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Existence and Uniqueness Results of Hadamard Fractional Volterra-Fredholm Integro-Differential Equations

Ahmed A. Hamoud[1*], Abdulrahman A. Sharif[2], Aishwary K. Ghadle[3] and Kirtiwant P. Ghadle[3]

[1]*Department of Mathematics, Taiz University, Taiz-380 015, Yemen.*
[2]*Department of Mathematics, Hodeidah University, AL-Hudaydah, Yemen.*
[3]*Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India.*
[*]*Corresponding author*

Abstract

In this paper, we establish some new conditions for the existence and uniqueness of solutions for a class of nonlinear Hadamard fractional Volterra-Fredholm integro-differential equations with initial conditions. The desired results are proved by using Arzelá-Ascoli theorem aid of fixed point theorems due to Banach and Krasnoselskii in Banach spaces.

Keywords: *Volterra-Fredholm integro-differential equation; Hadamard fractional derivative; Fixed point method.*
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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration of arbitrary order, which can be noninteger. Differential equations of fractional order have attracted the attention of several researchers; see the monographs [18, 20] and references therein. In the literature, there exist several definitions of fractional integrals and derivatives, from the most popular Riemann-Liouville and Caputo-type fractional derivatives to the other ones such as Hadamard fractional derivative, the Erdelyi-Kober fractional derivative, and so forth. Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering, and other applied sciences [16, 19].

Lately, there has been a developing interest for the fractional integro-differential equations (FIDEs). FIDEs have been recently used as effective tools in the modeling of many phenomena in various fields of applied sciences and engineering such as acoustic control, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, proteins, electromagnetics, optics, medicine, economics, astrophysics, chemical engineering, chaotic dynamics, statistical physics and so on [6, 8, 9, 10, 11, 12, 13, 14, 15, 19, 20, 21, 23, 24]. Many problems can be modeled by FIDE from various sciences and engineering applications.

The Hadamard fractional derivative, introduced by Hadamard in 1892, differs from the Riemann-Liouville and Caputo derivatives in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains a logarithmic function of arbitrary exponent [1, 7]. Recently, the study of Hadamard fractional derivative of differential equations is also of great importance. There has been a significant development in Hadamard derivative of differential equations in recent years for detail study on Hadamard fractional derivative, we refer to [4, 17].

Recently, Wang et al. [22] discussed the existence and stability of fractional differential equations with Hadamard derivative

$$\begin{aligned} &{}^{H}D^{\nu}_{1,t}u(t) = f(t,u(t)),\ t \in J = (1,b],\ 0 < \nu < 1,\\ &{}^{H}D^{\nu}_{1,t}u(1) = d,\ d \in \mathbb{R}. \end{aligned}$$

Ahmad and Ntouyas [5] studied the existence of solutions for an initial-value problem of hybrid fractional differential equations of Hadamard type given by

$$\begin{aligned} &{}^{H}D^{\nu}\frac{u(t)}{g(t,u(t))} = f(t,u(t)),\ t \in J = [1,T],\ 0 < \nu \leq 1,\\ &{}^{H}I^{1-\nu}u(t)|_{t=1} = d,\ d \in \mathbb{R}. \end{aligned}$$

Email addresses: **drahmedselwi985@gmail.com (A.A. Hamoud), abdul.sharef1985@gmail.com (A.A. Sharif), aishwariyghadle@gmail.com (A.K. Ghadle), drkp.ghadle@gmail.com (K.P. Ghadle)**

Abbas et al. [1] studied the stability for nonlinear Hilfer-Hadamard fractional differential equations of the form

$$ {}^{H}D_1^{\nu,\rho}u(t) = f(t,u(t),{}^{H}D_1^{\nu,\rho}u(t)),\ t\in J=[1,+\infty),\ 0<\nu<1,\ 0\le\rho\le1, $$
$$ {}^{H}I_1^{1-\gamma}u(t)|_{t=1} = u_0,\ \nu\le\gamma=\nu+\rho-\nu\rho. $$

Abbas et al. [2] discussed the existence and the Ulam stability of solutions for the following problem of fractional differential equations of the form

$$ {}^{H}D_1^{\nu,\rho}u(t) = f(t,u(t)),\ t\in J=[1,T],\ 0<\nu<1,\ 0\le\rho\le1, $$
$$ {}^{H}I_1^{1-\gamma}u(t)|_{t=1} = u_0,\ \nu\le\gamma=\nu+\rho-\nu\rho. $$

Ahmed et al. [3] discussed the existence of solutions by means of endpoint theory for initial value problem of Hadamard and Riemann-Liouville fractional integro-differential inclusions of the form

$$ {}^{H}D^{\nu}(u(t)-\sum_{i=1}^{m} I_i^{\rho}G_i(t,u(t))\in F(t,u(t)),\ t\in J:=[1,T],\ 0<\nu\le1, $$
$$ u(1)=0, $$

Motivated by the above works, we will study a more general problem of fractional integro-differential equations wich called Riemann-Liouville-Hadamard fractional Volterra-Fredholm integro-differential equations of the form

$$ {}^{H}D^{\nu}(u(t)-\sum_{i=1}^{m} I_i^{\rho}G_i(t,u(t))=F(t,u(t),Ku(t),Hu(t)),\ t\in J:=[1,T],\ 0<\nu\le1, \tag{1.1} $$
$$ u(1)=0, \tag{1.2} $$

where ${}^{H}D^{\nu}$ is the Hadamard fractional derivative of order ν, and I^{β} is the Riemann-Liouville fractional integral of order $\beta>0$, $\beta=\rho_i$, $i=1:m$. The $F:J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function, K and H are linear integral operators defined by $(Ku)(t)=\int_1^t k(t,s)u(s)ds$ and $(Hu)(t)=\int_1^T h(t,s)u(s)ds$, with $\varphi_1=\sup\{|k(t,s)|:(t,s)\in J\times J\}$ and $\varphi_2=\sup\{|h(t,s)|:(t,s)\in J\times J\}$.
The paper is organized as follows: Sect. 2, presents, as preliminaries, the definition of the Hadamard fractional derivative, Riemann-Liouville fractional integral, and some important results, given as theorems, as well as the spaces in which such operators and theorems are defined. In Sect. 3, we use the fixed point theorems due to Banach and Krasnoselskii to prove the existence and uniqueness results for the problem (1.1)-(1.2). In Sect. 4, concluding remarks close the paper.

2. Preliminaries

Let us first recall some basic definitions, propositions and lemmas, which will be used throughout the work.
Let $C(J,\mathbb{R})$ denotes the Banach space of continuous function on J with the norm

$$ \|u\|_C := \sup\{u(t): t\in J\}. $$

Definition 2.1. *[1, 20] The Hadamard derivative of fractional order ν for a function $h:[1,\infty)\longrightarrow\mathbb{R}$ is defined as*

$$ {}^{H}D^{\nu}h(t) = \frac{1}{\Gamma(n-\nu)}\left(t\frac{d}{dt}\right)^n\int_1^t\left(\log\frac{t}{s}\right)^{n-\nu-1}h(s)\frac{ds}{s},\ n-1<\nu<n. \tag{2.1} $$

where $n=[\nu]+1$, and $[\nu]$ denotes the integer part of real number ν and $\log(.)=\log_e(.)$.

Definition 2.2. *[1] The Hadamard fractional integral of order ν for a function h is defined as*

$$ {}^{H}I^{\nu}h(t) = \frac{1}{\Gamma(\nu)}\int_1^t\left(\log\frac{t}{s}\right)^{\nu-1}h(s)\frac{ds}{s},\ \nu>0, $$

provided the integral exists.

Definition 2.3. *[25] The Riemann-Liouville fractional integral of order $\nu>0$ of a function f is defined as*

$$ \begin{aligned} J^{\nu}h(t) &= \frac{1}{\Gamma(\nu)}\int_0^t(t-s)^{\nu-1}h(s)ds, \qquad t>0,\quad \nu\in\mathbb{R}^+, \\ J^0h(t) &= h(t), \end{aligned} \tag{2.2} $$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2.4. *[20] The Riemann-Liouville derivative of order ν with the lower limit zero for a function $h:[0,\infty)\longrightarrow\mathbb{R}$ can be written as*

$$ D^{\nu}h(t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t\frac{h(s)}{(t-s)^{\nu}}ds,\ t>0,\ 0<\nu<1. \tag{2.3} $$

Theorem 2.5. *[25] (Banach's fixed point theorem) Let (X,d) be a nonempty complete metric space with $T:X\longrightarrow X$ is a contraction mapping. Then map T has a fixed point $x^*\in X$such that $Tx^*=x^*$.*

Theorem 2.6. *[19] (Krasnoselskii's fixed point theorem) Let M be a closed convex and nonempty subset of a Banach space X. Let A,B be two operators such that:*
1. $Ax+By\in M$ whenever $x,y\in M$.
2. A is compact and continuous.
3. B is a contraction mapping.
Then there exists $z\in M$ such that $z=Az+Bz$.

3. Existence and uniqueness results

In this section, we shall give an existence and uniqueness results of Eq.(1.1), with the conditions (1.2). Before starting and proving the main results, we introduce the following hypotheses:
(A1) There exists a constant $L_g > 0$, such that

$|G_i(t,u(t)) - G_i(t,v(t))| \leq L_g|u(t)-v(t)|, \ \forall t \in J, \ u,v \in \mathbb{R}.$

(A2) There exist functions $\mu(t), \mu_1(t), \mu_2(t)$ and $\theta_i(t) \in C(J,\mathbb{R})$ such that

$$|F(t,u,v,y)| \leq \mu(t) + \mu_1(t)|v| + \mu_2(t)|y|, \ \forall (t,u,v,y) \in J \times \mathbb{R}^3,$$
$$|G_i(t,u)| \leq \theta_i(t), \ \forall (t,u) \in J \times \mathbb{R}.$$

Setting $\sup_{t\in J}|\mu(t)| = \|\mu\|, \ \sup_{t\in J}|\mu_1(t)| = \|\mu_1\|, \ \sup_{t\in J}|\mu_2(t)| = \|\mu_2\|, \sup_{t\in J}|\theta_i(t)| = \|\theta_i\|, \ i = 1:m.$
(A3) There exist constants $L_1, L_2, L_3 > 0$ such that

$$|F(t,u_1,v_1,y_1) - F(t,u_2,v_2,y_2)| \leq L_1|u_1-u_2| + L_2|v_1-v_2| + L_3|y_1-y_2|, \ \forall t \in J, u_i,v_i,y_i \in \mathbb{R}, i = 1,2.$$

Lemma 3.1. *Let $0 < \nu \leq 1$. Assume that $F(.,u(.),Hu(.),Ku(.)) \in C[J,X]$. Then u satisfies the problem (1.1)-(1.2) if and only if u satisfies the mixed type integral equation*

$$u(t) = \frac{1}{\Gamma(\nu)}\int_1^t \left(\log\frac{t}{s}\right)^{\nu-1} F(\tau,u(\tau),(Ku)(\tau),(Hu)(\tau))\frac{d\tau}{\tau} + \sum_{i=1}^m I^{\rho_i}G_i(t,u(t)), \ t \in J. \tag{3.1}$$

Theorem 3.2. *Assume that the hypothesis (A1)-(A2) are fulfilled, and if*

$$L_g\sum_{i=1}^m \frac{(T-1)^{\rho_i}}{\Gamma(\rho_i+1)} < 1.$$

Then there exists at least one solution for the problem (1.1)-(1.2).

Proof. Consider the operator $T : C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ defined by

$$(Tu)(t) = \frac{1}{\Gamma(\nu)}\int_1^t \left(\log\frac{t}{s}\right)^{\nu-1} F(\tau,u(\tau),(Ku)(\tau),(Hu)(\tau))\frac{d\tau}{\tau} + \sum_{i=1}^m I^{\rho_i}G_i(t,u(t)).$$

Consider the ball $B_r = \{u \in C(J,\mathbb{R}) : \|u\| \leq r\}$ with $r > 0$, where

$$\frac{\left(\sum_{i=1}^m \frac{(T-1)^{\rho_i}}{\Gamma(\rho_i+1)}\|\theta_i\| + \frac{(\log T)^\nu}{\Gamma(\nu+1)}\|\mu\|\right)}{\left(1-(\varphi_1\|\mu_1\|+\varphi_2\|\mu_2\|)\left[\frac{\gamma}{\Gamma(\nu)}+\frac{(\log T)^\nu}{\Gamma(\nu+1)}\right]\right)} \leq r.$$

$\gamma = T\int_0^{\log T} u^{\nu-1}e^{-u}$ and $(\varphi_1\|\mu_1\|+\varphi_2\|\mu_2\|)\|\left[\frac{\gamma}{\Gamma(\nu)}+\frac{(\log T)^\nu}{\Gamma(\nu+1)}\right] < 1$. We define the operators P and Q such that $T = P+Q$, by

$$(Pu)(t) = \frac{1}{\Gamma(\nu)}\int_1^t \left(\log\frac{t}{s}\right)^{\nu-1} F(\tau,u(\tau),(Ku)(\tau),(Hu)(\tau))\frac{d\tau}{\tau},$$
$$(Qu)(t) = \sum_{i=1}^m \frac{1}{\Gamma(\rho_i)}\int_1^t (t-s)^{\rho_i-1}G_i(\tau,u(\tau))d\tau, \ t \in J. \tag{3.2}$$

For any $u,v \in B_r$, we have

$$\begin{aligned}
|(Pu)(t)+(Qu)(t)| &\leq \frac{1}{\Gamma(\nu)}\int_1^t \left(\log\frac{t}{s}\right)^{\nu-1}|F(\tau,u(\tau),Ku(\tau),Hu(\tau))|\frac{d\tau}{\tau} + \sum_{i=1}^m\frac{1}{\Gamma(\rho_i)}\int_1^t (t-s)^{\rho_i-1}|G_i(\tau,u(\tau))|d\tau \\
&\leq \frac{1}{\Gamma(\nu)}\int_1^t \left(\log\frac{t}{s}\right)^{\nu-1}(|\mu(\tau)|+|\mu_1(\tau)||Ku(\tau)|+|\mu_2(\tau)||Hu(\tau)|)\frac{d\tau}{\tau} + \sum_{i=1}^m\frac{1}{\Gamma(\rho_i)}\int_1^t (t-s)^{\rho_i-1}|\theta_i(\tau)|d\tau \\
&\leq \sum_{i=1}^m\frac{(T-1)^{\rho_i}}{\Gamma(\rho_i+1)}\|\theta_i\| + \frac{(\log T)^\nu}{\Gamma(\nu+1)}\|\mu\| + (\varphi_1\|\mu_1\|+\varphi_2\|\mu_2\|)r\left[\frac{\gamma}{\Gamma(\nu)}+\frac{(\log T)^\nu}{\Gamma(\nu+1)}\right] \\
&\leq r.
\end{aligned}$$

Next we will show that P is continuous and compact. The operator P is obviously continuous. Also, P is uniformly bounded on B_r as

$$\|Pu\| \leq \frac{(\log T)^\nu}{\Gamma(\nu+1)}\|\mu\| + (\varphi_1\|\mu_1\|+\varphi_2\|\mu_2\|)r\left[\frac{\gamma}{\Gamma(\nu)}+\frac{(\log T)^\nu}{\Gamma(\nu+1)}\right].$$

Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $u \in B_r$. Then we have

$$\begin{aligned}
|(Pu)(t_2)-(Pu)(t_1)| &= |\frac{1}{\Gamma(\nu)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\nu-1}F(\tau,u(\tau),(Ku)(\tau),(Hu)(\tau))\frac{d\tau}{\tau} - \frac{1}{\Gamma(\nu)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\nu-1}F(\tau,u(\tau),(Ku)(\tau),(Hu)(\tau))\frac{d\tau}{\tau} \\
&\leq \frac{\|\mu\|+(\varphi_1\|\mu_1\|+\varphi_2\|\mu_2\|)r(T-1)}{\Gamma(\nu+1)}\left[(\log t_1)^\nu - (\log t_2)^\nu\right] \\
&\longrightarrow 0 \ \text{ as } \ t_2 \longrightarrow t_1.
\end{aligned}$$

Thus, P is equicontinuous. So P is relatively compact on B_r. Hence, by the Arzelá-Ascoli theorem, P is compact on B_r. Now we show that Q is a contraction mapping. Let $u,v\in B_r$. Then, for $t\in J$, we have

$$\begin{aligned}\Big|(Qu)(t)-(Qv)(t)\Big| &= \Big|\sum_{i=1}^{m}\frac{1}{\Gamma(\rho_i)}\int_1^t (t-s)^{\rho_i-1}G_i(\tau,u(\tau))d\tau-\sum_{i=1}^{m}\frac{1}{\Gamma(\rho_i)}\int_1^t (t-s)^{\rho_i-1}G_i(\tau,v(\tau))d\tau\Big|\\ &\leq \sum_{i=1}^{m}\frac{1}{\Gamma(\rho_i)}\int_1^t (t-s)^{\rho_i-1}\Big|G_i(\tau,u(\tau))-G_i(\tau,v(\tau))\Big|d\tau\\ &\leq \sum_{i=1}^{m}\frac{1}{\Gamma(\rho_i)}\int_1^t (t-s)^{\rho_i-1}L_g\|u-v\|d\tau\\ &\leq \sum_{i=1}^{m}\frac{L_g(T-1)^{\rho_i}}{\Gamma(\rho_i+1)}\|u-v\|.\end{aligned}$$

Hence, T is a contraction mapping. Thus all the assumptions of Krasnoselskii fixed point theorem are satisfied, which implies that the problem (1.1)-(1.2) has at least one solution on J, and the proof is completed. □

Theorem 3.3. *Assumes that (A1) and (A3) hold, and if*

$$\phi:=\sum_{i=1}^{m}\frac{L_g(T-1)^{\rho_i}}{\Gamma(\rho_i+1)}+L_1\frac{(\log T)^{\nu}}{\Gamma(\nu+1)}+(\varphi_1L_1+\varphi_2L_2)\Big[\frac{\gamma}{\Gamma(\nu)}+\frac{(\log T)^{\nu}}{\Gamma(\nu+1)}\Big]<1. \tag{3.3}$$

Then there exists a unique solution to the problem (1.1)-(1.2).

Proof. Consider the operator T defined by Theorem 3.2 and we show that T has a unique fixed point, which is a unique solution of the problem (1.1)-(1.2).
Let us fix $N=\sup_{t\in J}|F(t,0,0,0)|, g_i=\sup_{t\in J}|G_I(t,0)|$, and choose $M=\sum_{i=1}^{m}g_i\frac{(T-1)^{\rho_i}}{\Gamma(\rho_i+1)}+N\frac{(\log T)^{\nu}}{\Gamma(\nu+1)},\ \frac{M}{1-\phi}\leq r$. Then we show that $TB_r\subset B_r$. For $u\in B_r$, we have

$$\begin{aligned}|(Tu)(t)| &= \Big|\frac{1}{\Gamma(\nu)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\nu-1}F(\tau,u(\tau),(Ku)(\tau),(Hu)(\tau))\frac{d\tau}{\tau}+\sum_{i=1}^{m}\frac{1}{\Gamma(\rho_i)}\int_1^t(t-s)^{\rho_i-1}G_i(\tau,u(\tau))d\tau\Big|\\ &\leq \sup_{t\in J}\Big[\frac{1}{\Gamma(\nu)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\nu-1}|F(\tau,u(\tau),(Ku)(\tau),(Hu)(\tau))|\frac{d\tau}{\tau}+\sum_{i=1}^{m}\frac{1}{\Gamma(\rho_i)}\int_1^t(t-s)^{\rho_i-1}|G_i(\tau,u(\tau))|d\tau\Big]\\ &\leq \sup_{t\in J}\Big[\frac{1}{\Gamma(\nu)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\nu-1}(|F(\tau,u(\tau),(Ku)(\tau),(Hu)(\tau))-F(\tau,0,0,0)|+|F(\tau,0,0,0)|)\frac{d\tau}{\tau}\\ &\quad+\sum_{i=1}^{m}\frac{1}{\Gamma(\rho_i)}\int_1^t(t-s)^{\rho_i-1}(|G_i(\tau,u(\tau))-G_i(\tau,0)|+|G_i(\tau,0)|)d\tau\Big]\\ &\leq \sum_{i=1}^{m}\frac{(T-1)^{\rho_i}}{\Gamma(\rho_i+1)}(rL_g+g_i)+(rL_1+N)\frac{(\log T)^{\nu}}{\Gamma(\nu+1)}+(\varphi_1L_1+\varphi_2L_2)r\Big[\frac{\gamma}{\Gamma(\nu)}+\frac{(\log T)^{\nu}}{\Gamma(\nu+1)}\Big]\\ &= r\phi+M\\ &\leq r,\end{aligned}$$

which implies that $TB_r\subset B_r$.
Now, for $u,v\in C(J,\mathbb{R})$ and for each $t\in J$, we obtain

$$\begin{aligned}&|(Tu)(t)-(Tv)(t)|\\ &\leq \sup_{t\in J}\Big[\frac{1}{\Gamma(\nu)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\nu-1}|F(\tau,u(\tau),(Ku)(\tau),(Hu)(\tau))-F(\tau,v(\tau),(Kv)(\tau),(Hv)(\tau))|\frac{d\tau}{\tau}\\ &\quad+\sum_{i=1}^{m}\frac{1}{\Gamma(\rho_i)}\int_1^t(t-s)^{\rho_i-1}|G_i(\tau,u(\tau))-G_i(\tau,v(\tau))|d\tau\Big]\\ &\leq \Big(\sum_{i=1}^{m}\frac{L_g(T-1)^{\rho_i}}{\Gamma(\rho_i+1)}+L_1\frac{(\log T)^{\nu}}{\Gamma(\nu+1)}+(\varphi_1L_1+\varphi_2L_2)\Big[\frac{\gamma}{\Gamma(\nu)}+\frac{(\log T)^{\nu}}{\Gamma(\nu+1)}\Big]\Big)\|u-v\|\\ &\leq \phi\|u-v\|.\end{aligned}$$

This gives $\|Tu-Tv\|\leq\phi\|u-v\|$. By inequality (3.3), the operator T is a contracting mapping. Hence, we conclude that the operator T has a unique fixed point $u\in C(J,\mathbb{R})$. □

4. Conclusions

In this paper, we establish some new conditions for the existence and uniqueness of solutions for a class of nonlinear Hadamard fractional Volterra-Fredholm integro-differential equations with initial conditions. The desired results are proved by using Arzelá-Ascoli theorem, aid of fixed point theorems due to Banach and Krasnoselskii in Banach spaces.

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