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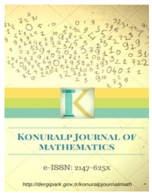

The Conharmonic Curvature Tensor on $N(\kappa)$-Paracontact Metric Manifold

K. K. Mirji[1*] and D. G. Prakasha[2]

[1]*Department of Mathematics, KLS Gogte Institute of Technology, Gnana Ganga, Udyambag Belagavi-590008, Karnataka, India.*
[2]*Department of Mathematics, Davangere University, Shivagangothri, Davangere 577007, Karnataka, India.*
[*]*Corresponding Author*

Abstract

The object of the paper is to study $N(k)$-paracontact metric manifolds satisfying certain curvature conditions on conharmonic curvature tensor. Specially, we study the symmetric properties of conharmonic curvature tensor on $N(k)$-paracontact metric manifolds such as conharmonically φ-symmetric, 3-dimensional locally conharmonically φ-symmetric $N(k)$-paracontact metric manifolds and φ-conharmonically semisymmetric $N(k)$-paracontact metric manifolds and get some new results.

Keywords: *$N(\kappa)$-paracontact metric manifolds, Conharmonic curvature tensor, Einstein manifolds.*
2010 Mathematics Subject Classification: *Primary 53B30; Secondary 53C15, 53C25.*

1. Introduction

The study of nullity distribution on paracontact geometry is one among the most interesting topics in modern paracontact geometry which was initiated by Kaneyuki [15]. A systematic study of paracontact metric manifolds was carried out by Zamkovoy [31]. The importance of paracontact geometry interplays with the theory of para-Kahler manifolds and its role in pseudo-Riemannian geometry and mathematical physics. Recently, many authors studied paracontact geometry and emphasize the similarities and differences with respect to the well known contact cases [3, 4, 6, 7, 11, 14, 19].

In [18], Montano et. al., introduced the class of paracontact metric manifolds for which the characteristic vector field ξ belongs to the (κ,μ)-nullity condition (or distribution) for some real constant κ and μ. Such manifolds are known as (κ,μ)-paracontact metric manifolds. If $\mu = 0$, then the notion of (κ,μ)-nullity distribution reduces to κ-nullity distribution. A paracontact metric manifold with ξ belongs to κ-nullity distribution is called as $N(\kappa)$-paracontact metric manifold. The study of these manifolds is recently carried out in [10, 22, 23, 24, 27, 16].

Takahashi [28] introduced the notion of locally φ-symmetric Sasakian manifold as a weaker version of local symmetry of the above said manifolds. In regard of contact geometry, the thought of φ-symmetry was presented and considered by Boeckx et. al., [5] with cases. De et. al., [8, 9] considered the idea of φ-symmetry and talked about a few cases for Kenmotsu manifolds and almost contact metric manifolds of dimension 3. Venkatesha et. al., [30, 20] and Shukla et. al., [26] considered para-Sasakian manifolds and produced interesting results.

Significant interest attached to a special type of conformal transformations is known as conharmonic transformations (i.e., conformal transformations that keep the property of smooth harmonic functions). This type of transformations was introduced by Ishii in 1957 [13] and is studied now from various points of view. It is easy to verify that this tensor is an algebraic curvature tensor which has classical symmetry properties of the Riemann curvature tensor. The completion of a Riemannian structure to almost Hermitian structures allows additional symmetry properties of the conharmonic curvature tensor. In this paper, we study the geometric sense of these properties of symmetry in the case of $N(k)$-paracontact metric manifolds. The conhamonic curvature tensor L of type $(1,3)$ on a Reimannian manifold M of dimension $(2n+1)$ is given by [13]

$$\begin{aligned} L(X,Y)Z &= R(X,Y)Z - \frac{1}{(2n-1)}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{1.1}$$

for all vector fields X, Y, Z on M, where R is Riemannian curvature tensor, S is Ricci tensor and Q is Ricci-operator defined by $g(QX,Y) = S(X,Y)$. The conharmonic curvature tensor has been studied by Abdussattar [1], Özgür [2], Siddiqui and Ahsan [25], Praksaha et. al., [21], Ghosh et. al., [12], Taleshian et. al., [29] and many others.

Email addresses: **mirjikk@gmail.com, kkmirji@git.edu (K. K. Mirji), prakashadg@gmail.com (D. G. Prakasha)**

The paper is organised as follows. In section 2, we give brief account on $N(\kappa)$-paracontact metric manifolds. Section 3 is devoted to the study of conharmonically φ-symmetric and 3-dimensional locally conharmonically φ-symmetric $N(\kappa)$-paracontact metric manifolds. Finally, we study the properties of φ-conharmonically semisymmetric $N(\kappa)$-paracontact metric manifolds.

2. Preliminaries

An almost paracontact structure on a $(2n+1)$-dimensional smooth manifold M is a triple (φ,ξ,η,g), where φ is a $(1,1)$ tensor field, ξ a global vector field and η a 1-form, such that ([17, 31]):

$$\eta(X)=g(X,\xi),\ \eta(\xi)=1,\ \eta\circ\varphi=0,\ \varphi^2X=X-\eta(X)\xi, \tag{2.1}$$

and the restriction J of φ on the horizontal distribution $ker\eta$ is an almost paracomplex structure. A pseudo-Riemannian metric g on M is compatible with the almost paracontact structure (φ,ξ,η) when

$$g(\varphi X,\varphi Y)=-g(X,Y)+\eta(X)\eta(Y), \tag{2.2}$$

for any vector fields X, Y on M. Any almost paracontact structure admits compatible metrics, which is necessarily have signature $(n+1,n)$. The fundamental 2-form Φ of an almost paracontact metric structure (φ,ξ,η,g) is defined by $\Phi(X,Y)=g(X,\varphi Y)$, for all tangent vector fields X, Y on M. If $\Phi=d\eta$, then the manifold (M,φ,ξ,η,g) is called a paracontact metric manifold, where g being associated metric. On a paracontact metric manifold (M,φ,ξ,η,g), we define a $(1,1)$ tensor field h by $2h=\pounds_\xi g$, where $\pounds$ denotes the operator of Lie differentiation. Then h is symmetric and satisfies $h\xi=0,\ h\varphi=-\varphi h,\ Tr\cdot h=Tr\cdot\varphi h=0$. For any vector fields X and Y on M, if ∇ denotes the Levi-Civita connection of g, then we have the following relation

$$\nabla_X\xi=-\varphi X+\varphi hX, \tag{2.3}$$

and

$$(\nabla_X\eta)Y \quad = \quad g(X-hX,\varphi Y). \tag{2.4}$$

A paracontact metric manifolds (M,φ,ξ,η,g) is said to be a (κ,μ)-space if its curvature tensor R satisfies

$$R(X,Y)\xi=\kappa[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY],$$

for all tangent vector fields X, Y, where κ and μ are smooth functions on M. In particular, if $\mu=0$, then the notion of (κ,μ)-nullity distribution reduces to κ-nullity distribution. Then the curvature tensor R reduces to the following form:

$$R(X,Y)\xi=\kappa[\eta(Y)X-\eta(X)Y]. \tag{2.5}$$

If (M,φ,ξ,η,g) be an $N(\kappa)$-paracontact metric manifold of dimension $(2n+1)$, then for any vector field X on M, the following identities hold [18]:

$$h^2 \quad = \quad (1+k)\varphi^2, \tag{2.6}$$
$$S(X,\xi) \quad = \quad 2nk\eta(X). \tag{2.7}$$

A $(2n+1)$-dimensional $N(\kappa)$-paracontact metric manifold is called an η-Einstein manifold if its Ricci curvature tensor S satisfies the condition

$$S(X,Y)=\lambda g(X,Y)+\mu\eta(X)\eta(Y),$$

where λ and μ are scalars. If $\mu=0$, then the manifold reduces to an Einstein manifold.

Theorem 2.1. *[32] Let M^{2n+1} be a paracontact metric manifold and suppose that $R(X,Y)\xi=0$ for all vector fields X and Y. Then locally M^{2n+1} is the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature equal to -4.*

3. Conharmonically φ-symmetric and 3-dimensional locally φ-symmetric $N(\kappa)$-paracontact metric manifolds

A $N(\kappa)$-paracontact metric manifolds M with $\kappa\neq-1$ is said to be conharmonically φ-symmetric if conharmonic curvature tensor L satisfies the condition

$$\varphi^2((\nabla_WL)(X,Y)Z)=0, \tag{3.1}$$

for all vector fields X, Y, Z on M. If M is conharmonically φ-symmetric $N(\kappa)$-paracontact metric manifold, then by virtue of (2.1) and (3.1), it follows that

$$(\nabla_WL)(X,Y)Z-\eta((\nabla_WL)(X,Y)Z)\xi=0. \tag{3.2}$$

In view of (1.1) and (3.2), we obtain

$$\begin{aligned} & g((\nabla_WR)(X,Y)Z,U)-\frac{1}{2n-1}[g(X,U)(\nabla_WS)(Y,Z) \\ - \ & g(Y,U)(\nabla_WS)(X,Z)+g(Y,Z)g((\nabla_WQ)X,U) \\ - \ & g(X,Z)g((\nabla_WQ)Y,U)-(\nabla_WS)(Y,Z)\eta(X)\eta(U) \\ + \ & (\nabla_WS)(X,Z)\eta(Y)\eta(U)-g(Y,Z)g((\nabla_WQ)X,\xi)\eta(U) \\ + \ & g(X,Z)g((\nabla_WQ)Y,\xi)\eta(U)]-\eta((\nabla_WR)(X,Y)Z)\eta(U)=0, \end{aligned} \tag{3.3}$$

and replacing $X = U = e_i$ in (3.3) and taking summation over i, where $\{e_i\}$, $i = 1, 2, ..., (2n+1)$ is an orthonormal basis of the tangent space at any point of the manifold, we get the following result

$$\begin{aligned} & \frac{1}{2n-1}\sum_{i=1}^{2n+1}[\eta((\nabla_W Q)e_i)\eta(e_i) - g((\nabla_W Q)e_i, e_i)]g(Y,Z) \\ + & \frac{1}{2n-1}[(\nabla_W S)(Y,Z) - (\nabla_W S)(Z,\xi)\eta(Y)] \\ - & \sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) = 0. \end{aligned} \tag{3.4}$$

Taking $Z = \xi$ in (3.4), we have

$$\begin{aligned} & \frac{1}{2n-1}(\nabla_W S)(Y,\xi) \\ + & \frac{1}{2n-1}\left[\left\{\sum_{i=1}^{2n+1}\eta((\nabla_W Q)e_i)\eta(e_i) - dr(W) - (\nabla_W S)(\xi,\xi)\right\}\eta(Y)\right] \\ - & \sum_{i=1}^{2n+1}\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = 0. \end{aligned} \tag{3.5}$$

Considering the second term of (3.5) and using the properties of Riemannian metric g, we get

$$\begin{aligned} \sum_{i=1}^{2n+1}\eta((\nabla_W Q)e_i)\eta(e_i) &= \sum_{i=1}^{2n+1}g((\nabla_W Q)e_i,\xi)g(e_i,\xi) \\ &= g((\nabla_W Q)\xi,\xi) \\ &= 2nkg(\nabla_W\xi,\xi) - S(\nabla_W\xi,\xi) \\ &= 0. \end{aligned} \tag{3.6}$$

Also, considering the last term of (3.5), we have

$$\eta((\nabla_W R)(e_i,Y)\xi)\eta(e_i) = g((\nabla_W R)(e_i,Y)\xi,\xi)g(e_i,\xi). \tag{3.7}$$

Next,

$$\begin{aligned} g((\nabla_W R)(e_i,Y)\xi,\xi) &= g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(\nabla_W e_i,Y)\xi,\xi) \\ &- g(R(e_i,\nabla_W Y)\xi,\xi) - g(R(e_i,Y)\nabla_W\xi,\xi). \end{aligned}$$

As $\{e_i\}$ is an orthonormal basis, $g(R(\nabla_W e_i,Y)\xi,\xi) = 0$ and also, since Riemannian curvature tensor R is skew-symmetric, $g(R(e_i,\nabla_W Y)\xi,\xi) = 0$. Hence,

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = g(\nabla_W R(e_i,Y)\xi,\xi) - g(R(e_i,Y)\nabla_W\xi,\xi). \tag{3.8}$$

We know that, $g(R(e_i,Y)\xi,\xi) = -g(R(\xi,\xi)Y,e_i) = 0$. So we get the following

$$g(\nabla_W R(e_i,Y)\xi,\xi) + g(R(e_i,Y)\xi,\nabla_W\xi) = 0. \tag{3.9}$$

By virtue of (3.9) and (3.8), we obtain

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = -g(R(e_i,Y)\xi,\nabla_W\xi) - g(R(e_i,Y)\nabla_W\xi,\xi),$$

and since R is skew-symmetric

$$g((\nabla_W R)(e_i,Y)\xi,\xi) = 0. \tag{3.10}$$

Substituting (3.6) and (3.10) in (3.5), it follows that

$$\frac{1}{2n-1}(\nabla_W S)(Y,\xi) - \frac{1}{2n-1}dr(W)\eta(Y) = 0. \tag{3.11}$$

Setting $Y = \xi$ in (3.11) yields the assertion

$$dr(W) = 0, \tag{3.12}$$

which implies that, the scalar curvature tensor r is constant and hence it follows the theorem:

Theorem 3.1. *In a $(2n+1)$-dimensional conharmonically φ-symmetric $N(\kappa)$-paracontact metric manifold M with $\kappa \neq -1$, the scalar curvature tensor r is constant.*

Therefore, in view of (3.11) and (3.12), we get

$$(\nabla_W S)(Y,\xi)=0. \tag{3.13}$$

Also, we know that

$$(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W \xi). \tag{3.14}$$

From (2.3), (2.4), (2.7) and (3.13), (3.14) takes the following form:

$$2nkg(W-hW,\varphi Y)+S(Y,-\varphi W+\varphi hW)=0. \tag{3.15}$$

Replacing W by φW in (3.15) and then using (2.1), (2.2), (2.7), we have

$$S(Y,W)=2nkg(Y,W)+2nkg(Y,hW)-S(Y,hW) \tag{3.16}$$

and replacing W by hW in (3.16) and using (2.1) and (2.6), we deduce

$$2nkg(Y,hW)-S(Y,hW)=(1+k)[S(Y,W)-2nkg(Y,W)]. \tag{3.17}$$

By virtue of (3.16) and (3.17), we get

$$k[S(Y,W)-2nkg(Y,W)]=0,$$

which leads either $k=0$ or M is an Einstein manifold.
If $k=0$, then from (2.5) it is clear that $R(X,Y)\xi=0$. Therefore, from Threorem 2.1 we conclude that, the manifold is locally the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature equal to -4. Hence we can state the following theorem:

Theorem 3.2. *Let M be a $(2n+1)$-dimensional conharmonically φ-symmetric $N(\kappa)$-paracontact metric manifold with $\kappa\neq -1$. Then we have the following:*

1. *M is locally the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature equal to -4; or*
2. *M is an Einstein manifold.*

Again, substituting $S(Y,W)=2nkg(Y,W)$ in (3.3) followed by a simple calculation gives,

$$g((\nabla_W R)(X,Y)Z,U)-\eta((\nabla_W R)(X,Y)Z)\eta(U)=0,$$

and this implies that

$$\varphi^2((\nabla_W R)(X,Y)Z)=0. \tag{3.18}$$

From (3.18), it is clear that the manifold M is φ-symmetric and this leads the following:

Theorem 3.3. *A $(2n+1)$-dimensional conharmonically φ-symmetric Einstein $N(\kappa)$-paracontact metric manifold M with $\kappa\neq -1$ is φ-symmetric $N(\kappa)$-paracontact metric manifold.*

Further, for 3-dimensional semi-Riemannian manifolds, the Riemannian curvature tensor is

$$\begin{aligned} R(X,Y)Z &= [S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY] \\ &- \frac{r}{2}[g(Y,Z)X-g(X,Z)Y], \end{aligned} \tag{3.19}$$

for any vector fields X, Y, Z on M, being r the scalar curvature, by putting $Y=Z=\xi$ in (3.19) and using (2.5) and (2.7), we obtain:

$$QX=\left(\frac{r}{2}-k\right)X+\left(3k-\frac{r}{2}\right)\eta(X)\xi. \tag{3.20}$$

Taking the inner product of (3.20) with any vector field Y, we get

$$S(X,Y)=\left(\frac{r}{2}-k\right)g(X,Y)+\left(3k-\frac{r}{2}\right)\eta(X)\eta(Y). \tag{3.21}$$

Substitution of (3.20) and (3.21) into (3.19) gives

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2}-2k\right)[g(Y,Z)X-g(X,Z)Y] \\ &+ \left(3k-\frac{r}{2}\right)[g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y], \end{aligned} \tag{3.22}$$

where R is the Riemannian curvature tensor, S is the Ricci curvature tensor and Q is Ricci operator. Hence we can state the following theorem:

Theorem 3.4. *In a 3-dimensional $N(\kappa)$-paracontact metric manifold M^3 with $\kappa\neq -1$, the Ricci operator, the Ricci tensor and the Riemannian curvature tensor are given by (3.20), (3.21) and (3.22) respectively.*

A 3-dimensional $N(\kappa)$-paracontact metric manifold M^3 with $\kappa \neq -1$ is said to be locally conharmonically φ-symmetric if it satisfies the following condition

$$\varphi^2((\nabla_W L)(X,Y)Z) = 0, \tag{3.23}$$

for all vector fields X, Y, Z and W orthogonal to ξ on M.

Suppose $N(\kappa)$-paracontact metric manifold M^3 of dimension 3 is locally conharmonically φ-symmetric. Then in view of (3.20), (3.21) and (3.22), (1.1) takes the following form:

$$L(X,Y)Z = \frac{r}{2}[g(X,Z)Y - g(Y,Z)X].$$

Covariant differentiation of the above relation on both sides gives

$$(\nabla_W L)(X,Y)Z = \frac{dr(W)}{2}[g(X,Z)Y - g(Y,Z)X]. \tag{3.24}$$

Applying φ^2 on both sides of (3.24) and then using (2.1), we have

$$\begin{aligned} \varphi^2((\nabla_W L)(X,Y)Z) &= \frac{dr(W)}{2}[g(X,Z)Y - g(Y,Z)X \\ &- g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi]. \end{aligned} \tag{3.25}$$

Let us assume that all the vector fields X, Y, Z and W are orthogonal to ξ. Then the above relation (3.25) becomes

$$\varphi^2((\nabla_W L)(X,Y)Z) = \frac{dr(W)}{2}[g(X,Z)Y - g(Y,Z)X].$$

This proves the assertion of our theorem:

Theorem 3.5. *A 3-dimensional $N(\kappa)$-paracontact metric manifold M^3 with $\kappa \neq -1$ is locally conharmonically φ-symmetric if and only if the scalar curvature tensor r is constant.*

In [23], Prakasha and Mirji has proved that

Theorem 3.6. *[23] A 3-dimensional $N(k)$-paracontact metric manifold $(M^3, \phi, \xi, \eta, g)$ is locally ϕ-symmetric if and only if the scalar curvature tensor r of g is constant.*

In view of Theorem 3.5 and Theorem 3.6, we get the following result:

Theorem 3.7. *A 3-dimensional $N(\kappa)$-paracontact metric manifold M^3 with $\kappa \neq -1$ is locally conharmonically φ-symmetric if and only if it is locally ϕ-symmetric.*

Also, from (3.24), it is clear that $N(\kappa)$-paracontact metric manifold is locally conharmonically symmetric if and only if the scalar curvature tensor r is constant and this leads the theorem stated below:

Theorem 3.8. *A 3-dimensional $N(\kappa)$-paracontact metric manifold M^3 with $\kappa \neq -1$ is locally conharmonically symmetric if and only if the scalar curvature tensor r is constant.*

4. φ-conharmonically semisymmetric $N(\kappa)$-paracontact metric manifolds

A $N(\kappa)$-paracontact metric manifold M is said to be φ-conharmonically semisymmetric if the conharmonic curvature tensor L satisfies

$$L(X,Y)\cdot\varphi = 0, \tag{4.1}$$

for all vector fields X, Y on M.

If M be a φ-conharmonically semisymmetric $N(\kappa)$-paracontact metric manifold, then condition (4.1) holds for all vector field X, Y, Z and implies that

$$(L(X,Y)\cdot\varphi)Z = L(X,Y)\varphi Z - \varphi L(X,Y)Z = 0, \tag{4.2}$$

By virtue of (1.1) in (4.2), we obtain

$$\begin{aligned} &[R(X,Y)\varphi Z - \varphi R(X,Y)Z] - \frac{1}{2n+1}[S(Y,\varphi Z)X \\ - \ &S(X,\varphi Z)Y + g(Y,\varphi Z)QX - g(X,\varphi Z)QY \\ - \ &S(Y,Z)\varphi X + S(X,Z)\varphi Y - g(Y,Z)\varphi QX + g(X,Z)\varphi QY] = 0. \end{aligned} \tag{4.3}$$

If $N(\kappa)$-paracontact metric manifold M with $k \neq -1$ and ξ belongs to κ-nullity distribution, then for any vector fields X, Y and Z on M, the following relation holds [18].

$$\begin{aligned} &R(X,Y)\varphi Z - \varphi R(X,Y)Z \\ = \ &[(1+k)(g(\varphi X,Z)\eta(Y) - g(\varphi Y,Z)\eta(X)) - (g(\varphi hX,Z)\eta(Y) \\ - \ &g(\varphi hY,Z)\eta(X))]\xi + g(Y - hY,Z)(\varphi X - \varphi hX) \\ - \ &g(X - hX,Z)(\varphi Y - \varphi hY) - g(\varphi X - \varphi hX,Z)(Y - hY) \\ + \ &g(\varphi Y - \varphi hY,Z)(X - hX) + \eta(Z)[(1+k)(\eta(X)\varphi Y - \eta(Y)\varphi X) \\ - \ &(\eta(X)\varphi hY - \eta(Y)\varphi hX)]. \end{aligned} \tag{4.4}$$

Using (4.4) in (4.3), it follows that

$$\begin{aligned} & [(1+k)(g(\varphi X,Z)\eta(Y)-g(\varphi Y,Z)\eta(X))-(g(\varphi hX,Z)\eta(Y) \\ - & g(\varphi hY,Z)\eta(X))]\xi+g(Y-hY,Z)(\varphi X-\varphi hX) \\ - & g(X-hX,Z)(\varphi Y-\varphi hY)-g(\varphi X-\varphi hX,Z)(Y-hY) \\ + & g(\varphi Y-\varphi hY,Z)(X-hX) \\ + & \eta(Z)[(1+k)(\eta(X)\varphi Y-\eta(Y)\varphi X)-(\eta(X)\varphi hY-\eta(Y)\varphi hX)] \\ - & \frac{1}{2n+1}[S(Y,\varphi Z)X-S(X,\varphi Z)Y+g(Y,\varphi Z)QX \\ - & g(X,\varphi Z)QY-S(Y,Z)\varphi X+S(X,Z)\varphi Y \\ - & g(Y,Z)\varphi QX+g(X,Z)\varphi QY]=0. \end{aligned} \tag{4.5}$$

Replacing X by φX in (4.5) and using (2.1) and skew symmetric property of φ gives

$$\begin{aligned} & [(1+k)\{g(X,Z)\eta(Y)-\eta(X)\eta(Y)\eta(Z)\}+g(hX,Z)\eta(Y)]\xi \\ + & g(Y-hY,Z)(X-\eta(X)\xi+hX)-g(\varphi X-h\varphi X,Z)(\varphi Y-\varphi hY) \\ - & g(X-\eta(X)\xi+hX,Z)(Y-hY)+g(\varphi Y-\varphi hY,Z)(\varphi X-h\varphi X) \\ + & \eta(Z)[(1+k)\eta(Y)(\eta(X)\xi-X)-\eta(Y)hX]) \\ - & \frac{1}{2n-1}[S(Y,\varphi Z)\varphi X-S(\varphi X,\varphi Z)Y+g(Y,\varphi Z)Q\varphi X \\ - & g(\varphi X,\varphi Z)QY-S(Y,Z)(X-\eta(X)\xi)+S(\varphi X,Z)\varphi Y \\ - & g(Y,Z)\varphi Q\varphi X+g(\varphi X,Z)\varphi QY]=0, \end{aligned} \tag{4.6}$$

and taking the inner product of (4.6) with ξ, we deduce

$$\begin{aligned} & k[(g(X,Z)\eta(Y)-\eta(X)\eta(Y)\eta(Z)] \\ + & \frac{1}{2n-1}[S(\varphi X,\varphi Z)\eta(Y)+2nkg(\varphi X,\varphi Z)\eta(Y)]=0. \end{aligned} \tag{4.7}$$

Putting $Y=\xi$ in (4.7) and using (2.1), we have

$$S(\varphi X,\varphi Z) \quad = \quad k[g(X,Z)-\eta(X)\eta(Z)]. \tag{4.8}$$

Again replacing X by φX and Z by φZ in (4.8) and using (2.1), (2.2), it follows that

$$S(X,Z) \quad = \quad -kg(X,Z)+(2n+1)k\eta(X)\eta(Z). \tag{4.9}$$

This proves the assertion of our theorem:

Theorem 4.1. *A $(2n+1)$-dimensional φ-conharmonically semisymmetric $N(\kappa)$-paracontact metric manifold M with $\kappa \neq -1$ is an η-Einstein manifold.*

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