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On New Generalized Fractional Integral Operators and Related Fractional Inequalities

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Abstract

In this paper, we define the generalized k-fractional integrals of a function with respect to the another function which generalizes many different types of fractional integrals such as Riemann-Liouville fractional, Hadamard fractional integrals, Katugampola fractional integral, (k,s)-fractional integral operators. Moreover, we obtain Hermite-Hadamard inequalities utilizing k-fractional integrals of a function with respect to the another function. We also investigate trapezoid inequalities for the functions whose derivatives in absolute value are convex. Finally, some special cases of these inequalities are given.

Keywords: *Fractional integral operators, Hermite-Hadamard inequality, midpoint inequality, convex function.*
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1. Introduction

The Hermite-Hadamard inequality is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. Numerous mathematicians have devoted their efforts to generalise, refine, counterpart and extend it for different classes of functions such as using convex mappings.
The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[13, p.137], [7]). These inequalities state that if $f: I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [3]-[5], [8]-[10], [16]-[20]).
In [6] Diaz and Pariguan have defined k -gamma function Γ_k that is generalization of the classical gamma. Γ_k is given by formula

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \quad k > 0.$$

It has shown that Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, clearly given by

$$\Gamma_k(\alpha) := \int_0^\infty e^{-\frac{t^k}{k}} t^{\alpha-1} dt.$$

Obviously, $\Gamma_k(x+k) = x\Gamma_k(x)$, $\Gamma(x) = \lim_{k\to 1}\Gamma_k(x)$ and $\Gamma_k(x) = k^{\frac{x}{k}-1}\,\Gamma(\frac{x}{k})$.
The overall structure of the study takes the form of four sections including introduction. The remaining part of the paper proceeds as follows: In Section 2, we introduce generalized k-fractional integrals of a function with respect to the another function which generalizes different types of fractional integrals, including Riemann-Lioville fractional, Hadamard fractional integrals, Katugampola fractional integral, (k,s)-fractional integral operators and many others. In section 3, the Hermite-Hadamard type inequalities for convex functions via generalized k-fractional integrals of a function with respect to the another function are presented while in section 4 trapezoid type inequalities for functions whose derivatives in absolute value are convex with this type fractional integral operators are obtained and we also provide some corollary for theorems.

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2. New Generalized Fractional Integral Operators

In this section we present the concept of the generalized k-fractional integrals of a function with respect to the another function. We first define the function

$$\mathscr{F}^{\sigma,k}_{\rho,\lambda}(x) := \sum_{m=0}^{\infty} \frac{\sigma(m)}{k\Gamma_k(\rho k m+\lambda)} x^m \quad (\rho,\lambda>0; |x|<\mathscr{R}),$$

where the coefficients $\sigma(m)$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers and $\mathscr{R}$ is the set of real numbers.

Definition 2.1. *For $k>0$, let $g:[a,b]\to\mathbb{R}$ be an increasing and positive monoton function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b). The left and right sided generalized k-fractional integrals of f with respect to the function g on $[a,b]$ are defined, respectively, as follows:*

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}f(x) = \int_a^x \frac{g'(t)}{(g(x)-g(t))^{1-\frac{\lambda}{k}}} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega(g(x)-g(t))^{\rho}\right] f(t)dt,\ \ x>a, \tag{2.1}$$

and

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}f(x) = \int_x^b \frac{g'(t)}{(g(t)-g(x))^{1-\frac{\lambda}{k}}} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega(g(t)-g(x))^{\rho}\right] f(t)dt,\ \ x<b, \tag{2.2}$$

where $\lambda,\rho>0$, $\omega\in\mathbb{R}$.

The significant special cases of the integral operators (2.1) and (2.2) are mentioned below:
1) For $k=1$, operator in (2.1) leads to generalized fractional integral of f with respect to the function g on $[a,b]$. This relation is given by

$$\mathscr{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}f(x) = \int_a^x \frac{g'(t)}{(g(x)-g(t))^{1-\lambda}} \mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega(g(x)-g(t))^{\rho}\right] f(t)dt,\ \ x>a.$$

2) For $g(t)=t$, operator in (2.1) leads to generalized k-fractional integral of f. This relation is given by

$$\mathscr{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}f(x) = \int_a^x (x-t)^{\frac{\lambda}{k}-1} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega(x-t)^{\rho}\right] f(t)dt,\ \ x>a.$$

3) For $g(t)=\ln t$, operator in (2.1) leads to generalized Hadamard k-fractional integral of f. This relation is given by

$$\mathscr{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}f(x) = \int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\lambda}{k}-1} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(\ln\frac{x}{t}\right)^{\rho}\right] f(t)\frac{dt}{t},\ \ x>a.$$

4) For $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$ operator in (2.1) leads to generalized (k,s)-fractional integral of f. This relation is given by

$$\mathscr{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}f(x) = (s+1)^{1-\frac{\lambda}{k}} \int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\lambda}{k}-1} t^r \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(\frac{x^{s+1}-t^{s+1}}{s+1}\right)^{\rho}\right] f(t)dt,\ \ x>a.$$

Remark 2.2. *Similarly, all above special cases can also be seen for operator (2.2).*

Remark 2.3. *For $k=1$ and $g(t)=t$, operators in (2.1) and (2.2) reduce to the following generalized fractional integral operators defined by Raina [14] and Agarwal et. al [1], respectively:*

$$\mathscr{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(x) = \int_a^x (x-t)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega(x-t)^{\rho}\right] f(t)dt,\ \ x>a, \tag{2.3}$$

$$\mathscr{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(x) = \int_x^b (t-x)^{\lambda-1} \mathscr{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-x)^{\rho}\right] f(t)dt,\ \ x<b, \tag{2.4}$$

Remark 2.4. *One can obtain other new generalized fractional integral operators with different choices of g.*

Remark 2.5. *For $\lambda=\alpha$, $\sigma(0)=1$, $w=0$ in Definition 2.1, then we have the generalized fractional operators defined by Akkurt et al. in [2].*

Remark 2.6. *Let $\lambda=\alpha$, $\sigma(0)=1$, $w=0$ in Definition 2.1.*

1) Choosing $k=1$, then we have fractional integrals of a function f with respect to function g. [11].
2) Choosing $g(t)=t$, then we have k-fractional integrals [12].
3) Choosing $k=1$ and $g(t)=\ln t$, then we have Hadamard fractional integrals [11].
4) Choosing $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$, then we have (k,s)-fractional integral operators [15].
5) Choosing $k=1$ and $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$, then we have Katugampola fractional integral operators [10].
6) Choosing $k=1$ and $g(t)=t$, then we have Riemann-Lioville fractional integral operators [11].

3. Hermite-Hadamard Inequalities for Generalized Fractional Integral Operators

In this part, we obtain Hermite-Hadamard inequalities for the generalized k-fractional integrals of a function with respect to the another function.
Firstly, let us start with some notations given in [9]. Let $f : I^{\circ} \to \mathbb{R}$ be a function such that $a,b \in I^{\circ}$ and $0 < a < b < \infty$. We suppose that $f \in L^{\infty}(a,b)$ in such a way that $I^{\alpha}_{a^{+};g} f(x)$ and $I^{\alpha}_{b^{-};g} f(x)$ are well defined. We define the function

$$\widetilde{f}(x) = f(a+b-x),\ x \in [a,b]$$

$$F(x) = f(x) + \widetilde{f}(x),\ x \in [a,b].$$

In Definition 2.1, using the change of variable $s = \frac{\tau - a}{x-a}$, we have

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega} f(x) = \int_0^1 \frac{(x-a)\, g'((1-s)a+sx)}{[g(x)-g((1-s)a+sx)]^{1-\frac{\lambda}{k}}} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\,(g(x)-g((1-s)a+sb))^{\rho}\right] f(sx+(1-s)a)\,ds. \tag{3.1}$$

Similarly, using the change of variable $s = \frac{\tau - x}{b-x}$, we have

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega} f(x) = \int_0^1 \frac{(b-x)\, g'((1-s)x+sb)}{[g((1-s)x+sb)-g(x)]^{1-\frac{\lambda}{k}}} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\,(g((1-s)x+sb)-g(x)))^{\rho}\right] f(sb+(1-s)x)\,ds \tag{3.2}$$

Theorem 3.1. *Let $g : [a,b] \to \mathbb{R}$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $g'(x)$ on (a,b). If f is a convex on $[a,b]$, then the following Hermite-Hadamard type inequalities for generalized k-fractional integrals of f with respect to the function g on $[a,b]$ hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{4k\,(g(b)-g(a))^{\frac{\lambda}{k}}\, \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\,(g(b)-g(a))^{\rho}\right]} \left[\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega} F(a) + \mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega} F(b)\right] \le \frac{f(a)+f(b)}{2} \tag{3.3}$$

where $\lambda, \rho > 0$, $\omega \in \mathbb{R}$ and the coefficients $\sigma(m)$ $(m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers.

Proof. Since f is an convex mapping on $[a,b]$, we have

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \tag{3.4}$$

for $x,y \in [a,b]$. Now, for $s \in [0,1]$, let $x = sa+(1-s)b$ and $y = (1-s)a+sb$. Then we have

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2} f(sa+(1-s)b) + \frac{1}{2} f((1-s)a+sb). \tag{3.5}$$

Multiplying both sides of (3.5) by

$$(b-a)\frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\,(g(b)-g((1-s)a+sb))^{\rho}\right]$$

and integrating over $(0,1)$ with respect to s, we find that

$$f\left(\frac{a+b}{2}\right)(b-a)\int_0^1 \frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\,(g(b)-g((1-s)a+sb))^{\rho}\right] ds \tag{3.6}$$

$$\le \frac{b-a}{2}\int_0^1 \frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\,(g(b)-g((1-s)a+sb))^{\rho}\right] f(sa+(1-s)b)\,ds$$

$$+\frac{b-a}{2}\int_0^1 \frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\,(g(b)-g((1-s)a+sb))^{\rho}\right] f((1-s)a+sb)\,ds.$$

Using the change of the variable $x = g(b)-g((1-s)a+sb)$, we deduce that

$$\int_0^1 \frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\,(g(b)-g((1-s)a+sb))^{\rho}\right] ds = \frac{1}{b-a}\int_0^{g(b)-g(a)} x^{\frac{\lambda}{k}-1} \mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega x^{\rho}\right] ds \tag{3.7}$$

$$= \frac{k}{b-a} x^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega x^{\rho}\right]\Big|_0^{g(b)-g(a)}$$

$$= \frac{k}{b-a}\,(g(b)-g(a))^{\frac{\lambda}{k}}\, \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\,(g(b)-g(a))^{\rho}\right].$$

Using the (3.1), we have

$$(b-a)\int_0^1 \frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g(b)-g((1-s)a+sb)\right)^{\rho}\right]f(sa+(1-s)b)\,ds=\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}\widetilde{f}(b) \tag{3.8}$$

and

$$(b-a)\int_0^1 \frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g(b)-g((1-s)a+sb)\right)^{\rho}\right]f((1-s)a+sb)\,ds=\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}f(b). \tag{3.9}$$

Substituting the equalities (3.7)-(3.9) into (3.6), we find that

$$kf\left(\frac{a+b}{2}\right)(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right]\leq\frac{1}{2}\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}f(b)+\frac{1}{2}\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}\widetilde{f}(b).$$

That is

$$kf\left(\frac{a+b}{2}\right)(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right]\leq\frac{1}{2}\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}F(b). \tag{3.10}$$

Similarly, multiplying both sides of (3.5) by

$$(b-a)\frac{g'((1-s)a+sb)}{[g((1-s)a+sb)-g(a)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g((1-s)a+sb)-g(a)\right)^{\rho}\right]$$

and integrating over $(0,1)$ with respect to s, we find that

$$f\left(\frac{a+b}{2}\right)(b-a)\int_0^1 \frac{g'((1-s)a+sb)}{[g((1-s)a+sb)-g(a)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g((1-s)a+sb)-g(a)\right)^{\rho}\right]ds \tag{3.11}$$

$$\leq\frac{b-a}{2}\int_0^1 \frac{g'((1-s)a+sb)}{[g((1-s)a+sb)-g(a)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g((1-s)a+sb)-g(a)\right)^{\rho}\right]f(sa+(1-s)b)\,ds$$

$$+\frac{b-a}{2}\int_0^1 \frac{g'((1-s)a+sb)}{[g((1-s)a+sb)-g(a)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g((1-s)a+sb)-g(a)\right)^{\rho}\right]f((1-s)a+sb)\,ds.$$

Using the change of the variable $x=g((1-s)a+sb)-g(b)$, we find that

$$\int_0^1 \frac{g'((1-s)a+sb)}{[g((1-s)a+sb)-g(a)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g((1-s)a+sb)-g(a)\right)^{\rho}\right]ds=\frac{k}{b-a}(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right]. \tag{3.12}$$

Using the (3.2), we have

$$(b-a)\int_0^1 \frac{g'((1-s)a+sb)}{[g((1-s)a+sb)-g(a)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g((1-s)a+sb)-g(a)\right)^{\rho}\right]f(sa+(1-s)b)\,ds=\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}\widetilde{f}(a) \tag{3.13}$$

and

$$(b-a)\int_0^1 \frac{g'((1-s)a+sb)}{[g((1-s)a+sb)-g(a)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega\left(g((1-s)a+sb)-g(a)\right)^{\rho}\right]f((1-s)a+sb)\,ds=\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}f(a). \tag{3.14}$$

Substituting the equalities (3.12)-(3.14) into (3.11), we find that

$$kf\left(\frac{a+b}{2}\right)(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right]\leq\frac{1}{2}\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}f(a)+\frac{1}{2}\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}\widetilde{f}(a)$$

which yields

$$kf\left(\frac{a+b}{2}\right)(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right]\leq\frac{1}{2}\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}F(a). \tag{3.15}$$

Combining (3.10) and (3.15), we obtain

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{4k(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right]}\left[\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}F(b)\right].$$

Thus, the proof of first inequality is completed.
For the proof of the second inequality in (3.3), since f is convex, we have

$$f(sa+(1-s)b) \leq sf(a)+(1-s)f(b)$$

and

$$f((1-s)a+sb) \leq (1-s)f(a)+sf(b).$$

By adding these inequalities, we have

$$f(sa+(1-s)b)+f((1-s)a+sb) \leq f(a)+f(b). \tag{3.16}$$

Multiplying both sides of (3.16) by

$$(b-a)\frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega(g(b)-g((1-s)a+sb))^{\rho}\right]$$

and integrating the resulting inequality with respect to s over $(0,1)$, we have

$$(b-a)\int_0^1 \frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega(g(b)-g((1-s)a+sb))^{\rho}\right]f(sa+(1-s)b)\,ds \tag{3.17}$$

$$+(b-a)\int_0^1 \frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega(g(b)-g((1-s)a+sb))^{\rho}\right]f((1-s)a+sb)\,ds$$

$$\leq [f(a)+f(b)](b-a)\int_0^1 \frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega(g(b)-g((1-s)a+sb))^{\rho}\right]ds.$$

Substituting the equalities (3.7)-(3.9) into (3.17), we find that

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}F(b) \leq k[f(a)+f(b)](g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(b)-g(a))^{\rho}\right]. \tag{3.18}$$

Similarly, multiplying both sides of (3.16) by

$$(b-a)\frac{g'((1-s)a+sb)}{[g((1-s)a+sb)-g(a)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega((1-s)a+sb-g(a))^{\rho}\right]$$

and integrating the resulting inequality with respect to s over $(0,1)$, we have

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}F(a) \leq k[f(a)+f(b)](g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(b)-g(a))^{\rho}\right]. \tag{3.19}$$

Adding the inequalities of (3.18) and (3.19), we deduce that

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}F(b) \leq 2k[f(a)+f(b)](g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(b)-g(a))^{\rho}\right]$$

which yields

$$\frac{1}{4k(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(b)-g(a))^{\rho}\right]}\left[\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}F(b)\right] \leq \frac{f(a)+f(b)}{2}.$$

The proof is completely completed. □

Corollary 3.2. *If we choose $k=1$ in Theorem 3.1, then we obtain following inequality for generalized fractional integrals with respect to the another function*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{4(g(b)-g(a))^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(g(b)-g(a))^{\rho}\right]}\left[\mathscr{J}^{\sigma,g}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}F(b)\right] \leq \frac{f(a)+f(b)}{2}$$

where $\lambda,\rho>0$, $\omega\in\mathbb{R}$ and the coefficients $\sigma(m)$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Corollary 3.3. *If we choose $g(t)=t$ in Theorem 3.1, then we obtain following inequality for generalized k-fractional integrals*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{4k(b-a)^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(b-a)^{\rho}\right]}\left[\mathscr{J}^{\sigma,k}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}F(b)\right] \leq \frac{f(a)+f(b)}{2}$$

where $\lambda,\rho>0$, $\omega\in\mathbb{R}$ and the coefficients $\sigma(m)$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Corollary 3.4. *If we choose $g(t)=\ln t$ in Theorem 3.1, then we obtain following inequality for generalized Hadamard fractional integrals*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{4k\left(\ln\frac{b}{a}\right)^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(\ln\frac{b}{a}\right)^{\rho}\right]}\left[\mathscr{H}^{\sigma,k}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}F(b)\right]\leq\frac{f(a)+f(b)}{2}$$

where $\lambda,\rho>0$, $\omega\in\mathbb{R}$ and the coefficients $\sigma(m)$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Corollary 3.5. *If we choose $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$ in Theorem 3.1, then we obtain following inequality for generalized (k,s)-fractional integrals*

$$f\left(\frac{a+b}{2}\right)\leq\frac{(s+1)^{\frac{\lambda}{k}}}{4k\left(b^{s+1}-a^{s+1}\right)^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(\frac{b^{s+1}-a^{s+1}}{s+1}\right)^{\rho}\right]}\left[{}^{s}\mathscr{J}^{\sigma,k}_{\rho,\lambda,b-;\omega}F(a)+{}^{s}\mathscr{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}F(b)\right]\leq\frac{f(a)+f(b).}{2}$$

where $\lambda,\rho>0$, $\omega\in\mathbb{R}$ and the coefficients $\sigma(m)$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Remark 3.6. *If we choose $k=1$ and $g(t)=t$ in Theorem 3.1, then we obtain following inequality for generalized fractional integral*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2(b-a)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]}\left[\mathscr{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)+\mathscr{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)\right]\leq\frac{f(a)+f(b)}{2}$$

given by Yaldiz and Sarikaya in [20].

Remark 3.7. *If we choose $k=1$, $\lambda=\alpha$, $\sigma(0)=1$, $w=0$ in Theorem 3.1, then Theorem 3.1 reduces to Theorem 2.1 proved by Jleli and Samet in [9].*

Remark 3.8. *Choosing $\lambda=\alpha$, $\sigma(0)=1$, $w=0$ in Corollary 3.3, Corollary 3.4 and Corollary 3.5, one can obtain new result.*

4. Trapezoid Type Inequalities for Generalized Fractional Integral Operators

In this section, we establish trapezoid type inequalities for the mappings whose derivatives in absolute value are convex involving generalized k-fractional integrals of a function with respect to the another function.

Lemma 4.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{4k(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(b)-g(a))^{\rho}\right]}\left[\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}F(b)\right]$$

$$=\frac{b-a}{4(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(b)-g(a))^{\rho}\right]}\int_0^1\Xi^{\sigma,k}_{\rho,\lambda,g}(s)f'(sa+(1-s)b)\,ds$$

where $\Xi^{\sigma,k}_{\rho,\lambda,g}:[0,1]\to\mathbb{R}$ be a function defined by

$$\Xi^{\sigma,k}_{\rho,\lambda,g}(s)=[g(sa+(1-s)b)-g(a)]^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(sa+(1-s)b)-g(a))^{\rho}\right]-[g(sb+(1-s)a)-g(a)]^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(sb+(1-s)a)-g(a))^{\rho}\right]$$

$$-[g(b)-g(sa+(1-s)b)]^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(b)-g(sa+(1-s)b))^{\rho}\right]+[g(b)-g(sb+(1-s)a)]^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(b)-g(sb+(1-s)a))^{\rho}\right].$$

Proof. Since relation (3.1), we have

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}F(b)=(b-a)\int_0^1\frac{g'((1-s)a+sb)}{[g(b)-g((1-s)a+sb)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega(g(b)-g((1-s)a+sb))^{\rho}\right]F(sb+(1-s)a)\,ds.$$

Using integration by parts

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}F(b)=k[g(b)-g(a)]^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(b)-g(a)^{\rho}\right]F(a) \tag{4.1}$$

$$+(b-a)k\int_0^1[g(b)-g((1-s)a+sb)]^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega(g(b)-g((1-s)a+sb))^{\rho}\right]F'(sb+(1-s)a)\,ds.$$

Similarly, using (3.2), we have

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}F(a)=\int_0^1\frac{(b-a)g'((1-s)a+sb)}{[g((1-s)a+sb)-g(a)]^{1-\frac{\lambda}{k}}}\mathscr{F}^{\sigma,k}_{\rho,\lambda}\left[\omega(g((1-s)a+sb)-g(a)))^{\rho}\right]F(sb+(1-s)a)\,ds.$$

Using integration by parts

$$\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}F(a) = k\left[g(b)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g(b)-g(a)^{\rho}\right]F(b)\right.$$
$$-(b-a)k\int_{0}^{1}\left[g\left((1-s)a+sb\right)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g\left((1-s)a+sb\right)-g(a)\right)\right)^{\rho}\right]F'\left(sb+(1-s)a\right)ds. \tag{4.2}$$

Adding (4.1) and (4.2) and using the relation $F(x) = f(x)+f(a+b-x)$, we find that

$$\frac{4\left(g(b)-g(a)\right)^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right]}{b-a}\left[\frac{f(a)+f(b)}{2}-\frac{1}{4k\left(g(b)-g(a)\right)^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right]}\left[\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{J}^{\sigma,k,g}_{\rho,\lambda,a+;\omega}F(b)\right]\right]$$
$$=\int_{0}^{1}\left[g\left((1-s)a+sb\right)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g\left((1-s)a+sb\right)-g(a)\right)\right)^{\rho}\right]F'\left(sb+(1-s)a\right)ds$$
$$-\int_{0}^{1}\left[g(b)-g\left((1-s)a+sb\right)\right]^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g(b)-g((1-s)a+sb)\right)^{\rho}\right]F'\left(sb+(1-s)a\right)ds. \tag{4.3}$$

On the other hand, since $F'(x) = f'(x)-f'(a+b-x)$, we have

$$F'\left(sb+(1-s)a\right) = f'\left(sb+(1-s)a\right)-f'\left(sa+(1-s)b\right).$$

Then we obtain,

$$\int_{0}^{1}\left[g\left((1-s)a+sb\right)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g\left((1-s)a+sb\right)-g(a)\right)\right)^{\rho}\right]F'\left(sb+(1-s)a\right)ds \tag{4.4}$$
$$=\int_{0}^{1}\left[g\left((1-s)a+sb\right)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g\left((1-s)a+sb\right)-g(a)\right)\right)^{\rho}\right]f'\left(sb+(1-s)a\right)ds$$
$$-\int_{0}^{1}\left[g\left((1-s)a+sb\right)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g\left((1-s)a+sb\right)-g(a)\right)\right)^{\rho}\right]f'\left(sa+(1-s)b\right)ds$$
$$=\int_{0}^{1}\left[g\left((1-s)a+sa\right)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g\left((1-s)b+sa\right)-g(a)\right)\right)^{\rho}\right]f'\left(sa+(1-s)b\right)ds$$
$$-\int_{0}^{1}\left[g\left((1-s)a+sb\right)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(g\left((1-s)a+sb\right)-g(a)\right)\right)^{\rho}\right]f'\left(sa+(1-s)b\right)ds.$$

and

$$\int_0^1 [g(b)-g((1-s)a+sb)]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega(g(b)-g((1-s)a+sb))^{\rho}\right] F'(sb+(1-s)a)\,ds \tag{4.5}$$

$$= \int_0^1 [g(b)-g((1-s)a+sb)]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega(g(b)-g((1-s)a+sb))^{\rho}\right] f'(sb+(1-s)a)\,ds$$

$$-\int_0^1 [g(b)-g((1-s)a+sb)]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega(g(b)-g((1-s)a+sb))^{\rho}\right] f'(sa+(1-s)b)\,ds$$

$$= \int_0^1 [g(b)-g((1-s)b+sa)]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega(g(b)-g((1-s)b+sa))^{\rho}\right] f'(sa+(1-s)b)\,ds$$

$$-\int_0^1 [g(b)-g((1-s)a+sb)]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega(g(b)-g((1-s)a+sb))^{\rho}\right] f'(sa+(1-s)b)\,ds.$$

If we substitute (4.4) and (4.5) into (4.3), we obtain desired result. □

Theorem 4.2. *Let g be as the above. If $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $|f'|$is an convex function on $[a,b]$, then we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{4k(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega(g(b)-g(a))^{\rho}\right]}\left[\mathscr{J}_{\rho,\lambda,b-;\omega}^{\sigma,k,g}F(a)+\mathscr{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}F(b)\right]\right|$$

$$\leq \frac{I_{\rho,\lambda,g}^{\sigma,k}}{4(b-a)(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega(g(b)-g(a))^{\rho}\right]}\left[|f'(a)|+|f'(b)|\right].$$

where

$$I_{\rho,\lambda,g}^{\sigma,k}=L_{\rho,\lambda,g}^{\sigma,k}(b,b)+L_{\rho,\lambda,g}^{\sigma,k}(a,b)-L_{\rho,\lambda,g}^{\sigma,k}(b,a)-L_{\rho,\lambda,g}^{\sigma,k}(a,a)$$

and $L_{\rho,\lambda}^{\sigma,k}(x,y)$ is defined as

$$L_{\rho,\lambda,g}^{\sigma,k}(x,y)=\int_a^{\frac{a+b}{2}}|x-u|\,|g(y)-g(u)|^{\frac{\lambda}{k}}\mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega|g(y)-g(u)|^{\rho}\right]du-\int_{\frac{a+b}{2}}^{b}|x-u|\,|g(y)-g(u)|^{\frac{\lambda}{k}}\mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega|g(y)-g(u)|^{\rho}\right]du.$$

Proof. From Lemma 4.1 and using convexity of $|f'|$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{4k(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega(g(b)-g(a))^{\rho}\right]}\left[\mathscr{J}_{\rho,\lambda,b-;\omega}^{\sigma,k,g}F(a)+\mathscr{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g}F(b)\right]\right| \tag{4.6}$$

$$\leq \frac{b-a}{4(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega(g(b)-g(a))^{\rho}\right]}\int_0^1\left|\Xi_{\rho,\lambda,g}^{\sigma,k}(s)\right|\left|f'(sa+(1-s)b)\right|ds$$

$$\leq \frac{b-a}{4(g(b)-g(a))^{\frac{\lambda}{k}}\mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega(g(b)-g(a))^{\rho}\right]}\left[\left|f'(a)\right|\int_0^1 s\left|\Xi_{\rho,\lambda,g}^{\sigma,k}(s)\right|ds+\left|f'(b)\right|\int_0^1(1-s)\left|\Xi_{\rho,\lambda,g}^{\sigma,k}(s)\right|ds\right].$$

Here, we have

$$\int_0^1 s\left|\Xi_{\rho,\lambda,g}^{\sigma,k}(s)\right|ds=\frac{1}{(b-a)^2}\int_a^b(b-u)\left|\varphi_{\rho,\lambda,g}^{\sigma,k}(u)\right|dt$$

where

$$\varphi_{\rho,\lambda,g}^{\sigma,k}(u) = [g(u)-g(a)]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(u)-g(a)\right)^{\rho}\right] - [g(a+b-u)-g(a)]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(a+b-u)-g(a)\right)^{\rho}\right]$$

$$-[g(b)-g(u)]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(b)-g(u)\right)^{\rho}\right] + [g(b)-g(a+b-u)]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(b)-g(a+b-u)\right)^{\rho}\right].$$

Since g is an increasing function, $\varphi_{\rho,\lambda,g}^{\sigma,k}$ is a non-decreasing function on $[a,b]$. Additionally,

$$\varphi_{\rho,\lambda,g}^{\sigma,k}(a) = -2\left[g(b)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right] < 0$$

and

$$\varphi_{\rho,\lambda,g}^{\sigma,k}\left(\frac{a+b}{2}\right) = 0.$$

Consequently, we get

$$\begin{cases} \varphi_{\rho,\lambda,g}^{\sigma,k}(u) \leq 0 & \text{if } a \leq u \leq \frac{a+b}{2} \\ \varphi_{\rho,\lambda,g}^{\sigma,k}(u) > 0 & \text{if } \frac{a+b}{2} < u \leq b. \end{cases}$$

Therefore, we have

$$(b-a)^2 \int_0^1 s\left|\Xi_{\rho,\lambda,g}^{\sigma,k}(s)\right| ds = \int_a^b (b-u)\left|\varphi_{\rho,\lambda,g}^{\sigma,k}(u)\right| dt = I_1+I_2+I_3+I_4$$

where

$$I_1 = \int_a^{\frac{a+b}{2}} (b-u)\left[g(b)-g(u)\right]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(b)-g(u)\right)^{\rho}\right] du - \int_{\frac{a+b}{2}}^{b} (b-u)\left[g(b)-g(u)\right]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(b)-g(u)\right)^{\rho}\right] du = L_{\rho,\lambda,g}^{\sigma,k}(b,b),$$

$$I_2 = -\int_a^{\frac{a+b}{2}} (b-u)\left[g(u)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(u)-g(a)\right)^{\rho}\right] du + \int_{\frac{a+b}{2}}^{b} (b-u)\left[g(u)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(u)-g(a)\right)^{\rho}\right] du = -L_{\rho,\lambda,g}^{\sigma,k}(b,a),$$

$$I_3 = \int_a^{\frac{a+b}{2}} (b-u)\left[g(a+b-u)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(a+b-u)-g(a)\right)^{\rho}\right] du$$

$$-\int_{\frac{a+b}{2}}^{b} (b-u)\left[g(a+b-u)-g(a)\right]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(a+b-u)-g(a)\right)^{\rho}\right] du = -L_{\rho,\lambda,g}^{\sigma,k}(a,a),$$

and

$$I_4 = -\int_a^{\frac{a+b}{2}} (b-u)\left[g(b)-g(a+b-u)\right]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(b)-g(a+b-u)\right)^{\rho}\right] du$$

$$+\int_{\frac{a+b}{2}}^{b} (b-u)\left[g(b)-g(a+b-u)\right]^{\frac{\lambda}{k}} \mathscr{F}_{\rho,\lambda+k}^{\sigma,k}\left[\omega\left(g(b)-g(a+b-u)\right)^{\rho}\right] du = L_{\rho,\lambda,g}^{\sigma,k}(a,b).$$

Thus, from the previous equalities it follows that

$$\int_0^1 s\left|\Xi_{\rho,\lambda,g}^{\sigma,k}(s)\right| ds = \frac{L_{\rho,\lambda,g}^{\sigma,k}(b,b)+L_{\rho,\lambda,g}^{\sigma,k}(a,b)-L_{\rho,\lambda,g}^{\sigma,k}(b,a)-L_{\rho,\lambda,g}^{\sigma,k}(a,a)}{(b-a)^2}. \tag{4.7}$$

Similarly, it is clear that

$$\int_0^1 (1-s)\left|\Xi_{\rho,\lambda,g}^{\sigma,k}(s)\right| dt = \frac{L_{\rho,\lambda,g}^{\sigma,k}(b,b)+L_{\rho,\lambda,g}^{\sigma,k}(a,b)-L_{\rho,\lambda,g}^{\sigma,k}(b,a)-L_{\rho,\lambda,g}^{\sigma,k}(a,a)}{(b-a)^2}. \tag{4.8}$$

If we substitute equalities (4.7) and (4.8) in (4.6), we obtain the desired result. □

Corollary 4.3. *If we choose $k=1$ in Theorem 4.2, then we obtain following inequality for generalized fractional integral with respect to the function*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{4k\left(g(b)-g(a)\right)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right]}\left[\mathscr{J}^{\sigma,g}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{J}^{\sigma,g}_{\rho,\lambda,a+;\omega}F(b)\right]\right|$$

$$\leq\frac{I^{\sigma}_{\rho,\lambda,g}}{4\left(b-a\right)\left(g(b)-g(a)\right)^{\frac{\lambda}{k}}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(g(b)-g(a)\right)^{\rho}\right]}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

where $\lambda,\ \rho>0,\ \omega\in\mathbb{R}$ and the coefficients $\sigma(m)$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Corollary 4.4. *If we choose $g(t)=t$ in Theorem 4.2, then we obtain following inequality for generalized k-fractional integrals*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{4k\left(b-a\right)^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(b-a\right)^{\rho}\right]}\left[\mathscr{J}^{\sigma,k}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}F(b)\right]\right|$$

$$\leq\frac{I^{\sigma,k}_{\rho,\lambda}}{4\left(b-a\right)^{\frac{\lambda}{k}+1}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(b-a\right)^{\rho}\right]}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

where $\lambda,\ \rho>0,\ \omega\in\mathbb{R}$ and the coefficients $\sigma(m)$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Corollary 4.5. *If we choose $g(t)=\ln t$ in Theorem 4.2, then we obtain following inequality for generalized Hadamard fractional integrals*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{4k\left(\ln\frac{b}{a}\right)^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(\ln\frac{b}{a}\right)^{\rho}\right]}\left[\mathscr{H}^{\sigma,k}_{\rho,\lambda,b-;\omega}F(a)+\mathscr{H}^{\sigma,k}_{\rho,\lambda,a+;\omega}F(b)\right]\right|$$

$$\leq\frac{I^{\sigma,k}_{\rho,\lambda,\ln}}{4\left(b-a\right)\left(\ln\frac{b}{a}\right)^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(\ln\frac{b}{a}\right)^{\rho}\right]}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

where $\lambda,\ \rho>0,\ \omega\in\mathbb{R}$ and the coefficients $\sigma(m)$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Corollary 4.6. *If we choose $g(t)=\frac{t^{s+1}}{s+1}$, $s\in\mathbb{R}-\{-1\}$ in Theorem 4.2, then we obtain following inequality for generalized (k,s)-fractional integrals*

$$\left|\frac{f(a)+f(b)}{2}-\frac{(s+1)^{\frac{\lambda}{k}}}{4k\left(b^{s+1}-a^{s+1}\right)^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(\frac{b^{s+1}-a^{s+1}}{s+1}\right)^{\rho}\right]}\left[{}^{s}\mathscr{J}^{\sigma,k}_{\rho,\lambda,b-;\omega}F(a)+{}^{s}\mathscr{J}^{\sigma,k}_{\rho,\lambda,a+;\omega}F(b)\right]\right|$$

$$\leq\frac{(s+1)^{\frac{\lambda}{k}}I^{\sigma,k}_{\rho,\lambda,g}}{4\left(b-a\right)\left(b^{s+1}-a^{s+1}\right)^{\frac{\lambda}{k}}\mathscr{F}^{\sigma,k}_{\rho,\lambda+k}\left[\omega\left(\frac{b^{s+1}-a^{s+1}}{s+1}\right)^{\rho}\right]}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]$$

where $\lambda,\ \rho>0,\ \omega\in\mathbb{R}$ and the coefficients $\sigma(m)$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Corollary 4.7. *If we choose $k=1$ and $g(t)=t$ in Theorem 4.2, then we obtain following inequality for generalized fractional integrals*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{2\left(b-a\right)^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(b-a\right)^{\rho}\right]}\left[\mathscr{J}^{\sigma}_{\rho,\lambda,b-;\omega}f(a)+\mathscr{J}^{\sigma}_{\rho,\lambda,a+;\omega}f(b)\right]\right|$$

$$\leq(b-a)\frac{2^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(b-a\right)^{\rho}\right]-\mathscr{F}^{\sigma}_{\rho,\lambda+2}\left[w\left(\frac{b-a}{2}\right)^{\rho}\right]}{2^{\lambda}\mathscr{F}^{\sigma}_{\rho,\lambda+1}\left[w\left(b-a\right)^{\rho}\right]}\left[\frac{\left|f'(a)\right|+\left|f'(b)\right|}{2}\right]$$

where $\lambda,\ \rho>0,\ \omega\in\mathbb{R}$ and the coefficients $\sigma(m)$ $(m\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$ is a bounded sequence of positive real numbers.

Remark 4.8. *If we choose $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ in Corollary 4.7, then Corollary 4.7 reduces to Theorem 3 proved by Sarikaya et. al in [16].*

Remark 4.9. *If we choose $k=1,\ \lambda=\alpha,\ \sigma(0)=1,\ w=0$ in Theorem 4.2, then Theorem 4.2 reduces to Theorem 2.5 proved by Jleli and Samet in [9].*

Remark 4.10. *Choosing $\lambda=\alpha,\ \sigma(0)=1,\ w=0$ in, Corollary 4.4, Corollary 4.5 and Corollary 4.6, one can obtain new result.*

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