$

Proof. By the relation (3.25), the tangent vector of p_2- indicatrix curve is written as

$$T_{p_2} = \frac{-\theta'}{\sqrt{\|W\|^2+\theta'}}p_1 + \frac{\|W\|}{\sqrt{\|W\|^2+\theta'}}p_3.$$

We may express this by considering (3.22) as

$$T_{p_2} = -\cos\theta_2 p_1 + \sin\theta_2 p_3.$$

If we take the derivative of the above expression with respect to s and consider the relations (3.19) and (3.20), then we get

$$\frac{dT_{p_2}}{ds_{p_2}}\frac{ds_{p_2}}{ds} = \theta_2{}'\sin\theta_2 p_1 - \|W_2\|\,p_2 + \theta_2{}'\cos\theta_2 p_3$$

$$D_{T_{p_2}}T_{p_2} = \frac{\theta_2{}'\sin\theta_2 p_1 - \|W_2\|\,p_2 + \theta_2{}'\cos\theta_2 p_3}{\|W_2\|}. \tag{3.28}$$

Taking the norm as a last step, we get

$$k_{p_2} = \sqrt{1+\left(\frac{\theta_2{}'}{\|W_2\|}\right)^2},$$

and complete the proof. □

Theorem 3.25. *Let k_{p_3} denote the geodesic curvature of the p_3- indicatrix, then it is defined by*

$$k_{p_3} = \csc\theta_2.$$

Proof. As similar before it is clear that $T_{p_3} = -p_2$ by the relation (3.26). Now taking the derivative of this and considering the relations given in (3.19), we have

$$\frac{dT_{p_3}}{ds_{p_3}}\frac{ds_{p_3}}{ds} = -p_2{}'$$

$$D_{T_{p_3}}T_{p_3} = \frac{\theta'}{\|W\|}p_1 - p_3. \tag{3.29}$$

By taking the norm and using the relations in (3.22), we complete the proof by

$$\begin{aligned} k_{p_3} &= \sqrt{1+\left(\frac{\theta'}{\|W\|}\right)^2} \\ &= \sqrt{1+\cot^2\theta_2} \\ &= \csc\theta_2. \end{aligned}$$

□

Theorem 3.26. *If μ_{p_1}, μ_{p_2} and μ_{p_3} denote the geodesic curvatures of p_1, p_2 and p_3 indicatrices according to S^2, then they are defined as following:*

$$\mu_{p_1} = \tan\theta_2, \quad \mu_{p_2} = \frac{\theta_2{}'}{\|W_2\|}, \quad \mu_{p_3} = \cot\theta_2, \tag{3.30}$$

respectively.

Proof. By using the Gauss equation in (2.7) and the relation (3.27), we can write

$$\begin{aligned} \bar{D}_{T_{p_1}}T_{p_1} &= D_{T_{p_1}}T_{p_1} + \langle S(T_{p_1}), T_{p_1}\rangle p_1, \\ &= \frac{\|W\|}{\theta'}p_3. \end{aligned}$$

Now taking the norm of this and using (3.22) result

$$\mu_{p_1} = \frac{\|W\|}{\theta'} = \tan\theta_2.$$

By referring this time, the relation (3.28) with again the Gauss equation (2.7), we have the following:

$$\begin{aligned} \bar{D}_{T_{p_2}}T_{p_2} &= D_{T_{p_2}}T_{p_2} + \langle S(T_{p_2}), T_{p_2}\rangle p_2, \\ &= \frac{\theta_2{}'\sin\theta_2 p_1 + \theta_2{}'\cos\theta_2 p_3}{\|W_2\|}. \end{aligned}$$

Here, if we take the norm, then

$$\mu_{p_2}=\frac{\theta_2{}'}{\|W_2\|}.$$

Lastly, when considered (3.29) with (2.7), we obtain

$$\begin{aligned}\bar{D}_{T_{p_3}}T_{p_3}&=D_{T_{p_3}}T_{p_3}+\left\langle S\left(T_{p_3}\right),T_{p_3}\right\rangle p_3,\\&=\frac{\theta'}{\|W\|}p_1.\end{aligned}$$

By the norm of this and the relation (3.22), we find

$$\mu_{p_3}=\frac{\theta'}{\|W\|}=\cot\theta_2,$$

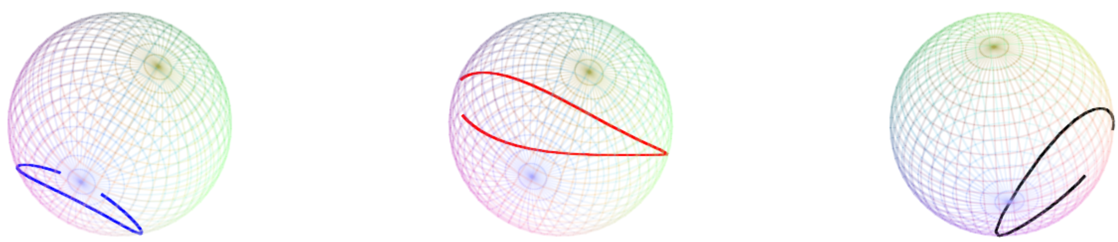
which completes the proof. □

Example 3.27. *Let us consider a simple twisted cubic curve as $\alpha(s)=\left(s,s^2,s^3\right)$. The corresponding Frenet apparatus and the Darboux vector of $\alpha=\alpha(s)$ are as fallows*

$$T(s)=\frac{\left(1,2s,3s^2\right)}{\sqrt{9s^4+4s^2+1}},\qquad N(s)=\frac{\left(-s\left(9s^2+2\right),-9s^4+1,3s\left(2s^2+1\right)\right)}{\sqrt{\left(9s^4+4s^2+1\right)\left(9s^4+9s^2+1\right)}},\qquad B(s)=\frac{\left(3s^2,-3s,1\right)}{\sqrt{9s^4+9s^2+1}},$$

$$\kappa(s)=\frac{2\sqrt{9s^4+9s^2+1}}{\left(9s^4+4s^2+1\right)^{\frac{3}{2}}},\qquad \tau(s)=\frac{3}{9s^4+9s^2+1},\qquad W(s)=\frac{\left(3,6s,9s^2\right)}{\left(9s^4+9s^2+1\right)\sqrt{9s^4+4s^2+1}}+\frac{\left(6s^2,-6s,2\right)}{\left(9s^4+4s^2+1\right)^{\frac{3}{2}}}.$$

According to the propositions (3.1) and (3.13), the spherical indicatrix curves of $\{e_1,\ e_2,\ e_3\}$ and $\{g_1,\ g_2,\ g_3\}$ Sannia frames and $\{T,\ N,\ B\}$ Frenet frame are same and these are illustrated in figure 3.1.



(a) e_1-, $\mathbf{g_3}-$ and $T-$ indicatrix curve (b) e_2-, $\mathbf{g_2}-$ and $N-$ indicatrix curve (c) e_3-, $\mathbf{g_1}-$ and $B-$ indicatrix curve

Figure 3.1: Spherical indicatrix curves of $\{e_1,\ e_2,\ e_3\}$ and $\{g_1,\ g_2,\ g_3\}$ Sannia frames and $\{T,\ N,\ B\}$ Frenet frame

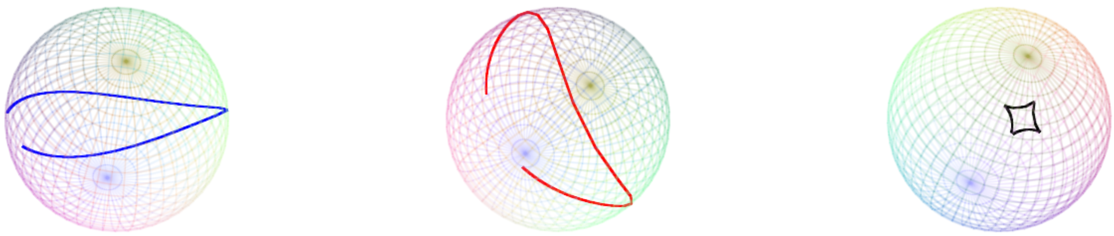
On the other hand, according to the propositions (3.3) and (3.17), the parametric form of the spherical indicatrix curves by $\{f_1,f_2,f_3\}$ and $\{p_1,p_2,p_3\}$ Sannia frames on the striction curve of the principal normal and unit Darboux ruled surface of α are given in the following:

$$f_1(s)=p_3(s)=\frac{\left(-s\left(9s^2+2\right),-9s^4+1,3s\left(2s^2+1\right)\right)}{\sqrt{9s^4+4s^2+1}\sqrt{9s^4+9s^2+1}},$$

$$f_2(s)=-p_2(s)=\frac{\begin{pmatrix}729s^{10}+486s^8-18s^6-126s^4-27s^2-2,-s\left(1053s^8+1296s^6+702s^4+144s^2+13\right),\\486s^{10}+729s^8+378s^6+6s^4-18s^2-3\end{pmatrix}}{\sqrt{\left(9s^4+9s^2+1\right)\left(9s^4+4s^2+1\right)\left(9477s^{12}+17496s^{10}+15795s^8+7380s^6+1755s^4+216s^2+13\right)}},$$

$$f_3(s)=p_1(s)=\frac{\left(3\left(18s^6+27s^4+6s^2+1\right),-30s^3,81s^6+54s^4+27s^2+2\right)}{\sqrt{9477s^{12}+17496s^{10}+15795s^8+7380s^6+1755s^4+216s^2+13}},$$

and the illustration of these curves are presented in figure 3.2.



(a) f_1- and p_3- indicatrix curve (b) f_2- and p_2- indicatrix curve (c) f_3- and p_1- indicatrix curve

Figure 3.2: Spherical indicatrix curves of both $\{f_1,\ f_2,\ f_3\}$ and $\{p_1, p_2, p_3\}$ Sannia frame

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