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Lacunary Statistical Convergence in Measure for Sequences of Fuzzy Valued Functions

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Abstract

In this study, we examine the concepts of outer and inner lacunary statistical convergence in measure for sequences of fuzzy-valued measurable functions and show that both kinds of convergence are equivalent in a finite measurable set. Also, we investigate the notion of lacunary statistical convergence in measure for sequences of fuzzy-valued measurable functions and establish interesting results. Furthermore, we give the lacunary statistical version of Egorov's theorem for sequences of fuzzy-valued measurable functions in a finite measurable space.

Keywords: *Statistical convergence, Pointwise convergence; Uniformly convergence; Lacunary convergence; Fuzzy-valued function.*
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1. Introduction and Definitions

Throughout the paper $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers. The concept of convergence of a real sequence was extended to statistical convergence independently by Fast [10] and Schoenberg [28]. It became a notable topic in summability theory after the work of Fridy [11] and Šalát [26]. Lacunary statistical convergence was examined by Fridy and Orhan [12]. Balcerzak et al. [6] investigated different kinds of statistical convergence and ideal convergence of sequences of functions namely pointwise, uniform and equi-statistical (or, ideal) convergence. For recent work on these types of convergence, we refer to Belen and Mohiuddine [7], Mohiuddine and Alamri [20]. For the statistical convergence of function sequences, Duman and Orhan [9] introduced convergence in μ-density and μ-statistical convergence of sequences of functions defined on a subset of real numbers, and proposed the concepts of μ-statistical uniform convergence and μ-statistical pointwise convergence.
Among various developments of the theory of fuzzy sets a progressive development has been made to find the fuzzy analogues of the classical set theory by Zadeh [35]. In fact the fuzzy set theory has become an area of active research for the last 40 years. The notion of fuzzyness are using by many persons for Cybernetics, Artificial Intelligence, Expert System and Fuzzy control, Pattern recognition, Operation research, Decision making, Image analysis, Projectiles, Probability theory, Agriculture, Weather forecasting. The fuzzy set theory has been used widely in many engineering applications, such as, in bifurcation of non-linear dynamical systems, in the control of chaos, in the non-linear operator, in population dynamics. The fuzzyness of all the subjects of mathematical sciences has been investigated. It attracted many workers on sequence spaces and summability theory to introduce different types of sequence spaces and study their different properties.
The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka [18] and proved some basic theorems for sequences of fuzzy numbers. Nanda [19] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Considering the uncertainty of data and information in a specific modeling process, this uncertainty was usually represented by a fuzzy number by Negoita [23]. Savaş [27] proved the characterization theorem for the sequence of fuzzy numbers. Statistical convergence in the setting of sequences of fuzzy numbers was discussed by Nuray and Savaş [24], and recently, this notion via difference operators together with weighted mean has been defined and studied by Mohiuddine et al. [21].
Aytar and Pehlivan [4] discussed the statistical convergence of sequences of fuzzy numbers and sequences of α-cuts. Later, Aytar et al. [5] extended the concepts of statistical superior limit and inferior limit to statistically bounded sequences of fuzzy numbers and obtained some fuzzy-analogues of properties of superior statistical limit and inferior limit for real numbers. Altın et al. [1] introduced the concept of pointwise statistical convergence sequences of fuzzy mapping and established some basic properties of fuzzy mappings. Recently, Gong et al. [13] studied statistical convergence, uniformly statistical convergence and equi-statistical convergence for sequences of fuzzy-valued functions and established some basic properties of sequences of fuzzy-valued functions based on sequences of α-level cuts. Some useful results on related topic may be found in Altinok et al. [2], Altinok and Et [3], Çınar et al. [8]. Hazarika et al. [14] investigated outer and

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inner statistical convergence, for double sequences of fuzzy-valued measurable functions, also defined statistical convergence in measure for double sequences of fuzzy-valued measurable functions and establish several interesting results.
Nuray [25] investigated lacunary statistical convergence of sequences of fuzzy numbers. Şençimen and Pehlivan [29] examined the concept of statistically convergent sequence in a fuzzy normed linear space. Türkmen and Çınar [30] studied lacunary statistical convergence in fuzzy normed linear space. Also, lacunary statistical convergence of double sequences in fuzzy normed spaces was investigated by Türkmen and Dündar [31].
Now, we recall some concepts and basic definitions used in the paper. (see [1–4, 6, 9, 10, 12–25, 27, 32–35])
Let K be a nonempty set. A fuzzy subset of K is a nonempty subset $\{t,\bar{x}(t) : t \in K\}$ of $K \times J(=[0,1])$ for some function $\bar{x} : K \to J(=[0,1])$.
A function $\bar{x} : \mathbb{R} \to J(=[0,1])$ is called a fuzzy number if the function $\bar{x}$ satisfies the following properties:
(i) $\bar{x}$ is convex, i.e., $\bar{x}(t) \geq \bar{x}(s) \wedge \bar{x}(r) = \min\{\bar{x}(s),\bar{x}(r)\}$, where $s < t < r$.
(ii) $\bar{x}$ is normal, i.e., there exists an $t_0 \in \mathbb{R}$ such that $\bar{x}(t_0) = 1$.
(iii) $\bar{x}$ is upper semi-continuous, i.e., for each $\varepsilon > 0$, $\bar{x}^{-1}((0,a+\varepsilon])$, for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R}$.
(iv) $[\bar{x}]^0 = cl(\{t \in \mathbb{R} : \bar{x}(t) \geq 0\})$ is compact, where cl is the closure operator.
We denote the set of all fuzzy numbers by $F(\mathbb{R})$. The set $\mathbb{R}$ can be embedded in $F(\mathbb{R})$ if we define $\bar{r} \in F(\mathbb{R})$ by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r, \\ 0, & \text{if } t \neq r. \end{cases}$$

For $0 < \alpha \leq 1$, α-cut of $\bar{x}$ is defined by $[\bar{x}]_\alpha = \{t \in \mathbb{R} : \bar{x}(t) \geq \alpha\} = [\bar{x}_\alpha^-, \bar{x}_\alpha^+]$ is a closed and bounded interval of $\mathbb{R}$. As in [23], the Hausdorff distance between two fuzzy numbers $\bar{x}$ and $\bar{y}$ given by $D : F(\mathbb{R}) \times F(\mathbb{R}) \to [0,\infty)$

$$D(\bar{x},\bar{y}) = \sup_{\alpha\in[0,1]} d([\bar{x}]_\alpha,[\bar{y}]_\alpha) = \sup_{\alpha\in[0,1]} \max\{|\bar{x}_\alpha^- - \bar{x}_\alpha^+|,|\bar{y}_\alpha^- - \bar{y}_\alpha^+|\},$$

where d is the Hausdorff metric. For any $\bar{x},\bar{y},\bar{z},\bar{u} \in F(\mathbb{R})$ we know that
(i) $(F(\mathbb{R}),D)$ is a complete metric space.
(ii) $D(\gamma\bar{x},\gamma\bar{y}) = |\gamma| D(\bar{x},\bar{y})$; $\gamma \in \mathbb{R}$.
(iii) $D(\bar{x}+\bar{u},\bar{y}+\bar{u}) = D(\bar{x},\bar{y})$.
(iv) $D(\bar{x}+\bar{z},\bar{y}+\bar{u}) \leq D(\bar{x},\bar{y}) + D(\bar{z},\bar{u})$.

Lemma 1.1. *Let $\bar{x} \in F(\mathbb{R})$ and $[\bar{x}]_\alpha = [\bar{x}_\alpha^-, \bar{x}_\alpha^+]$. Then, the following conditions are satisfied:*
(i) $\bar{x}_\alpha^-$ is a left continuous monotone nondecreasing function on $(0,1]$.
(ii) $\bar{x}_\alpha^+$ is a right continuous monotone nonincreasing function on $(0,1]$.
(iii) $\bar{x}_\alpha^-$ and $\bar{x}_\alpha^+$ are right continuous at $\alpha = 0$.
(iv) $\bar{x}_1^- \leq \bar{x}_1^+$.

For $K \subset \mathbb{N}$ and $j \in \mathbb{N}$, $\delta_j(K)$ is called jth partial density of K, if

$$\delta_j(K) = \frac{|K \cap \{1,2,...,j\}|}{j}.$$

If

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|, \left(\text{i.e., } \delta(K) = \lim_{j\to\infty} \delta_j(K)\right)$$

exists, it is called the natural density of K, where $|\{k \leq n : k \in K\}|$ denotes the number of elements of K not exceeding n. $\Psi = \{K \subset \mathbb{N} : \delta(K) = 0\}$ is called the zero density set.
A sequence of fuzzy numbers $(\bar{x}_n)$ is said to be statistically convergent to a fuzzy number $\bar{x}_0$ if for every $\varepsilon > 0$

$$\delta(\{n \in \mathbb{N} : D(\bar{x}_n,\bar{x}_0) \geq \varepsilon\}) = 0,$$

i.e., $\{n \in \mathbb{N} : D(\bar{x}_n,\bar{x}_0) \geq \varepsilon\} \in \Psi$. We write $st - \lim \bar{x}_n = \bar{x}_0$ or $\bar{x}_n \xrightarrow{st} \bar{x}_0$, $(n \to \infty)$.
A sequence of fuzzy number valued functions $\{\bar{f}_n\}$ is said to be pointwise statistically convergent to a fuzzy-number-valued function $\bar{f}$, if $\bar{f}_n(x) \xrightarrow{st} \bar{f}(x)$ for each $x \in [a,b]$, i.e.,

$$\forall x \in [a,b],\ \forall \varepsilon > 0,\ \exists M_x \in \Psi,\ \forall n \in \mathbb{N} \setminus M_x,\ D(\bar{f}_n(x),\bar{f}(x)) < \varepsilon.$$

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1},k_r]$.
Throughout this study, let $\theta = \{k_r\}$ be a lacunary sequence.
Let $A \subset \mathbb{N}$ and $r \in \mathbb{N}$. $\delta_\theta^r(A)$ is called the rth partial lacunary density of A, if

$$\delta_\theta^r(A) = \frac{|A \cap I_r|}{h_r}.$$

Let $A \subset \mathbb{N}$. The number $\delta_\theta(A)$ is called the lacunary density or θ-density of A if

$$\delta_\theta(A) = \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : k \in A\}|, \left(i.e., \delta_\theta(A) = \lim_{r\to\infty} \delta_\theta^r(A)\right)$$

exists. Also,

$\Lambda=\{A\subset\mathbb{N}:\delta_\theta(A)=0\}$

is said to be zero density set.
A sequence of fuzzy numbers $(\bar{x}_n)$ is said to be lacunary statistically convergent to a fuzzy number $\bar{x}_0$ if for every $\varepsilon>0$

$\delta_\theta\left(\{n\in\mathbb{N}:D(\bar{x}_n,\bar{x}_0)\geq\varepsilon\}\right)=0.$

We write $S_\theta-\lim_{n\to\infty}\bar{x}_n=\bar{x}_0$ or $\bar{x}_n\overset{S_\theta}{\to}\bar{x}_0$.

2. Main Results

Throughout the paper, we will suppose that $\overline{h}:[a,b]\to F(\mathbb{R})$ and $\overline{h}_m:[a,b]\to F(\mathbb{R})$ are the fuzzy-valued function and a sequence of fuzzy-valued functions for all $m\in\mathbb{N}$, respectively. We will show SFVF and FVF instead of sequence of fuzzy-valued functions and fuzzy-valued function, respectively.

Definition 2.1. *A SFVF $\left(\overline{h}_m\right)$ is pointwise lacunary statistically convergent to FVF $\overline{h}$ on $[a,b]$, denoted by $pS_\theta-\lim\overline{h}_m(y)=\overline{h}(y)$ or $\overline{h}_m\overset{pS_\theta}{\to}\overline{h}$, if for every $y\in[a,b]$ and every $\varepsilon>0$ there exists $T_y\in\Lambda$ such that for all $m\in\mathbb{N}\setminus T_y$ we have $D\left(\overline{h}_m(y),\overline{h}(y)\right)<\varepsilon$. It is clear that $\overline{h}_m\overset{pS_\theta}{\to}\overline{h}$ if for every $y\in[a,b]$ and every $\varepsilon>0$*

$\delta_\theta\left(\left\{m\in\mathbb{N}:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq\varepsilon\right\}\right)=0.$

Here, $\overline{h}$ is called the lacunary statistical limit function of $(\overline{h}_m)$.

Definition 2.2. *A SFVF $\left(\overline{h}_m\right)$ is uniformly lacunary statistically convergent to FVF $\overline{h}$ on $[a,b]$, denoted by $uS_\theta-\lim\overline{h}_m(y)=\overline{h}(y)$ or $\overline{h}_m\overset{uS_\theta}{\rightrightarrows}\overline{h}$, if for every $\varepsilon>0$ there exists $T\in\Lambda$ such that for all $m\in\mathbb{N}\setminus T$ we have $D\left(\overline{h}_m(y),\overline{h}(y)\right)<\varepsilon$, which holds for all $y\in[a,b]$. It is clear that $\overline{h}_m\overset{uS_\theta}{\rightrightarrows}\overline{h}$ if for every $\varepsilon>0$,*

$\delta_\theta\left(\left\{m\in\mathbb{N}:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq\varepsilon\right\}\right)=0,$

for all $y\in[a,b]$.

Remark 2.3. *If $\overline{h}_m\overset{uS_\theta}{\rightrightarrows}\overline{h}$, then $\overline{h}_m\overset{pS_\theta}{\to}\overline{h}$*

Remark 2.4. *$\overline{h}_m\overset{uS_\theta}{\rightrightarrows}\overline{h}$ if and only if $\sup_{x\in[a,b]}D\left(\overline{h}_m(y),\overline{h}(y)\right)\overset{pS_\theta}{\to}0$.*

Theorem 2.5. *Assume that the sequence of fuzzy-valued functions $\overline{h}_m\overset{pS_\theta}{\to}\overline{h}$ on $[a,b]$, where $\left(\overline{h}_m\right)$ are equi-continuous on $[a,b]$, then $\overline{h}$ is continuous and $\overline{h}_m\overset{uS_\theta}{\rightrightarrows}\overline{h}$ on $[a,b]$.*

Proof. First we prove that $\overline{h}_m$ is continuous. Let $\varepsilon>0$ and $y_0\in[a,b]$. By the equi-continuity of $\overline{h}_m$, then there exists $\gamma>0$ such that

$D\left(\overline{h}_m(y),\overline{h}_m(y_0)\right)<\frac{\varepsilon}{3},$

for any $m\in\mathbb{N}$ and $y\in(y_0-\gamma,y_0+\gamma)$. For any $y\in(y_0-\gamma,y_0+\gamma)$, since $\overline{h}_m\overset{pS_\theta}{\to}\overline{h}$, the set

$\left\{m\in\mathbb{N}:D\left(\overline{h}_m(y_0),\overline{h}(y_0)\right)\geq\frac{\varepsilon}{3}\right\}\cup\left\{m\in\mathbb{N}:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq\frac{\varepsilon}{3}\right\}\in\Lambda.$

Hence, there exists $m\in\mathbb{N}$,

$D\left(\overline{h}_m(y_0),\overline{h}(y_0)\right)<\frac{\varepsilon}{3}$ and $D\left(\overline{h}_m(y),\overline{h}(y)\right)<\frac{\varepsilon}{3}$.

We have

$$\begin{aligned}D\left(\overline{h}(y_0),\overline{h}(y)\right)&\leq D\left(\overline{h}(y_0),\overline{h}_m(y_0)\right)+D\left(\overline{h}_m(y_0),\overline{h}_m(y)\right)+D\left(\overline{h}_m(y),\overline{h}(y)\right)\\&<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

and the continuity of $\overline{h}$ is proved.
Now, we will prove that $\overline{h}_m\overset{uS_\theta}{\rightrightarrows}\overline{h}$ on $[a,b]$. Let $\varepsilon>0$. Since $\overline{h}$ is continuous on $[a,b]$, it gives that $\overline{h}$ is uniformly continuous and $\left(\overline{h}_m\right)$ is uniformly equi-continuous on $[a,b]$. Hence, pick $\gamma>0$ such that $|y-y'|<\gamma$ for any $y,y'\in[a,b]$, we have

$D\left(\overline{h}_m(y),\overline{h}_m(y')\right)<\frac{\varepsilon}{3}$ and $D\left(\overline{h}(y),\overline{h}(y')\right)<\frac{\varepsilon}{3}$.

By the finite covering theorem, choose finite open coverings

$(y_1-\gamma,y_1+\gamma),\ (y_2-\gamma,y_2+\gamma),\ \cdots,\ (y_r-\gamma,y_r+\gamma)$

from the cover of $[a,b]$. Using $\overline{h}_m \overset{\mu S_\theta}{\to} \overline{h}$, there exists a set $M_{y_i} \in \Lambda$ such that

$$D\left(\overline{h}_m(y_i), \overline{h}(y_i)\right) < \frac{\varepsilon}{3},$$

for all $m \notin M_{y_i}$ and $i \in \{1,2,...,r\}$. Let $m \notin M_{y_i}$ and $y \in [a,b]$. Thus $y \in (y_i - \gamma, y_i + \gamma)$ for some $i \in \{1,2,...,r\}$. Hence

$$\begin{aligned} D\left(\overline{h}_m(y), \overline{h}(y)\right) &\leq D\left(\overline{h}_m(y), \overline{h}_m(y_i)\right) + D\left(\overline{h}_m(y_i), \overline{h}(y_i)\right) + D\left(\overline{h}(y_i), \overline{h}(y)\right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which yields that $\overline{h}_m \overset{uS_\theta}{\rightrightarrows} \overline{h}$ on $[a,b]$. □

Definition 2.6. *A SFVF $\left(\overline{h}_m\right)$ is lacunary equi-statistically convergent to FVF $\overline{h}$, denoted by $\overline{h}_m \overset{eS_\theta}{\twoheadrightarrow} \overline{h}$, if for given $\varepsilon > 0$,*

$$G_{r,\varepsilon} = \delta_\theta^r\left(\left\{m \in \mathbb{N} : D\left(\overline{h}_m(y), \overline{h}(y)\right) \geq \varepsilon\right\}\right)$$

with regards to $y \in [a,b]$ is uniformly convergent to zero function. Thus, $\overline{h}_m \overset{eS_\theta}{\twoheadrightarrow} \overline{h}$ if and only if for all $\varepsilon, \beta > 0$, $\exists k \in \mathbb{N}$, for all $r \geq k$ and all $y \in [a,b]$,

$$\delta_\theta^r\left(\left\{m \in \mathbb{N} : D\left(\overline{h}_m(y), \overline{h}(y)\right) \geq \varepsilon\right\}\right) < \beta.$$

Notice that, by monotonicity of δ_θ^r, we can also take $\beta = \varepsilon$.

Remark 2.7. *It is clear that $\overline{h}_m \overset{pS_\theta}{\to} \overline{h}$ if and only if for every $y \in Y$ and every $\varepsilon, \beta > 0$ $\exists k \in \mathbb{N}$, for all $r \geq k$,*

$$\delta_\theta^r\left(\left\{m \in \mathbb{N} : D\left(\overline{h}_m(y), \overline{h}(y)\right) \geq \varepsilon\right\}\right) < \beta.$$

In this case, we may take $\beta = \varepsilon$. Clearly, $\overline{h}_m \overset{eS_\theta}{\twoheadrightarrow} \overline{h}$ implies $\overline{h}_m \overset{pS_\theta}{\to} \overline{h}$. Morever, we can see that $\overline{h}_m \overset{uS_\theta}{\rightrightarrows} \overline{h}$ implies $\overline{h}_m \overset{eS_\theta}{\twoheadrightarrow} \overline{h}$.

Theorem 2.8. *A SFVMF $\left(\overline{h}_m\right)$ is uniformly lacunary statistically convergent to FVMF $\overline{h}$ if and only if $\left[\overline{h}_m(y)\right]_\alpha$ is uniformly lacunary statistically convergent to $\left[\overline{h}(y)\right]_\alpha$ uniformly with regards to α and y.*

Proof. Let $\varepsilon > 0$. Given $\overline{h}_m \overset{uS_\theta}{\rightrightarrows} \overline{h}$, there exists $M \in \Lambda$ such that $D\left(\overline{h}_m(y), \overline{h}(y)\right) < \varepsilon$, for any $m \in \mathbb{N} \setminus M$ and $y \in [a,b]$, i.e.,

$$\sup_{\alpha \in [0,1]} \max\left\{\left|\left(\overline{h}_m(y)\right)_\alpha^- - \overline{h}_\alpha^-(y)\right|, \left|\left(\overline{h}_m(y)\right)_\alpha^+ - \overline{h}_\alpha^+(y)\right|\right\} < \varepsilon.$$

That is, there are

$$\left|\left(\overline{h}_m(y)\right)_\alpha^- - \overline{h}_\alpha^-(y)\right| < \varepsilon \text{ and } \left|\left(\overline{h}_m(y)\right)_\alpha^+ - \overline{h}_\alpha^+(y)\right| < \varepsilon,$$

for any $m \in \mathbb{N} \setminus M$ and $y \in [a,b]$. In addition,

$$\left[\overline{h}_m(y)\right]_\alpha = \left[\left(\overline{h}_m(y)\right)_\alpha^-, \left(\overline{h}_m(y)\right)_\alpha^+\right] \text{ and } \left[\overline{h}(y)\right]_\alpha = \left[\overline{h}_\alpha^-(y), \overline{h}_\alpha^+(y)\right].$$

Therefore, we get $\left[\overline{h}_m(y)\right]_\alpha$ is uniformly lacunary statistically convergent to $\left[\overline{h}(y)\right]_\alpha$ uniformly with regards to α and y.
Conversely, for any $\alpha \in [0,1]$ and for any $y \in [a,b]$, $\left[\overline{h}_m(y)\right]_\alpha$ is uniformly lacunary statistically convergent to $\left[\overline{h}(y)\right]_\alpha$ with regards to α and y. Thus, for given $\varepsilon > 0$ there exists $M_1 \in \Lambda$ such that

$$\left|\left(\overline{h}_m(y)\right)_\alpha^- - \overline{h}_\alpha^-(y)\right| < \varepsilon,$$

for all $m \in \mathbb{N} \setminus M_1$, $y \in [a,b]$ and any $\alpha \in [0,1]$. Also, we can see that for given $\varepsilon > 0$ there exists $M_2 \in \Lambda$ such that

$$\left|\left(\overline{h}_m(y)\right)_\alpha^+ - \overline{h}_\alpha^+(y)\right| < \varepsilon,$$

for all $m \in \mathbb{N} \setminus M_2$, any $y \in [a,b]$ and any $\alpha \in [0,1]$. Let $M = M_1 \cup M_2 \in \Lambda$. We have

$$\left|\left(\overline{h}_m(y)\right)_\alpha^- - \overline{h}_\alpha^-(y)\right| < \varepsilon \text{ and } \left|\left(\overline{h}_m(y)\right)_\alpha^+ - \overline{h}_\alpha^+(y)\right| < \varepsilon,$$

for all $m \in \mathbb{N} \setminus M$, any $y \in [a,b]$ and any $\alpha \in [0,1]$. Hence, we get

$$\sup_{\alpha \in [0,1]} \max\left\{\left|\left(\overline{h}_m(y)\right)_\alpha^- - \overline{h}_\alpha^-(y)\right|, \left|\left(\overline{h}_m(y)\right)_\alpha^+ - \overline{h}_\alpha^+(y)\right|\right\} < \varepsilon,$$

that is,

$$D\left(\overline{h}_m(y), \overline{h}(y)\right) < \varepsilon.$$

This completes the proof. □

Theorem 2.9. *Let $\overline{h}$ FVMF and $\left(\overline{h}_m\right)$ SFVMF. Fix $y_0 \in [a,b]$, if $\overline{h}_m \overset{eS_\theta}{\twoheadrightarrow} \overline{h}$ on $[a,b]$ and all $\left(\overline{h}_m\right)$ are continuous on y_0, then $\overline{h}$ is continuous at y_0.*

Proof. Let $\varepsilon > 0$. $\overline{h}_m \overset{eS_\theta}{\twoheadrightarrow} \overline{h}$, we can find a number $r \in \mathbb{N}$ such that for all $y \in [a,b]$

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : D\left(\overline{h}_m(y), \overline{h}(y)\right) \geq \frac{\varepsilon}{3} \right\} \right) < \frac{1}{2}.$$

Let $K(y) = \left\{ m \in \mathbb{N} : D\left(\overline{h}_m(y), \overline{h}(y)\right) < \frac{\varepsilon}{3} \right\}$, $y \in [a,b]$. Therefore, $\delta_\theta^r(K(y)) > \frac{1}{2}$, for all $y \in [a,b]$. By the continuity of $\overline{h}_1, \overline{h}_2, ..., \overline{h}_r$ at y_0, there is a neighborhood $(y_0 - \zeta, y_0 + \zeta)$ of y_0 such that

$$D\left(\overline{h}_i(y), \overline{h}_i(y_0)\right) < \frac{\varepsilon}{3},$$

for all $i = 1,2,...,r$, $y \in (y_0 - \zeta, y_0 + \zeta)$. Fix $y \in (y_0 - \zeta, y_0 + \zeta)$. Since $\delta_\theta^r(K(y)) > \frac{1}{2}$ and $\delta_\theta^r(K(y_0)) > \frac{1}{2}$, we find $p \in K(y) \cap K(y_0)$. Thus,

$$\begin{aligned} D\left(\overline{h}(y), \overline{h}(y_0)\right) &\leq D\left(\overline{h}(y), \overline{h}_p(y)\right) + D\left(\overline{h}_p(y), \overline{h}_p(y_0)\right) + D\left(\overline{h}_p(y_0), \overline{h}(y_0)\right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Thus, we have

$$D\left(\overline{h}(y), \overline{h}(y_0)\right) < \varepsilon,$$

for all $y \in \mathscr{U}(y_0, \zeta)$, i.e., $\overline{h}$ is continuous on y_0. □

Theorem 2.10. *A of SFVMF $\left(\overline{h}_m\right)$ is lacunary equi-statistically convergent to FVMF $\overline{h}$ if and only if $\left[\overline{h}_m(y)\right]_\alpha$ is lacunary equi-statistically convergent to $\left[\overline{h}(y)\right]_\alpha$ uniformly for any $\alpha \in [0,1]$ and any $y \in [a,b]$.*

Proof. $\overline{h}_m \overset{eS_\theta}{\twoheadrightarrow} \overline{h}$ shows that for any $\varepsilon > 0$ and $\sigma > 0$, there exists $k \in \mathbb{N}$, for all $r \geq k$ and any $y \in [a,b]$ such that

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : D\left(\overline{h}_m(y), \overline{h}(y)\right) \geq \varepsilon \right\} \right) < \sigma.$$

Thus, for any $\alpha \in [0,1]$ we get

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : \sup_{\alpha \in [0,1]} \max \left\{ \left| \left(\overline{h}_m(y)\right)_\alpha^- - \overline{h}_\alpha^-(y) \right|, \left| \left(\overline{h}_m(y)\right)_\alpha^+ - \overline{h}_\alpha^+(y) \right| \right\} \geq \varepsilon \right\} \right) < \sigma.$$

Therefore, for any $\alpha \in [0,1]$ we obtain

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : \left| \left(\overline{h}_m(y)\right)_\alpha^- - \overline{h}_\alpha^-(y) \right| \geq \varepsilon \right\} \right) < \sigma$$

and

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : \left| \left(\overline{h}_m(y)\right)_\alpha^+ - \overline{h}_\alpha^+(y) \right| \geq \varepsilon \right\} \right) < \sigma.$$

Then, note that

$$\left[\overline{h}_m(y)\right]_\alpha = \left[\left(\overline{h}_m(y)\right)_\alpha^-, \ \left(\overline{h}_m(y)\right)_\alpha^+ \right] \text{ and } \left[\overline{h}(y)\right]_\alpha = \left[\overline{h}_\alpha^-(y), \ \overline{h}_\alpha^+(y) \right].$$

Hence, $\left[\overline{h}_m(y)\right]_\alpha$ is uniformly lacunary statistically convergent to $\left[\overline{h}(y)\right]_\alpha$ for any $\alpha \in [0,1]$ and any $y \in [a,b]$.
Conversely, let $\varepsilon > 0$ and $\sigma > 0$, there exists $k_1 \in \mathbb{N}$ such that

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : \left| \left(\overline{h}_m(y)\right)_\alpha^- - \overline{h}_\alpha^-(y) \right| \geq \varepsilon \right\} \right) < \sigma,$$

for all $r \geq k_1$ and any $y \in [a,b]$ and for any $\alpha \in [0,1]$. Analogously, there exists $k_2 \in \mathbb{N}$ such that

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : \left| \left(\overline{h}_m(y)\right)_\alpha^+ - \overline{h}_\alpha^+(y) \right| \geq \varepsilon \right\} \right) < \sigma,$$

for all $r \geq k_2$ and any $y \in [a,b]$ and for any $\alpha \in [0,1]$. Then, pick $k = \max\{k_1, k_2\}$. We may get

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : \left| \left(\overline{h}_m(y)\right)_\alpha^- - \overline{h}_\alpha^-(y) \right| \geq \varepsilon \right\} \right) < \sigma,$$

and

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : \left| \left(\overline{h}_m(y)\right)_\alpha^+ - \overline{h}_\alpha^+(y) \right| \geq \varepsilon \right\} \right) < \sigma,$$

for all $r \geq k$ and any $y \in [a,b]$ and for any $\alpha \in [0,1]$. Thus, we have

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : \sup_{\alpha \in [0,1]} \max \left\{ \left| \left(\overline{h}_m(y)\right)_\alpha^- - \overline{h}_\alpha^-(y) \right|, \left| \left(\overline{h}_m(y)\right)_\alpha^+ - \overline{h}_\alpha^+(y) \right| \right\} \geq \varepsilon \right\} \right) < \sigma,$$

that is,

$$\delta_\theta^r \left(\left\{ m \in \mathbb{N} : D\left(\overline{h}_m(y), \overline{h}(y)\right) \geq \varepsilon \right\} \right) < \sigma.$$

This completes the proof. □

The generalization of Egorov's theorem, a classical and well known result of measure theory, has been studied by several authors in various directions. The following result is the lacunary statistical version of Egorov's theorem, a classical theorem of measure theory, for the SFVF.

Theorem 2.11. *Let $(\Omega,\mathscr{M},\mu)$ be a finite measurable space. Suppose that the FVF $\overline{h}$ and SFVF $\left(\overline{h}_m\right)$ are measurable and defined almost everywhere on Ω. Suppose also that $\overline{h}_m \overset{pS_\theta}{\rightarrow} \overline{h}$ almost everywhere on Ω. Then, for every $\varepsilon>0$ there exists $A\subset\mathscr{M}$ such that $\mu(\Omega\setminus A)<\varepsilon$ and $\overline{h}_{m|A} \overset{eS_\theta}{\twoheadrightarrow} \overline{h}_{|A}$ on A.*

Proof. We assume that each fuzzy-valued functions $\left(\overline{h}_m\right)$ and $\overline{h}$ are defined everywhere on Ω and also suppose that $\overline{h}_m(y) \overset{pS_\theta}{\rightarrow} \overline{h}(y)$ for all $y\in\Omega$. Now, for any fix $\sigma, r\in\mathbb{N}$, examine that the set

$$P=\left\{y\in\Omega:\delta_\theta^r\left(\left\{m\in\mathbb{N}:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq\frac{1}{\sigma}\right\}\right)<\frac{1}{\sigma}\right\}$$

is measurable. Then, the function $\Phi_m(y)=D\left(\overline{h}_m(y),\overline{h}(y)\right)$, $y\in\Omega$, is measurable. Let $H_m=\Phi_m^{-1}\left(\left[\frac{1}{\sigma},\infty\right)\right)$. For every $y\in\Omega$, we have $y\in P$ if and only if

$$\frac{1}{h_r}\sum_{m\in I_r}\chi_{H_m}(y)<\frac{1}{\sigma}.$$

Since the function

$$h=\frac{1}{h_r}\sum_{m\in I_r}\chi_{H_m}(y)$$

is measurable, so we have $P=h^{-1}\left(\left(-\infty,\frac{1}{\sigma}\right)\right)$. For $k\in\mathbb{N}$, one writes

$$\Theta_{\sigma,k}=\left\{y\in\Omega:\forall r\geq k,\delta_\theta^r\left(\left\{m\in\mathbb{N}:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq\frac{1}{\sigma}\right\}\right)<\frac{1}{\sigma}\right\}.$$

Then, from the previous observation, we conclude that $\Theta_{\sigma,k}$ is measurable and so, we have

$$\Theta_{\sigma,k}\subset\Theta_{\sigma,k+1}\ (\forall k\in\mathbb{N})\ \text{and}\ \Omega=\bigcup_{k=1}^{\infty}\Theta_{\sigma,k}.$$

As a result, $\mu(\Omega)=\lim_{m\to\infty}\mu\left(\Theta_{\sigma,k}\right)$. Let $\varepsilon>0$ be given. For every $k\in\mathbb{N}$, choose $k(\sigma)\in\mathbb{N}$ such that $\mu\left(\Omega\setminus\Theta_{\sigma,k(\sigma)}\right)<\frac{\varepsilon}{2^\sigma}$. Set

$$T_0=\bigcup_{\sigma=1}^{\infty}\left(\Omega\setminus\Theta_{\sigma,k(\sigma)}\right).$$

Then, we have

$$\mu(T_0)\leq\sum_{\sigma=1}^{\infty}\mu\left(\Omega\setminus\Theta_{\sigma,k(\sigma)}\right)<\varepsilon.$$

Let

$$T=\Omega\setminus T_0=\bigcap_{\sigma=1}^{\infty}\Theta_{\sigma,k(\sigma)}.$$

Thus, $\mu(\Omega\setminus T)=\mu(T_0)<\varepsilon$. Hence, we get $\forall\sigma\in\mathbb{N},\forall r\geq k(\sigma),\forall y\in T$,

$$\delta_\theta^r\left(\left\{m\in\mathbb{N}:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq\frac{1}{\sigma}\right\}\right)<\frac{1}{\sigma}.$$

This gives that $\overline{h}_{m|\mathscr{A}} \overset{eS_\theta}{\twoheadrightarrow} \overline{h}_{|\mathscr{A}}$ on $\mathscr{A}$. □

Corollary 2.12. *Let $(\Omega,\mathscr{M},\mu)$ be a finite measurable space. Suppose that the FVF $\overline{h}$ and SFVF $\left(\overline{h}_m\right)$ are measurable and defined almost everywhere on Ω. Then, $\overline{h}_m \overset{pS_\theta}{\rightarrow} \overline{h}$ almost everywhere on Ω iff there exists a sequence (A_m) of sets on $\mathscr{M}$ such that $\overline{h}_{m|A_m} \overset{eS_\theta}{\twoheadrightarrow} \overline{h}_{|A_m}$ on A_m for all m and $\mu\left(\Omega\setminus\bigcup_{m\in\mathbb{N}}A_m\right)=0$.*

Proof. Suppose that both FVF $\overline{h}$ and SFVF $\left(\overline{h}_m\right)$ are measurable and defined almost everywhere on Ω, and also $\overline{h}_m \overset{pS_\theta}{\rightarrow} \overline{h}$ almost everywhere on Ω. Then, the conclusion is obvious by considering $\varepsilon=\frac{1}{m}$ $(m\in\mathbb{N})$ in Theorem 2.11. Next, we suppose that $\overline{h}_{m|A_m} \overset{eS_\theta}{\twoheadrightarrow} \overline{h}_{|A_m}$ on A_m for all m. Thus, we get $\overline{h}_{m|A_m} \overset{pS_\theta}{\rightarrow} \overline{h}_{|A_m}$ on A_m for all m. Therefore, we conclude that $\overline{h}_m \overset{pS_\theta}{\rightarrow} \overline{h}$ almost everywhere on Ω. □

Now, we will define outer and inner lacunary statistical convergence in measure of SFVF and prove that these two concepts are equivalent. For our convenience, we shall use the notations SFVMF and FVMF instead of sequence of fuzzy-valued measurable function and fuzzy-valued measurable function, respectively.

Definition 2.13. *Let $(\Omega,\mathscr{M},\mu)$ be a measurable space. Suppose that the set $\mathscr{L}^0$ of all FVMF defined almost everywhere on Ω, and $(\overline{h}_m)$ and $\overline{h}$ in $\mathscr{L}^0$. The outer lacunary statistical convergence in measure of a SFVMF $(\overline{h}_m)$ to a FVMF $\overline{h}$ is defined by*

$$\delta_\theta^r\left(\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq\eta\right\}\right)\geq\zeta\right\}\right)\to 0,\ \text{if } r\to\infty, \tag{2.1}$$

for η, $\zeta>0$. We shall write $\overline{h}_m\overset{\delta_\theta,\mu}{\to}\overline{h}$. Notice that, by changing the order of δ_θ^r and μ in relation (2.1), one gets the inner statistical convergence in measure of a SFVMF $(\overline{h}_m)$ to a FVMF $\overline{h}$ as follows:

$$\mu\left(\left\{y\in\Omega:\delta_\theta^r\left(\left\{m\in\mathbb{N}:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq\eta\right\}\right)\geq\zeta\right\}\right)\to 0,\ \text{if } r\to\infty.$$

We shall show $\overline{h}_m\overset{\mu,\delta_\theta}{\to}\overline{h}$.

Theorem 2.14. *Let $(\Omega,\mathscr{M},\mu)$ be a measurable space. Suppose that $(\overline{h}_m)$ and $\overline{h}$ in $\mathscr{L}^0$.*
(i) If $\overline{h}_m\overset{\delta_\theta,\mu}{\to}\overline{h}$, then $\overline{h}_m\overset{\mu,\delta_\theta}{\to}\overline{h}$.
(ii) If $\overline{h}_m\overset{\mu,\delta_\theta}{\to}\overline{h}$, then $\overline{h}_m\overset{\delta_\theta,\mu}{\to}\overline{h}$, provided $\mu(\Omega)<\infty$.

Proof. Since $\delta_\theta^r:P_1\to[0,1]$ $(r\in\mathbb{N})$ is a probability measure, one can suppose the product measure $\mu\times\delta_\theta^r$ on the product algebra $\mathscr{M}\otimes P_1$ of subsets of $\Omega\times\mathbb{N}$. Now for fixed $\eta>0$, we write

$$S_\eta=\{(y,m)\in\Omega\times\mathbb{N}:D(\overline{h}_m(y),\overline{h}(y))\geq\eta\}.$$

We define a function $\Phi:\Omega\times\mathbb{N}\to\mathbb{R}$ as

$$\Phi((y,m))=D(\overline{h}_m(y),\overline{h}(y)),(y,m)\in\Omega\times\mathbb{N}$$

is $\mathscr{M}\otimes P_1$-measurable. Therefore, we have $S_\eta\in\mathscr{M}\otimes P_1$. Now for any $\mathscr{K}\subset\Omega\times\mathbb{N}$, one writes

$$\mathscr{K}(y)=\{m\in\mathbb{N}:(y,m)\in\mathscr{K}\}\ \text{if } y\in\Omega$$

and

$$\mathscr{K}(m)=\{y\in\Omega:(y,m)\in\mathscr{K}\}\ \text{if } m\in\mathbb{N}.$$

(i) In order to obtain this, we need to prove that

$$\forall\varepsilon,q>0,\ \exists r_0\in\mathbb{N},\ \forall r\geq r_0,\ \mu\left(\left\{y\in\Omega:\delta_\theta^r\left(S_\eta(y)\right)\geq q\right\}\right)<\varepsilon. \tag{2.2}$$

Fix $\varepsilon>0$ and $q>0$. Since $\overline{h}_m\overset{\delta_\theta,\mu}{\to}\overline{h}$, one may find $r_0\in\mathbb{N}$ such that $r\geq r_0$, one get the following:

$$\delta_\theta^r\left(\left\{m\in\mathbb{N}:\mu\left(S_\eta(m)\right)\geq 1\right\}\right)<\frac{q}{2} \tag{2.3}$$

and

$$\delta_\theta^r\left(\left\{m\in\mathbb{N}:\mu\left(S_\eta(m)\right)\geq\frac{q\varepsilon}{4}\right\}\right)<\frac{q\varepsilon}{4}. \tag{2.4}$$

Suppose that

$$P=\left\{m\in\mathbb{N}:\mu\left(S_\eta(m)\right)<1\right\}.$$

Then, we have from condition (2.3) that

$$\delta_\theta^r(\mathbb{N}\setminus P)<\frac{q}{2},\ \ (\forall r\geq r_0).$$

Hence, for all $\forall r\geq r_0$, one obtains

$$\begin{aligned}\mu\left(\left\{y\in\Omega:\delta_\theta^r\left(S_\eta(y)\right)\geq q\right\}\right)&\leq\mu\left(\left\{y\in\Omega:\delta_\theta^r\left(S_\eta(y)\cap P\right)\geq\frac{q}{2}\right\}\right)+\mu\left(\left\{y\in\Omega:\delta_\theta^r\left(S_\eta(y)\setminus P\right)\geq\frac{q}{2}\right\}\right)\\&\leq\mu\left(\left\{y\in\Omega:\delta_\theta^r\left(S_\eta(y)\cap P\right)\geq\frac{q}{2}\right\}\right).\end{aligned}$$

Let $S_\eta^*=S_\eta\cap(\Omega\times P)$. Therefore, we have

$$S_\eta^*(y)=S_\eta(y)\cap P\ \ (y\in\Omega)\ \text{and } S_\eta^*(m)=S_\eta(m),\ \ (m\in P).$$

To obtain relation (2.2), it is enough to prove that

$$\forall r\geq r_0,\ \mu\left(\left\{y\in\Omega:\delta_\theta^r\left(S_\eta^*(y)\right)\geq\frac{q}{2}\right\}\right)<\varepsilon. \tag{2.5}$$

For the set $S_\eta^*\subset\Omega\times P$ and for every fix $r\in\mathbb{N}$, we can apply the famous Fubini theorem for the characteristic function of S_η^* of the finite measure $\mu\times\delta_\theta^r$. Indeed,

$$S_\eta^*=\bigcup_{m\in P}\left(m\times S_\eta(m)\right),$$

where

$$\mu\left(S_{\eta}(m)\right)<1,\ (\forall m\in P) \text{ and } \delta_{\theta}^{r}(\{(m)\})=0\ (\forall m>r).$$

Thus,

$$\int_{P}\mu\left(S_{\eta}^{*}(m)\right)dm=\left(\mu\times\delta_{\theta}^{r}\right)\left(S_{\eta}^{*}\right)=\int_{\Omega}\delta_{\theta}^{r}\left(S_{\eta}^{*}(y)\right)dy.$$

Assume $r_0\in\mathbb{N}$ such that $r\geq r_0$, one obtains

$$\begin{aligned}
\frac{q\varepsilon}{2} &> \frac{q\varepsilon}{4}+\delta_{\theta}^{r}\left(\left\{m\in\mathbb{N}:\mu\left(S_{\eta}(m)\right)\geq\frac{q\varepsilon}{4}\right\}\right)\\
&\geq \int_{\left\{m\in P:\mu\left(S_{\eta}(m)\right)<\frac{q\varepsilon}{4}\right\}}\mu\left(S_{\eta}(m)\right)dm+\int_{\left\{m\in P:\mu\left(S_{\eta}(m)\right)\geq\frac{q\varepsilon}{4}\right\}}1.dm\\
&\geq \int_{P}\mu\left(S_{\eta}(m)\right)dm=\int_{P}\mu\left(S_{\eta}^{*}(m)\right)dm\\
&= \int_{\Omega}\delta_{\theta}^{r}\left(S_{\eta}^{*}(y)\right)dy\geq\int_{\left\{y\in\Omega:\mu\left(S_{\eta}^{*}(y)\right)\geq\frac{q}{2}\right\}}\delta_{\theta}^{r}\left(S_{\eta}^{*}(y)\right)dy\\
&\geq \frac{q}{2}\mu\left(\left\{y\in\Omega:\delta_{\theta}^{r}\left(S_{\eta}^{*}(y)\right)\geq\frac{q}{2}\right\}\right),
\end{aligned}$$

which shows that strict inequality (2.5) is valid.
(ii) Assume that $\mu(\Omega)<\infty$. Fix $\eta>0$. To prove our result, we need to show that

$$\forall\varepsilon,q>0,\ \exists r_0\in\mathbb{N},\ \forall r\geq r_0,\ \delta_{\theta}^{r}\left(\left\{m\in\mathbb{N}:\mu\left(S_{\eta}(m)\right)\geq q\right\}\right)<\varepsilon.$$

Let $\varepsilon>0$ and $q>0$ be given. Since $\overline{h}_m\overset{\mu;\delta_\theta}{\to}\overline{h}$, one may find $r_0\in\mathbb{N}$ such that for all $r\geq r_0$, we have

$$\mu\left(\left\{y\in\Omega:\delta_{\theta}^{r}\left(S_{\eta}(y)\right)\geq\frac{q\varepsilon}{2\mu(\Omega)}\right\}\right)<\frac{q\varepsilon}{2}.$$

By taking into account the well-known Fubini theorem for the characteristic function of $S_\eta\subset\Omega\times I_r$, we get

$$\int_{\Omega}\delta_{\theta}^{r}\left(S_{\eta}(y)\right)dy=\left(\mu\times\delta_{\theta}^{r}\right)\left(S_{\eta}\right)=\int_{\mathbb{N}}\mu\left(S_{\eta}(m)\right)dm.$$

Supposing r_0 such that for all $r\geq r_0$, we have

$$\begin{aligned}
q\varepsilon &> \frac{q\varepsilon\mu(\Omega)}{2\mu(\Omega)}+\mu\left(\left\{y\in\Omega:\delta_{\theta}^{r}\left(S_{\eta}(y)\right)\geq\frac{q\varepsilon}{2\mu(\Omega)}\right\}\right)\\
&\geq \int_{\left\{y\in\Omega:\ \delta_{\theta}^{r}\left(S_{\eta}(y)\right)<\frac{q\varepsilon}{2\mu(\Omega)}\right\}}\delta_{\theta}^{r}\left(S_{\eta}(y)\right)dy+\int_{\left\{y\in\Omega:\ \delta_{\theta}^{r}\left(S_{\eta}(y)\right)\geq\frac{q\varepsilon}{2\mu(\Omega)}\right\}}1dy\\
&\geq \int_{\Omega}\delta_{\theta}^{r}\left(S_{\eta}(y)\right)dy=\int_{\mathbb{N}}\mu\left(S_{\eta}(m)\right)dm\\
&\geq \int_{\left\{m\in\mathbb{N}:\mu\left(S_{\eta}(m)\right)\geq q\right\}}\mu\left(S_{\eta}(m)\right)dm\\
&\geq q\delta_{\theta}^{r}\left(\left\{m\in\mathbb{N}:\mu\left(S_{\eta}(m)\right)\geq q\right\}\right).
\end{aligned}$$

This completes the proof. □

Theorem 2.14 shows that the both kinds of convergence (in Definition 2.13) in measure are equivalent if Ω is finite measurable set. Hence, by considering finite measurable set Ω, we define lacunary statistical convergence in measure of SFVF as follows.

Definition 2.15. *Let $(\Omega,\mathscr{M},\mu)$ be a finite measurable space, Assume that $\left(\overline{h}_m\right)$ and $\overline{h}$ in $\mathscr{L}^0$. A SFVMF $\left(\overline{h}_m\right)$ is said to be lacunary statistical convergent in measure (shortly, LSCM) to a FVMF $\overline{h}$, in symbol, $\overline{h}_m\overset{\mu S_\theta}{\to}\overline{h}$, if $\mu\left(\left\{y\in\Omega:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)$ is lacunary statistically convergent to zero for any $q>0$ and all $m\in\mathbb{N}$. We give this notion is equivalent to the following formula:*

$$\forall\eta>0,\ \forall q>0,\ \left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)>\eta\right\}\in\Lambda.$$

Here, we can write $\eta=q$ or $q=\frac{1}{r}$, $r\in\mathbb{N}$.

Proposition 1. *Let $(\Omega,\mathscr{M},\mu)$ be a finite measurable space. Assume that $\left(\overline{h}_m\right)$ and $\overline{h}$ in $\mathscr{L}^0$. Then, $\overline{h}_m\overset{uS_\theta}{\rightrightarrows}\overline{h}\Rightarrow\overline{h}_m\overset{\mu S_\theta}{\to}\overline{h}$.*

Proof. We assume that $\overline{h}_m\overset{uS_\theta}{\rightrightarrows}\overline{h}$. Let $q>0$ be given. Then, there is a set $T\in\Lambda$ such that

$$D\left(\overline{h}_m(y),\overline{h}(y)\right)<q,\ \forall m\notin T,\ y\in\Omega.$$

Thus, we get

$$\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)>q\right\}\subset\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)\neq\emptyset\right\}\subset T\in\Lambda.$$

This shows that $\overline{h}_m(y)\overset{\mu S_\theta}{\to}\overline{h}(y)$. □

Theorem 2.16. *Let $(\Omega,\mathscr{M},\mu)$ be a measurable space. Assume that $(\overline{h}_m)$ and $\overline{h}$ in $\mathscr{L}^0$. If SFVMF $(\overline{h}_m)$ pointwise lacunary statistically convergent to a FVMF $\overline{h}$ almost everywhere on Ω, then $\overline{h}_m \overset{\mu S_\theta}{\to} \overline{h}$.*

Proof. Suppose that $\overline{h}_m(y) \overset{pS_\theta}{\to} \overline{h}(y)$ almost everywhere on Ω. We have from Theorem 2.14 that $\overline{h}_m \overset{\mu,\delta_\theta}{\to} \overline{h}$ is same as $\overline{h}_m(y) \overset{\mu S_\theta}{\to} \overline{h}(y)$. So, to prove our result, we will prove that $\overline{h}_m(y) \overset{\mu,\delta_\theta}{\to} \overline{h}(y)$. Assume that $\varepsilon > 0$ and $q > 0$. It follows from Theorem 2.11 that $\mathscr{A} \subset \mathscr{M}$ such that $\overline{h}_{m|\mathscr{A}} \overset{eS_\theta}{\twoheadrightarrow} \overline{h}_{|\mathscr{A}}$ and $\mu(\Omega\setminus\mathscr{A}) < \varepsilon$. Choose indexes k such that

$$\delta_\theta^r\left(\left\{m\in\mathbb{N}: D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right) < q$$

for all $r \geq k$ and $y \in \mathscr{A}$. Thus, for all $r \geq k$, we have

$$\left\{y\in\Omega: \delta_\theta^r\left(\left\{m\in\mathbb{N}: D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)\geq q\right\}\subset\Omega\setminus\mathscr{A}.$$

Hence, for all $r \geq k$,

$$\mu\left(\left\{y\in\Omega: \delta_\theta^r\left(\left\{m\in\mathbb{N}: D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)\geq q\right\}\right)<\varepsilon$$

as desired. □

Theorem 2.17. *Let $(\Omega,\mathscr{M},\mu)$ be a finite measurable space. Assume that both $(\overline{h}_m)$, $\overline{h}\in\mathscr{L}^0$. If $\overline{h}_m(y) \overset{pS_\theta}{\to} \overline{h}(y)$ almost everywhere on Ω, then $\overline{h}_m(y) \overset{\mu S_\theta}{\to} \overline{h}(y)$.*

Proof. Let $q,\varepsilon > 0$ be given. In view of Theorem 2.11, there is an $\mathscr{A} \subset \mathscr{M}$ such that $\overline{h}_{m|\mathscr{A}} \overset{eS_\theta}{\twoheadrightarrow} \overline{h}_{|\mathscr{A}}$ and $\mu(\Omega\setminus\mathscr{A}) < \varepsilon$. Consider $k\in\mathbb{N}$ such that

$$\delta_\theta^r\left(\left\{m\in\mathbb{N}: D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right) < q, \quad (\forall r\geq k \text{ and } y\in\mathscr{A})$$

which yields

$$\left\{y\in\Omega: \delta_\theta^r\left(\left\{m\in\mathbb{N}: D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)\geq q\right\}\subset\Omega\setminus\mathscr{A} \ (\forall r\geq k).$$

Therefore, one obtains

$$\mu\left(\left\{y\in\Omega: \delta_\theta^r\left(\left\{m\in\mathbb{N}: D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)\geq q\right\}\right)<\varepsilon.$$

□

Corollary 2.18. *Let $(\Omega,\mathscr{M},\mu)$ be a finite measurable space. Assume that both $(\overline{h}_m)$, $\overline{h}\in\mathscr{L}^0$. If $\overline{h}_m(y) \overset{\mu S_\theta}{\to} \overline{h}(y)$, then $\exists$ a subsequence $(\overline{h}_{m_k})$ of $(\overline{h}_m)$ such that $\overline{h}_{m_k}(y) \overset{pS_\theta}{\to} \overline{h}(y)$ almost everywhere on Ω.*

Proof. Suppose that $\overline{h}_m(y) \overset{\mu S_\theta}{\to} \overline{h}(y)$, so any subsequence $(\overline{h}_{m_k})$ of $(\overline{h}_m)$ also lacunary statistically convergent in measure to $\overline{h}$. Thus, $(\overline{h}_m)$ has a subsequence that lacunary statistically convergent in measure to $\overline{h}$ almost everywhere on Ω. This means that $\overline{h}_{m_k}(y) \overset{pS_\theta}{\to} \overline{h}(y)$ almost everywhere on Ω. □

Definition 2.19. *Let $(\Omega,\mathscr{M},\mu)$ be a finite measurable space. Assume that $(\overline{h}_m)^+_\alpha$, $(\overline{h}_m)^-_\alpha$, $\overline{h}^+_\alpha$, $\overline{h}^-_\alpha\in\mathscr{L}^0$. The double sequence $[\overline{h}_m(y)]_\alpha$ is uniformly lacunary statistically convergent in measure (in short, we shall write ULSCM) to $[\overline{h}(y)]_\alpha$ with regards to α if*

$$\mu\left(\left\{y\in\Omega: \sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)^+_\alpha-\left(\overline{h}(y)\right)^+_\alpha\right|\geq q\right\}\right)$$

and

$$\mu\left(\left\{y\in\Omega: \sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)^-_\alpha-\left(\overline{h}(y)\right)^-_\alpha\right|\geq q\right\}\right)$$

both lacunary statistically convergent in measure to zero for every $q > 0$. Note that this notion is equivalent to the following formula:

$$\forall\eta>0,\forall q>0,\ \left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega: \sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)^+_\alpha-\left(\overline{h}(y)\right)^+_\alpha\right|\geq q\right\}\right)\geq\eta\right\}\in\Lambda$$

and

$$\forall\eta>0,\forall q>0,\ \left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega: \sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)^-_\alpha-\left(\overline{h}(y)\right)^-_\alpha\right|\geq q\right\}\right)\geq\eta\right\}\in\Lambda,$$

in this case, one can write $\eta = q$ or $q=\frac{1}{r}$, $r\in\mathbb{N}$.

Theorem 2.20. *Let $(\Omega,\mathscr{M},\mu)$ be a finite measurable space. Assume that both $(\overline{h}_m)$, $\overline{h}\in\mathscr{L}^0$. A SFVMF $(\overline{h}_m)$ is LSCM to FVMF $\overline{h}$ if and only if $[\overline{h}_m(y)]_\alpha$ is ULSCM to $[\overline{h}(y)]_\alpha$ with regards to α.*

Proof. Assume that $\left(\overline{h}_m\right)$ is LSCM to $\overline{h}$. Then,

$$\mu\left(\left\{y\in\Omega:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)$$

is lacunary statistically convergent to zero for every $q>0$, i.e.,

$$\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)\geq q\right\}\in\Lambda.$$

Thus,

$$\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:\sup_{\alpha\in[0,1]}\max\left\{\left|\left(\overline{h}_m(y)\right)_\alpha^- -\left(\overline{h}(y)\right)_\alpha^-\right|,\left|\left(\overline{h}_m(y)\right)_\alpha^+ -\left(\overline{h}(y)\right)_\alpha^+\right|\right\}\geq q\right\}\right)\geq q\right\}\in\Lambda.$$

Therefore, for every $q>0$, one obtains

$$\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:\sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)_\alpha^- -\left(\overline{h}(y)\right)_\alpha^-\right|\geq q\right\}\right)\geq q\right\}\in\Lambda$$

and

$$\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:\sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)_\alpha^+ -\left(\overline{h}(y)\right)_\alpha^+\right|\geq q\right\}\right)\geq q\right\}\in\Lambda,$$

which yields that

$$\mu\left(\left\{y\in\Omega:\sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)_\alpha^- -\left(\overline{h}(y)\right)_\alpha^-\right|\geq q\right\}\right)\overset{\mu S_\theta}{\to}0$$

and

$$\mu\left(\left\{y\in\Omega:\sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)_\alpha^+ -\left(\overline{h}(y)\right)_\alpha^+\right|\geq q\right\}\right)\overset{\mu S_{\theta_2}}{\to}0.$$

Hence, $\left[\overline{h}_m(y)\right]_\alpha$ is ULSCM to $\left[\overline{h}(y)\right]_\alpha$ with regards to α.
Next, we suppose that $\left[\overline{h}_m(y)\right]_\alpha$ is ULSCM to $\left[\overline{h}(y)\right]_\alpha$ with regards to α. Then, for every $q>0$, one have

$$\mu\left(\left\{y\in\Omega:\sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)_\alpha^- -\left(\overline{h}(y)\right)_\alpha^-\right|\geq q\right\}\right)\overset{\mu S_\theta}{\to}0$$

and

$$\mu\left(\left\{y\in\Omega:\sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)_\alpha^+ -\left(\overline{h}(y)\right)_\alpha^+\right|\geq q\right\}\right)\overset{\mu S_\theta}{\to}0.$$

Thus,

$$\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:\sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)_\alpha^- -\left(\overline{h}(y)\right)_\alpha^-\right|\geq q\right\}\right)\geq q\right\}\in\Lambda$$

and

$$\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:\sup_{\alpha\in[0,1]}\left|\left(\overline{h}_m(y)\right)_\alpha^+ -\left(\overline{h}(y)\right)_\alpha^+\right|\geq q\right\}\right)\geq q\right\}\in\Lambda.$$

From the last two relations, we get

$$\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:\sup_{\alpha\in[0,1]}\max\left\{\left|\left(\overline{h}_m(y)\right)_\alpha^- -\left(\overline{h}(y)\right)_\alpha^-\right|,\left|\left(\overline{h}_m(y)\right)_\alpha^+ -\left(\overline{h}(y)\right)_\alpha^+\right|\right\}\geq q\right\}\right)\geq q\right\}\in\Lambda,$$

which gives that

$$\left\{m\in\mathbb{N}:\mu\left(\left\{y\in\Omega:D\left(\overline{h}_m(y),\overline{h}(y)\right)\geq q\right\}\right)\geq q\right\}\in\Lambda.$$

This means that $\left(\overline{h}_m\right)$ is LSCM to FVMF $\overline{h}$. □

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