Konuralp Journal of Mathematics, 8 (2) (2020) 284-286

Konuralp Journal of Mathematics

Journal Homepage: *www.dergipark.gov.tr/konuralpjournalmath*
e-ISSN: 2147-625X

Eigenvalue Problems with Interface Conditions

Oktay Mukhtarov[1,2], Hayati Olğar[1*] and Kadriye Aydemir[3]

[1] *Department of Mathematics, Faculty of Science, Gaziosmanpasa University, 60150, Tokat, Turkey.*
[2] *Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan.*
[3] *Department of Mathematics, Faculty of Science, Amasya University, 05100, Amasya, Turkey.*
[*] *Corresponding author*

Abstract

Sturm-Liouville type boundary value problems arise a result of using the Fourier's method of separation of variables to solve the classical partial differential equations of mathematical physics, such as the Laplace's equation, the heat equation and the wave equation. A large class of physical problems require the investigation of the Sturm-Liouville type problems with the eigen-parameter in the boundary conditions. Also, many physical processes, such as the vibration of loaded strings, the interaction of atomic particles, electrodynamics of complex medium, aerodynamics, polymer rheology or the earth's free oscillations yield. Sturm-Liouville eigenvalue problems(see, for example, [1, 6, 12, 13]). On the other hand, the Sturm-Liouville problems with transmission conditions (such conditions are known by various names including transmission conditions, interface conditions, jump conditions and discontinuous conditions) arise in problems of heat and mass transfer, various physical transfer problems [8], radio science [7], and geophysics [9]. In this work we shall investigate some spectral properties of a regular Sturm-Liouville problem on a finite interval with the transmission conditions at a point of interaction. We prove that the set of eigenfunctions for the problem under consideration forms a basis in the corresponding Hilbert space.

Keywords: *Sturm-Liouville type boundary value problems, transmission conditions.*
2010 Mathematics Subject Classification: *34B24, 34L10, 34B27, 34L20*

1. Introduction

Boundary value problems for Sturm-Liouville equations with eigen-parameter dependent boundary conditions were considered in different formulations by many authors (see [3, 4, 5, 10, 15]. Such type of problems have appeared in some physical and engineering problems such as heat conduction problems and vibrating string problems [4, 8, 14, 16]. Hence, it is an actual and important research topic in mathematical physics. In the case when the spectral parameter is contained in the boundary and transmission conditions the spectral properties of eigenvalues and eigenfunctions for the discontinuous Sturm-Liouville problems were studied in [1, 2, 11].

In this study, we shall examine Sturm-Liouville type boundary value problems when the potential of the differential equation may have discontinuity at one inner point and the eigenparameter appears not only in the differential equation, but also in the boundary conditions.

We shall extend some results of the classical Sturm-Liouville boundary value problems to the new type boundary value problems. Basically it has been investigated boundary value problems which consist of ordinary differential equations with continuous coefficients and end-point boundary conditions which are independent from the spectral parameter. But in this study we shall consider a new type discontinuous eigenvalue problem which consist of a Sturm-Liouville equation involving an abstract linear operator $\mathscr{A}$, namely the equation

$$\ell y \; := \; -y''(x) \; + \; q(x)y(x) + (\mathscr{A}y)(x) \; = \; \lambda y(x)\,, \quad x \in [a_1,a_2) \cup (a_2,a_3], \tag{1.1}$$

together with eigenparameter-dependent boundary conditions at the end-points $x = a_1$ and $x = a_3$, given by

$$B_1(y) \; := \; \left(\ln y\right)'(a_1) \; = \; \frac{\alpha_1 + \lambda\alpha_1'}{\alpha_2 + \lambda\alpha_2'}, \tag{1.2}$$

$$B_2(y) \; := \; \left(\ln y\right)'(a_3) \; = \; \frac{\beta_1 + \lambda\beta_1'}{\beta_2 + \lambda\beta_2'}, \tag{1.3}$$

and with the transmission conditions at the point of discontinuity $x = a_2$, given by

$$T_1(y) \; := \; y(a_2+0) - \delta_1 y(a_2-0) - \gamma_1 y'(a_2-0) = 0, \tag{1.4}$$

Email addresses: **omukhtarov@yahoo.com (Oktay Mukhtarov), hayatiolgar@gmail.com (Hayati Olğar), kadriyeaydemr@gmail.com (Kadriye Aydemir)**

$$T_2(y) := y'(a_2+0) - \delta_2 y(a_2-0) - \gamma_2 y'(a_2-0) = 0, \tag{1.5}$$

where $\alpha_i, \beta_i, \alpha_i', \beta_i', \delta_i, \gamma_i$ $(i = 1,2)$ are real numbers, $q(x)$ is the real-valued function which is continuous in each of (a_1,a_2) and (a_2,a_3) with the finite limits $q(a_1+0), q(c_2\pm 0), q(a_3-0)$, λ is a complex spectral parameter. Everywhere we will assume that $\alpha_1'.\alpha_2 - \alpha_1.\alpha_2' > 0$, $\beta_1'\beta_2 - \beta_1\beta_2' > 0$ and $\delta_1\gamma_2 - \delta_2\gamma_1 > 0$.

2. The Hilbert Space Formulation

For operator-theoretic treatment of the BVP (1.1)-(1.5) we shall introduce a new inner-product in the classical Hilbert space $H := L_2[a_1,a_3] \oplus \mathbb{C}^2$ and a differential operator $\mathscr{L}$ defined in this Hilbert space such a way that the problem (1.1)-(1.5) can be considered as the eigenvalue problem of this operator.
At first consider a Sturm Liouville equation

$$\ell_0 y := -y''(x) + q(x)y(x) = \lambda y(x) \tag{2.1}$$

on two disjoint intervals (a_1,a_2) and (a_2,a_3), together with the same boundary and transmission conditions (1.2)-(1.5). We shall introduce a new equivalent inner-product on $H := L_2(a_1,a_2) \oplus L_2(a_2,a_3) \oplus \mathbb{C}^2$ by the formula

$$\langle Y,Z\rangle_H = \Big(\delta_1\gamma_2 - \delta_2\gamma_1\Big)\int_{a_1}^{a_2-0} y(x)\overline{z(x)}dx + \int_{a_2+0}^{a_3} y(x)\overline{z(x)}dx + \left(\frac{\delta_1\gamma_2-\delta_2\gamma_1}{\alpha_1'.\alpha_2 - \alpha_1.\alpha_2'}\right)y_1\overline{z_1} + \left(\frac{1}{\beta_1'\beta_2 - \beta_1\beta_2'}\right)y_2\overline{z_2} \tag{2.2}$$

for $Y = \left(y(x), y_1, y_2 \right)^T$, $Z = \left(z(x), z_1, z_2 \right)^T \in H$. We can prove that $H := \left(L_2(a_1,a_2) \oplus L_2(a_2,a_3) \oplus \mathbb{C}^2, \langle\, ,\, \rangle_H\right)$ is a complete, i.e. H is Hilbert space.
At first we shall investigate the linear operator $\mathscr{L}_0 : H \to H$ with domain of definition

$$\begin{aligned} D(\mathscr{L}_0) = & \Big\{ Y = \left(y(x), y_1, y_2 \right)^T \in H : y \text{ and } y' \text{ are absolutely continuous on both } (a_1,a_2) \text{ and } (a_2,a_3), \\ & \text{there are finite limits } y(a_1+0), y(a_2\pm 0), y(a_3-0), y'(a_1+0), y'(a_2\pm 0), y'(a_3-0) \\ & \ell_0 y \in L_2(a_1,a_2)\oplus L_2(a_2,a_3),\ y_1 = \alpha_1' y(a_1) - \alpha_2' y'(a_1),\ y_2 = \beta_1' y(a_3) - \beta_2' y'(a_3) \Big\} \end{aligned} \tag{2.3}$$

and action low

$$\mathscr{L}_0 \begin{pmatrix} y(x) \\ \alpha_1' y(a_1) - \alpha_2' y'(a_1) \\ \beta_1' y(a_3) - \beta_2' y'(a_3) \end{pmatrix} = \begin{pmatrix} -y''(x) + q(x)y(x) \\ \alpha_1 y(a_1) - \alpha_2 y'(a_1) \\ \beta_1 y(a_3) - \beta_2 y'(a_3) \end{pmatrix} \tag{2.4}$$

Thus, the problem (2.1),(1.2)-(1.5) can be written in the form $\mathscr{L}_0 Y = \lambda Y$.

Definition 2.1. *The eigenvalues of the operator $\mathscr{L}_0$ is said to be the eigenvalues of the BVP consisting of the equation (2.1) and boundary-transmission conditions (1.2)-(1.5).*

Definition 2.2. *The first components of the eigenelements of $\mathscr{L}_0$ is said to be the eigenfunctions of the BVP (2.1),(1.2)-(1.5).*

We can prove the following theorems.

Theorem 2.3. *The domain $D(\mathscr{L}_0)$ of the linear differential operator $\mathscr{L}_0$ is dense in the Hilbert space H.*

Theorem 2.4. *The linear differential operator $\mathscr{L}_0$, which is definite by (2.3)-(2.4), is symmetric.*

Corollary 2.5. *The eigenvalues of $\mathscr{L}_0$ are real and the eigenfunctions corresponding to different eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal with respect to the inner-product (2.2).*

Theorem 2.6. *The eigenvalues of the problem (2.1),(1.2) – (1.5) is bounded below, i.e. there is $m \in \mathbb{R}$ such that $\lambda_n \geq m$ for all $n = 0,1,2,.....$*

3. Asymptotics of the Eigenvalues

We can prove that the following asymptotic approximation formulas for the eigenvalues of the considered problem (2.1),(1.2)-(1.5) are holds.

Theorem 3.1. *The BVTP consisting of the equation (2.1) with boundary-transmission conditions (1.2)-(1.5) has an precisely denumerable many real eigenvalues which may be expressed by two sequence $\lambda_{n,1} = \mu_{n,1}^2$ and $\lambda_{n,2} = \mu_{n,2}^2$ with following asymptotic as $n \to \infty$:*

Case 1. If $\beta_2' \neq 0$ and $\alpha_2' \neq 0$, then

$$\mu_{n,1} = \left(\frac{\pi}{a_3-a_2}\right)\left(n-\frac{5}{2}\right) + O\left(\frac{1}{n}\right) \quad , \quad \mu_{n,2} = \left(\frac{\pi}{a_2-a_1}\right)\left(n+\frac{1}{2}\right) + O\left(\frac{1}{n}\right)$$

Case 2. If $\beta_2' \neq 0$ and $\alpha_2' = 0$, then

$$\mu_{n,1} = \left(\frac{\pi}{a_3-a_2}\right)\left(n+\frac{1}{2}\right) + O\left(\frac{1}{n}\right) \quad , \quad \mu_{n,2} = \left(\frac{\pi}{a_2-a_1}\right)\left(n-2\right) + O\left(\frac{1}{n}\right)$$

Case 3. *If* $\beta_2' = 0$ *and* $\alpha_2' \neq 0$, *then*

$$\mu_{n,1} = \left(\frac{\pi}{a_3 - a_2}\right)\left(n-2\right) + O\left(\frac{1}{n}\right) \quad , \quad \mu_{n,2} = \left(\frac{\pi}{a_2 - a_1}\right)\left(n+\frac{1}{2}\right) + O\left(\frac{1}{n}\right)$$

Case 4. *If* $\beta_2' = 0$ *and* $\alpha_2' = 0$, *then*

$$\mu_{n,1} = \left(\frac{\pi}{a_3 - a_2}\right)\left(n-2\right) + O\left(\frac{1}{n}\right) \quad , \quad \mu_{n,2} = \left(\frac{\pi}{a_2 - a_1}\right) n + O\left(\frac{1}{n}\right)$$

We shall suppose that $D(\mathscr{A}) \supseteq D(\mathscr{L}_0)$ and define the main operator $\mathscr{L} : H \to H$ on the domain $D(\mathscr{L}) = D(\mathscr{L}_0)$ by the formula

$$\mathscr{L}\begin{pmatrix} y(x) \\ \alpha_1' y(a_1) - \alpha_2' y'(a_1) \\ \beta_1' y(a_3) - \beta_2' y'(a_3) \end{pmatrix} = \begin{pmatrix} -y''(x) + q(x)y(x) + (\mathscr{A}y)(x) \\ -\alpha_1 y(a_1) + \alpha_2 y'(a_1) \\ -\beta_1 y(a_3) + \beta_2 y'(a_3) \end{pmatrix}$$

Now we can write the main problem (1.1)-(1.5) in the operator form as $\mathscr{L}Y = \lambda Y$.
Let us define a new inner-product space $H_1 := (D(\mathscr{L}), \langle\, , \rangle_{H_1})$ on the subspace $D(\mathscr{L}) \subseteq H$ equipped with the inner-product

$$\langle Y,Z\rangle_{H_1} = \langle y,z\rangle_{W_2^2(a_1,a_2)} + \langle y,z\rangle_{W_2^2(a_2,a_3)}$$

for $Y = \left(y(x), y_1, y_2 \right)^T$, $Z = \left(z(x), z_1, z_2 \right)^T \in D(\mathscr{L})$.

Lemma 3.2. *H_1 is a Hilbert space.*

Let us define the following linear operators in the Hilbert space H_1 by the equalities

$$\mathscr{A}_1 Y := (\mathscr{A}y, 0, 0) \quad \text{and} \quad \mathscr{L}Y := \mathscr{L}_0 Y + \mathscr{A}_1 Y$$

for $Y = (y(x), y_1, y_2) \in D(\mathscr{L}_0)$.

Theorem 3.3. *Suppose that the linear operator $\mathscr{A}$ from $W_2^2(a_1,a_2) \oplus W_2^2(a_2,a_3)$ into $L_2(a_1,a_2) \oplus L_2(a_2,a_3)$ be compact. Then, for any sufficiently small $\theta > 0$ there exists constants $K_\theta > 0$ and $C_\theta > 0$ such that for all complex numbers λ satisfying*

$$\theta < \arg\lambda < 2\pi - \theta \quad and \quad |\lambda| > K_\theta,$$

the operator $\lambda I - \mathscr{L}$ from $\mathscr{H}_1$ onto $\mathscr{H}$ is an isomorphism.

Theorem 3.4. *Assume that all conditions of the previous theorem are satisfied. Then the coercive inequality*

$$||Y||_{H_1} + |\lambda|\, ||Y||_H \leq C_\theta\, ||F||_H$$

holds for the solution $Y = Y(\lambda)$ of the equation $(\lambda I - \mathscr{L})Y = F$, $F \in H$, where C_θ is a constant, which is independent from Y and λ and depend only of θ.

Theorem 3.5. *For any $\theta > 0$ (small enough) there exists $K_\theta > 0$ and $C_\theta > 0$ such that the estimate $\|(\lambda I - \mathscr{L})^{-1}\| \leq C_\theta |\lambda|^{-1}$ holds for all λ with $\theta < \arg\lambda < 2\pi - \theta$, $|\lambda| > K_\theta$.*

From these results it follows the next theorem.

Theorem 3.6. *The spectrum of the linear operator $\mathscr{L}$ is discrete.*

Now we are ready to formulate the following important result.

Theorem 3.7. *Let the operator $\mathscr{A}$ from $W_2^2(a_1,a_2) \oplus W_2^2(a_2,a_3)$ into $L_2(a_1,a_2) \oplus L_2(a_2,a_3)$ be compact. Then, the spectrum of the problem (1.1)-(1.5) is discrete and for the eigenvalues $\{\lambda_n\}$ ($n = 1,2,...$) arranged as $|\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \leq ....$, the following asymptotic formulas are valid:*

$$Im\lambda_n = o\left(n^2\right) \quad , \quad \lambda_n = \left(\frac{1}{a_3 - a_1}\right)^2 \pi^2 n^2 + o\left(n^2\right)$$

References

[1] K. Aydemir and O. Mukhtarov, Variational principles for spectral analysis of one Sturm-Liouville problem with transmission conditions, Advances in Difference Equations, 2016:76 (2016)
[2] E. Bairamov and E. Ugurlu, On the characteristic values of the real component of a dissipative boundary value transmission problem, Appl. Math. and Comp. 218, 9657-9663(2012).
[3] P. A. Binding and P. J. Browne, Oscillation theory for indefinite Sturm-Liouville problems with eigenparameter dependent boundary conditions, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), 1123-1136.
[4] C. T. Fulton, *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edin. 77A(1977), P. 293-308.
[5] B. Harmsen and A. Li, Discrete Sturm-Liouville problems with parameter in the boundary conditions, Journal of Difference Equations and Applications, 8(11)(2002), 969-981.
[6] Q. Kong, H. Wu and A. Zettl, Geometric aspects of Sturm-Liouville problems I. Structures on spaces of boundary conditions, Proceedings of the Royal Society of Edinburgh Section A Mathematics, 130(3), 2000, 561-589.
[7] F. R. Lapwood and T. Usami, Free Oscillations of the Earth, Cambridge University Press, Cambridge, 1981.
[8] A. V. Likov and Yu. A. Mikhailov, The Theory of Heat and Mass Transfer, Translated from Russian by I. Shechtman, Israel Program for Scientific Translations, Jerusalem, 1965.
[9] O. N. Litvinenko and V. I. Soshnikov, The Theory of Heteregenous Lines and their Applications in Radio Engineering (in Russian), Radio, Moscow, 1964.
[10] O. Sh. Mukhtarov and K. Aydemir, The Eigenvalue Problem with Interaction Conditions at One Interior Singular Point, Filomat 31:17 (2017), 5411-5420.
[11] O. Sh. Mukhtarov, H. Olğar and K. Aydemir, Resolvent Operator and Spectrum of New Type Boundary Value Problems. Filomat, 29(7) (2015), 1671-1680.
[12] J. D. Pryce, Numerical solution of Sturm-Liouville Problems, Oxford University Press, 1993.
[13] M. Rotenberg, Theory and application of Sturmian functions. Adv. Atomic. and Mol. Phys. 6(1970).
[14] A. N. Tikhonov and A. A. Samarskii, Equations Of Mathematical Physics, Oxford and New York, Pergamon (1963).
[15] J. Walter, Regular eigenvalue problems with eigenvalue parameter in the boundary conditions, Math. Z., 133(1973), 301-312.
[16] N. N. Voitovich, B. Z. Katsenelbaum and A. N. Sivov, Generalized Method Of Eigen-Vibration In The Theory Of Diffraction, Nakua, Moskow, 1997 (Russian).