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Some Properties of Lacunary Convergence and Lacunary Ideal Convergence in Fuzzy Normed Spaces

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Abstract

In this study, firstly lacunary convergence and lacunary ideal convergence is introduced in fuzzy normed spaces. Later, the relation between lacunary convergence and lacunary ideal convergence are investigated in fuzzy normed spaces. Finally, we have introduced the concept of $F\mathscr{I}_\theta-$limit point, $F\mathscr{I}_\theta-$cluster point, $F\theta-$Cauchy sequence and $F\mathscr{I}_\theta-$Cauchy sequences have introduced in fuzzy normed space.

Keywords: *Statistical convergence; Fuzzy normed spaces; Lacunary convergence; $\mathscr{I}-$convergence; $\mathscr{I}-$Cauhy sequence*

2010 Mathematics Subject Classification: *46A45, 40A35, 46S40*

1. Introduction and Background

The statistical convergence was derived from the convergence of real sequences by Fast [14] and Schoenberg [37]. After the studies of Šalát [34], Fridy [16] and Connor [4] in this area, many studies have been conducted. On the one hand, Kostyrko et al. [22] has introduced the concept of ideal convergence by expanding the concept of statistical convergence. On the other hand; Nuray and Ruckle [29] have worked on the same topic as generalized statistical convergence.

Matloka [24] was the first scholar who introduced the convergence of a sequence of fuzzy numbers and he showed evidence some basic Theorems. In next years, Nanda [27] made studies the sequences of fuzzy numbers again and Ṣençimen and Pehlivan [36] introduced the notions of statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. The concepts of $\mathscr{I}$-convergence, $\mathscr{I}^*$-convergence and $\mathscr{I}$-Cauchy sequence was studied by Hazarika [18] in a fuzzy normed linear space. Recently, on the one hand, Türkmen and Çınar [39] studied lacunary statistical convergence. On the other hand, Türkmen and Dündar [43] scrutinized same concepts for double sequences and Türkmen [41] reinterpreted these works in fuzzy n-normed spaces. In addition, many researchers have been working on these issues recently.[2, 3, 10, 11, 12, 13]

In this paper, we introduce and study the concepts of lacunary $\mathscr{I}-$convergence, lacunary $\mathscr{I}-$Cauchy, lacunary $\mathscr{I}^*-$convergence with respect to fuzzy norm. Also, we study some properties and relations of them. In addition, the fact that these definitions and theorems are parallel to the definitions and theorems given in different fuzzy norms [5, 6, 7, 8, 9, 21] supports that the definitions and theorems are correct.

Now, we recall the concept of ideal, convergence, statistical convergence, ideal convergence, lacunary convergence, fuzzy normed and some basic definitions (see[1, 14, 15, 16, 17, 19, 20, 23, 25, 26, 28, 29, 30, 31, 32, 33, 35, 38, 36, 34, 42, 39, 40, 43, 44].

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership.

According to Zadeh, a fuzzy subset of X is a nonempty subset $\{(x,u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \to [0,1]$. The function u itself is often used for the fuzzy set. And the function u is called fuzzy number under certain conditions. Also, we denote all fuzzy numbers as $L(\mathbb{R})$. We have written $L^*(\mathbb{R})$ by the set of all non-negative fuzzy numbers. For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ \sup p, & \text{if } \alpha = 0. \end{cases}$$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ is defined as

$$D(u,v) = \sup_{0\leq\alpha\leq1} \max\left\{\left|u_\alpha^- - v_\alpha^-\right|, \left|u_\alpha^+ - v_\alpha^+\right|\right\}.$$

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A sequence $x=(x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0 if for every $\varepsilon>0$, there exists a positive integer k_0 such that $D(x_k,x_0)<\varepsilon$, for $k>k_0$. And a sequence $x=(x_k)$ of fuzzy numbers convergens to levelwise to x_0 if and only if $\lim_{k\to\infty}[x_k]_\alpha=[x_0]_\alpha^-$ and $\lim_{k\to\infty}[x_k]_\alpha=[x_0]_\alpha^+$, where $[x_k]_\alpha=\left[(x_k)_\alpha^-,(x_k)_\alpha^+\right]$ and $[x_0]_\alpha=\left[(x_0)_\alpha^-,(x_0)_\alpha^+\right]$, for every $\alpha\in(0,1)$.
A sequence $x=(x_k)$ of fuzzy numbers is said to be statistically convergent to fuzzy numbers x_0 if every $\varepsilon>0$,

$$\lim_n \frac{1}{n}\left|\left\{k\leq n:\overline{d}(x_k,x_0)\geq\varepsilon\right\}\right|=0.$$

Later, many mathematicians studied statistical convergence of fuzzy numbers and extended to fuzzy normed spaces.
Let X be a vector space over $\mathbb{R}$, let $\|.\|:X\to L^*(\mathbb{R})$ and the mappings $L;R:[0,1]\times[0,1]\to[0,1]$ be symetric, nondecreasing in both arguments and satisfy $L(0,0)=0$ and $R(1,1)=1$. The quadruple $(X,\|.\|,L,R)$ is called fuzzy normed linear space (briefly $(X,\|.\|)$ FNS) and $\|.\|$ is a fuzzy norm if $\|.\|$ provide certain conditions.
Let $(X,\|.\|)$ be an fuzzy normed linear space. A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $x\in X$ with respect to the fuzzy norm on X and we denote by $x_n\overset{FN}{\to}x$, provided that $(D)-\lim_{n\to\infty}\|x_n-x\|=\widetilde{0}$; i.e. for every $\varepsilon>0$ there is an $N(\varepsilon)\in\mathbb{N}$ such that $D\left(\|x_n-x\|,\widetilde{0}\right)<\varepsilon$ for all $n>N(\varepsilon)$. This means that for every $\varepsilon>0$ there is an $N(\varepsilon)\in\mathbb{N}$ such that $\sup_{\alpha\in[0,1]}\|x_n-x\|_\alpha^+=\|x_n-x\|_0^+<\varepsilon$ for all $n\geq N(\varepsilon)$.
Let $(X,\|.\|)$ be an FNS. A sequence (x_k) in X is statistically convergent to $L\in X$ with respect to the fuzzy norm on X and we denote by $x_n\overset{FS}{\to}x$, provided that for each $\varepsilon>0$, we have $\delta\left(\left\{k\in\mathbb{N}:D\left(\|x_k-L\|,\tilde{0}\right)\geq\varepsilon\right\}\right)=0$. This implies that for each $\varepsilon>0$, the set

$$K(\varepsilon)=\left\{k\in\mathbb{N}:\|x_k-L\|_0^+\geq\varepsilon\right\}$$

has natural density zero; namely, for each $\varepsilon>0$, $\|x_k-L\|_0^+<\varepsilon$ for almost all k.
By a lacunary sequence we mean an increasing integer sequence $\theta=\{k_r\}$ such that $k_0=0$ and $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$. The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$.
Let $(X,\|.\|)$ be an FNS and $\theta=\{k_r\}$ be lacunary sequence. A sequence $x=(x_k)_{k\in\mathbb{N}}$ in X is said to be lacunary summable with respect to fuzzy norm on X if there is an $L\in X$ such that

$$\lim_{r\to\infty}\frac{1}{h_r}\left(\sum_{k\in I_r}D\left(\|x_k-L\|,\tilde{0}\right)\right)=0.$$

In this case, we can write $x_k\to L((N_\theta)_{FN})$ or $x_k\overset{(N_\theta)_{FN}}{\longrightarrow}L$ and

$$(N_\theta)_{FN}=\left\{x=(x_k):\lim_{r\to\infty}\frac{1}{h_r}\left(\sum_{k\in I_r}D\left(\|x_k-L\|,\tilde{0}\right)\right)=0,\text{for some}L\right\}.$$

A sequence $x=(x_k)$ in X is said to be lacunary statistically convergent or $FS_\theta-$convergent to $L\in X$ with respect to fuzzy norm on X if for each $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r:D\left(\|x_k-L\|,\tilde{0}\right)\geq\varepsilon\right\}\right|=0$$

where $|A|$ denotes the number of elements of the set $A\subseteq\mathbb{N}$. In this case, we write $x_k\overset{FS_\theta}{\longrightarrow}L$ or $x_k\to L(FS_\theta)$ or $FS_\theta-\lim_{k\to\infty}x_k=L$. This implies that for each $\varepsilon>0$, the set $K(\varepsilon)=\left\{k\in I_r:\|x_k-L\|_0^+\geq\varepsilon\right\}$ has natural density zero, namely, for each $\varepsilon>0$, $\|x_k-L\|_0^+<\varepsilon$, for almost all k.
Let $X\neq\emptyset$. A class $\mathscr{I}$ of subsets of X is said to be an ideal in X provided:
(i) $\emptyset\in\mathscr{I}$, (ii) $A,B\in\mathscr{I}$ implies $A\cup B\in\mathscr{I}$,(iii) $A\in\mathscr{I}$, $B\subset A$ implies $B\in\mathscr{I}$.
$\mathscr{I}$ is called a nontrivial ideal if $X\notin\mathscr{I}$. A nontrivial ideal $\mathscr{I}$ in X is called admissible if $\{x\}\in\mathscr{I}$ for each $x\in X$.
Let $X\neq\emptyset$. A non empty class $\mathscr{F}$ of subsets of X is said to be a filter in X provided:
(i) $\emptyset\notin\mathscr{F}$, (ii) $A,B\in\mathscr{F}$ implies $A\cap B\in\mathscr{F}$, (iii) $A\in\mathscr{F}$, $A\subset B$ implies $B\in\mathscr{F}$.
Let $\mathscr{I}$ is a nontrivial ideal in X, $X\neq\emptyset$, then the class $\mathscr{F}(\mathscr{I})=\{M\subset X:(\exists A\in\mathscr{I})(M=X\backslash A)\}$ is a filter on X, called the filter associated with $\mathscr{I}$.
Let $(X,\|.\|)$ be fuzzy normed space. A sequence $x=(x_m)_{m\in\mathbb{N}}$ in X is said to be $\mathscr{I}-$ convergent to $L\in X$ with respect to fuzzy norm on X if for each $\varepsilon>0$, the set $A(\varepsilon)=\left\{m\in\mathbb{N}:\|x_m-L\|_0^+\geq\varepsilon\right\}$ belongs to $\mathscr{I}$. In this case, we write $x_m\overset{F\mathscr{I}}{\longrightarrow}L$. The element L is called the $\mathscr{I}-$limit of (x_m) in X.
A sequence (x_m) in X is said to be $\mathscr{I}^*$- convergent to L in X with respect to the fuzzy norm on X if there exists a set $M\in F(\mathscr{I})$, $M=\{t_k:t_1<t_2<\cdots\}\subset\mathbb{N}$ such that $\lim_{k\to\infty}\|x_{t_k}-L\|=0$.

2. Lacunary $\mathscr{I}-$Convergence

In this section, we introduce the concepts of lacunary convergence and lacunary ideal convergence in fuzzy normed spaces. Also, we investigate some properties these concepts and examined the relationships between them.
Throughout the paper, we let $(X,\|.\|)$ be an FNS and $\mathscr{I}\subset 2^{\mathbb{N}}$ be an strongly admissible ideal.

Definition 2.1. *Let $x=(x_m)$ be a sequence in X. If for each $\varepsilon>0$, there exists $r_0\in\mathbb{N}$ such that*

$$\frac{1}{h_r}\sum_{m\in J_r}D\left(\|x_m-L_1\|,\tilde{0}\right)<\varepsilon,$$

for all $r\geq r_0$, then x is $F\theta-$convergent to $L_1\in X$. In this case, we write $x_m\overset{F\theta}{\longrightarrow}L_1$ or $x_m\to L_1(F\theta)$ or $F\theta-\lim_{m\to\infty}x_m=L_1$. The element L_1 is called the $F\theta-$limit of (x_m) in X.

Theorem 2.2. *Let $x=(x_m)_{m\in\mathbb{N}}$ be a sequence in X. If (x_m) is $\theta-$convergent, $F\theta-\lim x$ is unique.*

Proof. Suppose that $F\theta-\lim x=L_1$ and $F\theta-\lim x=L_2$. Then for any $\varepsilon>0$, there exists $r_1\in\mathbb{N}$ such that $\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\frac{\varepsilon}{2}$, for all $r\geq r_1$. Also there exists $r_2\in\mathbb{N}$ such that $\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_2\|_0^+<\frac{\varepsilon}{2}$, for all $r\geq r_2$. Now, consider $r_0=max\{r_1,r_2\}$. Then for $r\geq r_0$, we will get a $p\in\mathbb{N}$ such that

$$\left\|x_p-L_1\right\|_0^+<\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\frac{\varepsilon}{2}\text{ and }\left\|x_p-L_2\right\|_0^+<\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_2\|_0^+<\frac{\varepsilon}{2}.$$

Then, we have

$$\|L_1-L_2\|_0^+\leq\left\|x_p-L_1\right\|_0^++\left\|x_p-L_2\right\|_0^+<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we have $\|L_1-L_2\|_0^+=0$ which implies that $L_1=L_2$. Therefore, we conclude that $F\theta-\lim x$ is unique. □

Definition 2.3. *Let $x=(x_m)$ be a sequence in X. If for each $\varepsilon>0$, the set $\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}D\left(\|x_m-L_1\|,\tilde{0}\right)\geq\varepsilon\right\}$ belongs to $\mathscr{I}$, then $x=(x_m)$ is said to be lacunary $\mathscr{I}-$convergent to $L_1\in X$ with respect to fuzzy norm on X. In this case, we write $x_m\xrightarrow{F\mathscr{I}_\theta}L_1$ or $x_m\to L_1\,(F\mathscr{I}_\theta)$ or $F\mathscr{I}_\theta-\lim_{m\to\infty}x_m=L_1$. The element L_1 is called the $F\mathscr{I}_\theta-$limit of (x_m) in X.*

Lemma 2.4. *Let $x=(x_m)$ be a sequence in X. For each $\varepsilon>0$, the following statements are equivalent.*
a) $F\mathscr{I}_\theta-\lim_{m\to\infty}x_m=L_1$,
b) $\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+\geq\varepsilon\right\}\in\mathscr{I}$,
c) $\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\varepsilon\right\}\in F(\mathscr{I})$,
d) $F\mathscr{I}_\theta-\lim_{m\to\infty}\|x_m-L_1\|_0^+=0$.

Theorem 2.5. *Let $x=(x_m)$ be a sequence in X. If (x_m) is $F\mathscr{I}_\theta-$convergent, $F\mathscr{I}_\theta-\lim x$ is unique.*

Proof. Suppose that $F\mathscr{I}_\theta-\lim x=L_1$ and $F\mathscr{I}_\theta-\lim x=L_2$. For any $\varepsilon>0$, define the following sets;

$$A_1=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+\geq\frac{\varepsilon}{2}\right\}\text{ and }A_2=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_2\|_0^+\geq\frac{\varepsilon}{2}\right\}.$$

Since $F\mathscr{I}_\theta-\lim x=L_1$ and $F\mathscr{I}_\theta-\lim x=L_2$, using Lemma 2.4, we have $A_1\in\mathscr{I}$ and $A_2\in\mathscr{I}$ for all $\varepsilon>0$. Let $A_3=A_1\cup A_2$, then $A_3\in\mathscr{I}$. So its complement $(A_3)^c$ is a non-empty set in $F(\mathscr{I})$. If $r\in(A_3)^c$, we have

$$\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\frac{\varepsilon}{2}\text{ and }\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_2\|_0^+<\frac{\varepsilon}{2}.$$

Clearly, we will get a $p\in\mathbb{N}$ such that

$$\left\|x_p-L_1\right\|_0^+<\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\frac{\varepsilon}{2}\text{ and }\left\|x_p-L_2\right\|_0^+<\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_2\|_0^+<\frac{\varepsilon}{2}.$$

Then, we have

$$\|L_1-L_2\|_0^+\leq\left\|x_p-L_1\right\|_0^++\left\|x_p-L_2\right\|_0^+<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we have $\|L_1-L_2\|_0^+=0$ which implies that $L_1=L_2$. Therefore, we conclude that $F\mathscr{I}_\theta-\lim x$ is unique. □

Theorem 2.6. *Let (x_m), (y_m) be two sequences in X. Then,*
i) If $F\mathscr{I}_\theta-\lim x_m=L_1$ and $F\mathscr{I}_\theta-\lim y_m=L_2$, then $F\mathscr{I}_\theta-\lim(x_m+y_m)=L_1+L_2$;
ii) If $F\mathscr{I}_\theta-\lim x_m=L_1$ then $F\mathscr{I}_\theta-\lim cx_m=cL_1$ for $c\in\mathbb{R}-\{0\}$.

Proof. i) For any $\varepsilon>0$, define the following sets;

$$A_1=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+\geq\frac{\varepsilon}{2}\right\}\text{ and }A_2=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|y_m-L_2\|_0^+\geq\frac{\varepsilon}{2}\right\}.$$

Since $F\mathscr{I}_\theta-\lim x_m=L_1$ and $F\mathscr{I}_\theta-\lim y_m=L_2$, using Lemma 2.4, we have $A_1\in\mathscr{I}$ and $A_2\in\mathscr{I}$ for all $\varepsilon>0$. Let $A_3=A_1\cup A_2$, then $A_3\in\mathscr{I}$. So $(A_3)^c$ is a non-empty set in $F(\mathscr{I})$. We claim that

$$(A_3)^c\subset\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1+y_m-L_2\|_0^+<\varepsilon\right\}.$$

Let $r\in(A_3)^c$, then we have

$$\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\frac{\varepsilon}{2}\text{ and }\frac{1}{h_r}\sum_{m\in J_r}\|y_m-L_2\|_0^+<\frac{\varepsilon}{2}.$$

We will get a $p\in\mathbb{N}$ such that

$$\left\|x_p-L_1\right\|_0^+<\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\frac{\varepsilon}{2}\text{ and }\left\|y_p-L_2\right\|_0^+<\frac{1}{h_r}\sum_{m\in J_r}\|y_m-L_2\|_0^+<\frac{\varepsilon}{2}.$$

Then, we have

$$\left\|x_p-L_1+y_p-L_2\right\|_0^+\leq\left\|x_p-L_1\right\|_0^++\left\|y_p-L_2\right\|_0^+<\varepsilon.$$

Hence,

$$(A_3)^c\subset\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1+y_m-L_2\|_0^+<\varepsilon\right\}.$$

Since $(A_3)^c\in F(\mathscr{I})$, so

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|(x_m+y_m)-(L_1+L_2)\|_0^+\geq\varepsilon\right\}\in\mathscr{I}.$$

Therefore $F\mathscr{I}_\theta-\lim(x_m+y_m)=L_1+L_2$.
ii) Let $F\mathscr{I}_\theta-\lim x_m=L_1$. We define the set

$$A_1=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\frac{\varepsilon}{|c|}\right\},$$

$c\in\mathbb{R}-\{0\}$ for every $\varepsilon>0$. So $A_1\in F(\mathscr{I})$. Let $r\in A_1$, then we have

$$\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\frac{\varepsilon}{|c|}\Rightarrow\frac{|c|}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<|c|.\frac{\varepsilon}{|c|}\Rightarrow\frac{1}{h_r}\sum_{m\in J_r}|c|\,\|x_m-L_1\|_0^+<\varepsilon\Rightarrow\frac{1}{h_r}\sum_{m\in J_r}\|c.x_m-c.L_1\|_0^+<\varepsilon$$

Hence,

$$A_1\subset\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|cx_m-cL_1\|_0^+<\varepsilon\right\}$$

and

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|cx_m-cL_1\|_0^+<\varepsilon\right\}\in F(\mathscr{I}).$$

Hence $F\mathscr{I}_\theta-\lim cx_m=cL_1$.

□

Theorem 2.7. *Let $(X,\|.\|)$ be a fuzzy normed space and $x=(x_m)$ in X. If $F\theta-\lim x=L_1$, then $F\mathscr{I}_\theta-\lim x=L_1$.*

Proof. Let $F\theta-\lim x=L_1$. Then for every $\varepsilon>0$, there exists $r_0\in\mathbb{N}$ such that $\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\varepsilon$ for all $r\geq r_0$. Therefore the set

$$K=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+\geq\varepsilon\right\}\quad\subset\quad\{1,2,3,...,r_0-1\}.$$

So, we have $K\in\mathscr{I}$. Hence $F\mathscr{I}_\theta-\lim x=L_1$. □

Theorem 2.8. *Let $(X,\|.\|)$ be a fuzzy normed space and $x=(x_m)$ in X. If $F\theta-\lim x=L_1$, then there exists a subsequence (x_{k_m}) such that $x_{k_m}\overset{FN}{\rightarrow}L_1$.*

Proof. Let $F\theta-\lim x=L_1$.Then for every $\varepsilon>0$, there exists $r_0\in\mathbb{N}$ such that $\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\varepsilon$ for all $r\geq r_0$. Clearly, for each $r\geq r_0$, we can select a $k_m\in J_r$ such that

$$\left\|x_{k_m}-L_1\right\|_0^+<\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\varepsilon.$$

It follows that $x_{k_m}\overset{FN}{\rightarrow}L_1$. □

3. Limit Point and Cluster Point

In this section, we gave definition of $F\mathscr{I}_\theta-$limit point and $F\mathscr{I}_\theta-$cluster point and some properties.

Definition 3.1. *Let $x=(x_m)$ be a sequence in X. If there is a set $K=\{k_1<k_2...<k_m<...\}\subset\mathbb{N}$ such that the set $K'=\{r\in\mathbb{N}:k_m\in J_r\}\notin\mathscr{I}$ and $F\theta-\lim x_{k_m}=L$, then the element $L\in X$ is said to be $F\mathscr{I}_\theta-$limit point of $x=(x_m)$.*

Definition 3.2. *Let $x=(x_m)$ be a sequence in X. If for every $\varepsilon>0$, we have*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L\|_0^+<\varepsilon\right\}\notin\mathscr{I},$$

then the element $L\in X$ is said to be $F\mathscr{I}_\theta-$cluster point of $x=(x_m)$.
We denote the set of all $F\mathscr{I}_\theta-$limit points of x as $\Lambda_F^{\mathscr{I}_\theta}(x)$ and denote the set of all $F\mathscr{I}_\theta-$cluster point of x as $\Gamma_F^{\mathscr{I}_\theta}(x)$.

Theorem 3.3. *Let $x=(x_m)$ be a sequence in X. For each $x=(x_m)$, we have $\Lambda_F^{\mathscr{I}_\theta}(x)\subset\Gamma_F^{\mathscr{I}_\theta}(x)$.*

Proof. Let $L_1\in\Lambda_F^{\mathscr{I}_\theta}(x)$. There exists a set $K\subset\mathbb{N}$ such that $K'\notin\mathscr{I}$, where K and K' are as in definition 3.1, satisfies $F\theta-\lim x_{k_m}=L_1$. For every $\varepsilon>0$, there exists $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_r}\sum_{k_m\in J_r}\left\|x_{k_m}-L_1\right\|_0^+<\varepsilon$$

for all $r\geq r_0$. Therefore,

$$C=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-L_1\|_0^+<\varepsilon\right\}\supseteq K'\backslash\{k_1,k_2,k_3,...,k_{r_0}\}.$$

So, we must have $K'\backslash\{k_1,k_2,k_3,...,k_{r_0}\}\notin\mathscr{I}$ and as such $C\notin\mathscr{I}$. Hence $L_1\in\Gamma_F^{\mathscr{I}_\theta}(x)$. □

4. $F\theta-$Cauchy and $F\mathscr{I}_\theta-$Cauchy

Definition 4.1. *Let $x=(x_m)$ be a sequence in X. If there exist $r_0,n\in\mathbb{N}$ satisfying $\frac{1}{h_r}\sum\limits_{m\in J_r}\|x_m-x_n\|_0^+<\varepsilon$ for every $\varepsilon>0$ and all $r\geq r_0$, $x=(x_m)$ is said to be $F\theta-$Cauchy sequence.*

Definition 4.2. *Let $x=(x_m)$ be a sequence in X. If there exists $n\in\mathbb{N}$ satisfying*

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{m\in J_r}\|x_m-x_n\|_0^+<\varepsilon\right\}\in F(\mathscr{I})$$

for every $\varepsilon>0$, $x=(x_m)$ is said to be $F\mathscr{I}_\theta-$Cauchy sequence.

Definition 4.3. *Let $x=(x_m)$ be a sequence in X. If there exists a set $M=\{k_1<k_2...<k_m<...\}\subset\mathbb{N}$ such that the set $M'=\{r\in\mathbb{N}:k_m\in J_r\}\in F(\mathscr{I})$ and the subsequence (x_{k_m}) of $x=(x_m)$ is a $F\theta-$Cauchy sequence, x is said to be $F\mathscr{I}_\theta^*-$Cauchy sequence.*

Theorem 4.4. *Let $x=(x_m)$ be a sequence in X. If x is $F\theta-$Cauchy sequence, then it is $F\mathscr{I}_\theta-$Cauchy sequence.*

Proof. This Theorem is an analogue of Theorem 2.7; the proof follows easily. □

Theorem 4.5. *Let $x=(x_m)$ be a sequence in X. If x is $F\theta-$Cauchy sequence, then there is a subsequence of $x=(x_m)$ which is ordinary Cauchy sequence.*

Proof. The proof of theorem is similar to that of Theorem 2.8. □

Theorem 4.6. *Let $x=(x_m)$ be a sequence in X. If x is $F\mathscr{I}_\theta^*-$Cauchy sequence, then it is $F\mathscr{I}_\theta-$Cauchy sequence as well.*

Proof. The proof of theorem can be proved easily using similar techniques as in the proof of Theorem 2.8. □

5. Conclusion

In this paper, first of all, we have introduced consept of $F\theta-$*convergence* and $F\mathscr{I}_\theta-$*convergence*. So, we saw that these limits are unique and if $F\theta-\lim x=L_1$, then $F\mathscr{I}_\theta-\lim x=L_1$. Later, we gave definitions of $F\mathscr{I}_\theta-$limit point and $F\mathscr{I}_\theta-$cluster point and we proved that every limit point is cluster point. Finally, we introduced definitions of $F\theta-$Cauchy and $F\mathscr{I}_\theta-$Cauchy and gave some properties.

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