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Closure Operators in Constant Filter Convergence Spaces

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Abstract

In this paper, we define two notions of closure in the category of constant filter convergence spaces which satisfy productivity, idempotency, and hereditariness. Moreover, by using these closure operators, we characterize each of T_i constant filter convergence spaces, $i = 0,1,2$ and show that each of these subcategories consisting of T_i constant filter convergence spaces, $i = 0,1,2$, are epireflective. Finally, we investigate the relationship among these subcategories.

Keywords: *Topological category; closure operator; constant filter convergence spaces.*
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1. Introduction

In 1979, Schwarz [17] introduced the category of constant filter convergence spaces and showed that it is isomorphic to the category of grill spaces, which introduced by Robertson [16] in 1975.
Closure operators are intensively used to study topological concepts such as separatedness, regularity, normality, compactness, and connectedness in abstract categories [5, 7, 10, 11, 12, 13, 14, 15].

2. Preliminaries

Let B be a nonempty set, $F(B)$ be the set of filters (proper or improper).

If the map $L : B \to P(F(B))$ satisfies
(1) $[\{x\}] = [x] \in L$ for each $x \in B$, where for $U \subset B$ and $[U] = \{V \subset B : U \subset V\}$,
(2) if $\alpha \in L$ and $\beta \supset \alpha$, then $\beta \in L$,
then (B,L) is called a constant filter convergence space.

Let (B,K) and (C,L) be constant filter convergence spaces. If $f(\alpha) \in L$ for each $\alpha \in K$, then a map $f : (B,K) \to (C,L)$ is called continuous, where $f(\alpha) = [\{f(D) : D \in \alpha\}]$.

Let ***ConFCO*** be the category consisting of all constant filter convergence spaces and continuous maps [17].

2.1 Let $\{(B_i,K_i), i \in I\}$ in ***ConFCO***, B be a set, and $\{f_i : B \to (B_i,K_i), i \in I\}$ be a source in ***Set*** the category of sets and functions. $\{f_i : (B,K) \to (B_i,K_i), i \in I\}$ in ***ConFCO*** is an initial lift iff $\alpha \in K$ precisely when $f_i(\alpha) \in K_i$ for all $i \in I$ [4, 17].

2.2 An epi sink $\{f_i : (B_i,K_i) \to (B,K)\}$ in ***ConFCO*** is a final lift iff $\alpha \in K$ implies that there exist $i \in I$ and $\beta_i \in K_i$ such that $f_i(\beta_i) \subset \alpha$ [4].

3. Closed Subobjects

Let B be a set, $B^\infty = B \times B \times ...$ be the countable cartesian product of B, and $p \in B$. The infinite wedge $\vee_p^\infty B$ denote is formed by taking countably many disjoint copies of B and identifying them at the point p. Note that the map A_p^∞ is the unique map arising from the multiple

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pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, p, ...p, id, p, ...) : B \to B^\infty$, where the identity map, id, is in the j-th place and 1 is terminal object in the category of ***Set*** [7].

Define $A_p^\infty : \vee_p^\infty B \to B^\infty$ by

$$A_p^\infty(x_i) = (p, ..., p, x, p, p, ...)$$

where x_i is in the i-th component of $\vee_p^\infty B$ and $\bigtriangledown_p^\infty : \vee_p^\infty B \longrightarrow B$ by

$$\bigtriangledown_p^\infty(x_i) = x$$

for all $i \in I$ [2].
Let $\mathscr{U} : \mathscr{E} \to$ ***Set*** be a topological functor [1]. X an object in $\mathscr{E}$ with $p \in \mathscr{U}(X) = B$. Let $\emptyset \neq M \subset B$ and X/M the final lift of the map

$$q : \mathscr{U}(X) = B \to B/M = (B \backslash M) \cup \{*\}$$

identifying M with a point $*$.

Definition 3.1. *([2, 3])*
(1) *If the initial lift of the U-source*

$$\{A_p^\infty : \vee_p^\infty B \longrightarrow U(X^\infty) = B^\infty \quad \text{and} \quad \nabla_p^\infty : \vee_p^\infty B \longrightarrow UD(B) = B\}$$

is discrete, then $\{p\}$ is said to be closed.

(2) *If $\{*\}$ is closed in X/M, then $M \subset X$ is said to be closed.*

(3) *If X/M is T_1 at $\{*\}$, then M is said to be strongly closed.*

(4) *If $B = M = \emptyset$, then M is to be (strongly) closed.*

Remark 3.2. *([9]) Let $\alpha, \beta \in F(A)$, $\gamma \in F(B)$, and $f : A \to B$ be a function. Then*

(1) $f(\alpha \cap \beta) = f(\alpha) \cap f(\beta)$.

(2) $f(\alpha) \cup f(\beta) \subset f(\alpha \cup \beta)$.

(3) $f^{-1} f \alpha \subset \alpha$.

(4) $\gamma \subset f f^{-1} \gamma$.

Theorem 3.3. *([3]) Let $(B, K) \in$* ***ConFCO****, $p \in B$, and $\emptyset \neq M \subset B$. Then*

(1) *$\{p\}$ is closed iff $[x] \cap [p] \notin K$ for all $x \in B$ with $x \neq p$.*

(2) *The following are equivalent.*

(a) *M is strongly closed.*

(b) *M is closed.*

(c) *$\alpha \not\subset [a]$ or $\alpha \cup [M]$ is improper for every $\alpha \in K$.*

Theorem 3.4. *(A, S) and (B, K) be constant filter convergence spaces and $f : (A, S) \to (B, K)$ be continuous.*

(1) *If $M \subset B$ is closed, then $f^{-1}(M) \subset A$ is closed.*

(2) *If $M \subset N$ and $N \subset B$ is closed, then $M \subset B$ is closed.*

Proof. **(1)** Suppose $M \subset B$ is closed, $x \in A$, $a \notin f^{-1}(M)$, and $\alpha \in S$. Note that $f(a) \notin D, f(\alpha) \in K$, and $f(\alpha) \not\subset [f(a)]$ or $f(\alpha) \cup [M]$ is improper since M is closed. Note that, by Remark 3.2,

$$f(\alpha) \cup [M] \subset f(\alpha) \cup [f f^{-1}(M)] \subset f(\alpha \cup [f^{-1}(M)]).$$

If $\alpha \cup [f^{-1}(M)]$ is proper, then $f(\alpha \cup [f^{-1}(M)])$ is proper (otherwise, $\emptyset \supset U \cup [f^{-1}(M)]$ for some $U \in \alpha$). It follows $U \cup [f^{-1}(M)]$ a contradiction and consequently, $f(\alpha) \cup [M]$ is proper. If $\alpha \subset [a]$, then $f\alpha \subset [f(a)]$ contradicting to $M \subset B$ is being closed. Thus, $\alpha \not\subset [a]$ and by Theorem 3.3, $f^{-1}(M) \subset A$ is closed.

(2) Suppose $M \subset N$ and $N \subset B$ is closed $a \notin M$ with $a \in B$ and $\alpha \in K$. If $a \notin N$, then by Theorem 3.3, $\alpha \not\subset [a]$ or $\alpha \cup [N]$ is improper since $N \subset B$ is closed. Suppose $\alpha \cup [N]$ is improper. Since

$$M \subset [N], \alpha \cup [M] \subset \alpha \cup [N]$$

and consequently, $\alpha \cup [M]$ is improper.
Suppose $a \in N$. K_N be a subspace structure on N deduced by the inclusion map $i : N \to (B,K)$. Note that

$$i^{-1}(\alpha) = \alpha \cup [N]$$

and by Remark 3.2, $\alpha \subset i(i^{-1}(\alpha))$. Since $\alpha \in K$, it follows $i(i^{-1}(\alpha)) \in K$ and by 2.1, $i^{-1}(\alpha) \in K_N$. Note that $a \notin M$, $a \in N$, and that $i^{-1}(\alpha) \in K_N$, by Theorem 3.3,

$$i^{-1}(\alpha) \not\subset [a]$$

or

$$i^{-1}(\alpha) \cup [M]$$

is improper since $M \subset N$ is closed.
Notice that

$$i^{-1}(\alpha) \cup [M] = \alpha \cup [N] \cup [M] = \alpha \cup [M]$$

and

$$i^{-1}(\alpha) = \alpha \cup [N] \not\subset [a]$$

implies

$$\alpha \not\subset [a]$$

(otherwise, if $\alpha \subset [a]$, then $\alpha \cup [N] \subset [a]$ since $a \in N$). Hence, $\alpha \not\subset [a]$ or $\alpha \cup [M]$ is improper and by Theorem 3.3, $M \subset B$ is closed. □

4. Closure Operators

Let $\mathscr{E}$ be a set based topological category and X be an object in $\mathscr{E}$. Recall [12, 13] that a closure operator $\boldsymbol{cl}$ of $\mathscr{E}$ is an an assignment to each subset F of (the underlying set of) X of a subset $\boldsymbol{cl}(F)$ of X such that

(1) $F \subset \boldsymbol{cl}(F)$.

(2) $\boldsymbol{cl}(N) \subset \boldsymbol{cl}(F)$ whenever $N \subset F$.

(3) (Continuity condition) for each $f : X \to Y$ in $\mathscr{E}$ and $F \subset Y$, $f(\boldsymbol{cl}(F)) \subset \boldsymbol{cl}(f(F))$.

If $\boldsymbol{cl}(F) = F$, then $F \subset X$ is called $\boldsymbol{cl}$-closed in X [12, 13].
If $\boldsymbol{cl}(\boldsymbol{cl}(F)) = \boldsymbol{cl}(F)$, then a closure operator $\boldsymbol{cl}$ is called idempotent [12, 13].

Definition 4.1. *Let (B,K) be a constant filter convergence space and $F \subset B$.*

$\boldsymbol{cl}(F) = \bigcap\{U \subset B : F \subset U \quad \text{is} \quad \text{closed}\}$ *is called the closure of M.*

$\boldsymbol{scl}(F) = \bigcap\{U \subset B : F \subset U \quad \text{is} \quad \text{strongly} \quad \text{closed}\}$ *is called the strong closure of M.*

Theorem 4.2. *$\boldsymbol{cl}$ and $\boldsymbol{scl}$ are productive, idempotent, and (weakly) hereditary closure operators of* $\boldsymbol{ConFCO}$.

Proof. Combine Theorem 3.4, Definition 4.1, and Theorems 2.3, 2.4, Exercise 2.D, and Proposition 2.5 of [13]. □

Let $\mathscr{E}$ be a topological category and $\boldsymbol{cl}$ be a closure operator of $\mathscr{E}$.

(1) $\mathscr{E}_{0cl} = \{X \in \mathscr{E} : x \in \boldsymbol{cl}(\{y\})$ and $y \in \boldsymbol{cl}(\{x\}) \Longrightarrow x = y$ with $x, y \in X\}$ [13].

(2) $\mathscr{E}_{1cl} = \{X \in \mathscr{E} : \boldsymbol{cl}(\{x\}) = \{x\}$, for each $x \in X\}$ [13].

(3) $\mathscr{E}_{2cl} = \{X \in \mathscr{E} : \boldsymbol{cl}(\triangle) = \triangle$, the diagonal$\}$ [13].

If $\mathscr{E} = \boldsymbol{Top}$, and $\boldsymbol{cl} = K$, the ordinary closure, then $\boldsymbol{Top}_{icl}$ reduce to the class of T_i, $i = 0,1,2$ spaces.

Theorem 4.3. *The following are equivalent.*

(1) $(B,K) \in \boldsymbol{ConFCO}_{0cl}$,

(2) $(B,K) \in \boldsymbol{ConFCO}_{0scl}$,

(3) *For each $x, y \in B$ with $x \neq y$, there exists $M \subset B$ such that $x \notin M$, $y \in M$ either $\alpha \not\subset [x]$ or $\alpha \cup [M]$ is improper for every $\alpha \in K$ or there exists $N \subset B$ such that $x \in N$, $y \notin N$ either $\alpha \not\subset [y]$ or $\alpha \cup [N]$ is improper for every $\alpha \in K$.*

Proof. By Theorem 3.3 and Definition 4.1, $(B,K) \in \boldsymbol{ConFCO}_{0cl}$ if and only if $(B,K) \in \boldsymbol{ConFCO}_{0scl}$ which shows (1) is equivalent to (2).

Suppose $(B,K) \in \boldsymbol{ConFCO}_{0cl}$ and $x,y \in B$ with $x \neq y$. Since $(B,K) \in \boldsymbol{ConFCO}_{0cl}$, $x \notin \boldsymbol{cl}(\{y\})$ or $y \notin \boldsymbol{cl}(\{x\})$. By Definition 4.1, there exists a closed $M \subset B$ such that $x \notin M$ and $y \in M$. By Theorem 3.3, either $\alpha \not\subset [x]$ or $\alpha \cup [M]$ is improper for every $\alpha \in K$. By Definition 4.1, there exists a closed $N \subset B$ such that $x \in N$ and $y \notin N$. By Theorem 3.3, either $\alpha \not\subset [y]$ or $\alpha \cup [N]$ is improper for every $\alpha \in K$. This show (1) implies (3).

Suppose (3) holds and $x,y \in B$ with $x \neq y$. If the first condition in (3) holds, then by Theorem 3.3, $M \subset B$ is closed and by Definition 4.1, $x \notin \boldsymbol{cl}(\{y\})$. If the second condition in (3) holds, then by Theorem 3.3, $N \subset B$ is closed and by Definition 4.1, $y \notin \boldsymbol{cl}(\{x\})$. Hence, $(B,K) \in \boldsymbol{ConFCO}_{0cl}$ which shows (3) implies (1).

□

Theorem 4.4. *The following are equivalent.*

(1) $(B,K) \in \boldsymbol{ConFCO}_{1cl}$,

(2) $(B,K) \in \boldsymbol{ConFCO}_{1scl}$,

(3) $[x] \cap [y] \notin K$ *for all* $x,y \in B$ *with* $x \neq y$.

Proof. By Theorem 3.3 and Definition 4.1, $(B,K) \in \boldsymbol{ConFCO}_{1cl}$ if and only if $(B,K) \in \boldsymbol{ConFCO}_{1scl}$ which shows (1) is equivalent to (2).

Suppose $(B,K) \in \boldsymbol{ConFCO}_{1cl}$ and $x \in B$. Note that $\boldsymbol{cl}(\{x\}) = \{x\}$, i.e,. $\{x\}$ is closed ($\boldsymbol{cl}$-closed). By Theorem 3.3, $[x] \cap [y] \notin K$ for all $y \neq x$ which shows (1) implies (3).

Suppose $[x] \cap [y] \notin K$ for all $x,y \in B$ with $x \neq y$. By Theorem 3.3, in particular, $\{x\}$ is closed, i.e., $\boldsymbol{cl}(\{x\}) = \{x\}$ and consequently, $(B,K) \in \boldsymbol{ConFCO}_{1cl}$ which shows (3) implies (1).

□

Theorem 4.5. *The following are equivalent.*

(1) $(B,K) \in \boldsymbol{ConFCO}_{2cl}$,

(2) $(B,K) \in \boldsymbol{ConFCO}_{2scl}$,

(3) *For all* $x,y \in B$ *with* $x \neq y$ *and* $\alpha,\beta \in K$, *if* $\alpha \subset [x]$ *and* $\beta \subset [y]$, *then* $\alpha \cup \beta$ *is improper.*

Proof. By Theorem 3.3 and Definition 4.1, $(B,K) \in \boldsymbol{ConFCO}_{2cl}$ if and only if $(B,K) \in \boldsymbol{ConFCO}_{2scl}$ which shows (1) is equivalent to (2).

Suppose $(B,K) \in \boldsymbol{ConFCO}_{2cl}$ and for all $x,y \in B$ with $x \neq y$ and for any $\alpha,\beta \in K$, $\alpha \subset [x]$ and $\beta \subset [y]$. Let $\sigma = \pi_1^{-1}\alpha \cup \pi_2^{-1}\beta$, where π_1 and π_2 are the projection maps. Note that $\pi_1\sigma = \alpha \in K$ and $\pi_2\sigma = \beta \in K$ and by 2.1, $\sigma \in K^2$, the product structure on B^2. If $V \in \sigma$, then there exists $V_1 \in \alpha$ and $V_2 \in \beta$, $V \supset V_1 \times V_2$. Since $\alpha \subset [x]$ and $\beta \subset [y]$, $x \in V_1$ and $y \in V_2$ and consequently, $\sigma \subset [(x,y)]$. Since $\triangle$ is closed in B^2, by Theorem 3.3, $\alpha \cup [\triangle]$ is improper. Therefore, there exists $V \in \sigma$ such that $V \cap \triangle = \emptyset$. Thus,

$$(V_1 \times V_2) \cap \triangle = \emptyset$$

if and only if

$$V_1 \cap V_2 = \emptyset,$$

i.e., $\alpha \cup \beta$ is improper.

Conversely, suppose that for all $x,y \in B$ with $x \neq y$ and for any $\alpha,\beta \in K$, if $\alpha \subset [x]$ and $\beta \subset [y]$, then $\alpha \cup \beta$ is improper. We show $(B,K) \in \boldsymbol{ConFCO}_{2cl}$, i.e., $\triangle$ is $\boldsymbol{cl}$-closed, i.e., by Theorem 3.3, for any $(x,y) \in B^2$, $(x,y) \notin \triangle$ and every $\sigma \in K^2$, i.e., $\sigma \cup [\triangle]$ is improper or $\sigma \not\subset [(x,y)]$. Since $\sigma \in K^2$, the product structure on B^2, by 2.1, $\pi_1\sigma, \pi_2\sigma \in K$ and $x \neq y$. By assumption, $\pi_1\sigma \cup \pi_2\sigma$ is improper if $\pi_1\sigma \subset [x]$ and $\pi_2\sigma \subset [y]$.

Let $\sigma_0 = \pi_1^{-1}\pi_1\sigma \cup \pi_2^{-1}\pi_2\sigma$. By Remark 3.2 (3), we have

$$\sigma_0 \subset \sigma,$$

$$\pi_1\sigma_0 = \pi_1\sigma \in K$$

and

$$\pi_2\sigma_0 = \pi_2\sigma \in K$$

and by 2.1, $\sigma_0 \in K^2$ and $\sigma_0 \subset [(x,y)]$. Since

$$\pi_1\sigma_0 \cup \pi_2\sigma_0 = \pi_1\sigma \cup \pi_2\sigma$$

is improper, there exists $V_1 \in \pi_1\sigma_0$ and $V_2 \in \pi_2\sigma_0$ such that $V_1 \cap V_2 = \emptyset$. It follows

$$(V_1 \times V_2) \cap \triangle = \emptyset,$$

which means

$$\sigma_0 \cup [\triangle]$$

is improper. By Theorem 3.3, $\triangle$ is closed, i.e., $(B,K) \in \boldsymbol{ConFCO}_{2cl}$. □

Let $\mathscr{U}: \mathscr{E} \to \boldsymbol{Set}$ be a topological functor, and X be an object of $\mathscr{E}$ with $\mathscr{U}(X) = B$.

If the initial lift of the $\mathscr{U}$-source

$$\{A : B^2 \vee_{\triangle} B^2 \to \mathscr{U}(X^3) = B^3 \quad and \quad \nabla : B^2 \vee_{\triangle} B^2 \to \mathscr{U}(\mathscr{D}(B^2)) = B^2\}$$

is discrete, then X is called $\overline{T}_0$[2].

If the initial lift of the $\mathscr{U}$-source

$$\{S : B^2 \vee_{\triangle} B^2 \to \mathscr{U}(X^3) = B^3 \quad and \quad \nabla : B^2 \vee_{\triangle} B^2 \to \mathscr{U}(\mathscr{D}(B^2)) = B^2\}$$

is discrete, then X is called T_1 [2], where A, S, and ∇ are the Principal axis, Skewed axis, and Folding maps defined in [2].

Theorem 4.6. *([4]) $(B,K) \in \boldsymbol{ConFCO}$ is T_1 if and only if it is $\overline{T}_0$ if and only if $[x] \cap [y] \notin K$ for all $x,y \in B$, $x \neq y$.*

Theorem 4.7. **(1)***The following categories are isomorphic.*

(i) $\boldsymbol{ConFCO}_{1cl}$,

(ii) $\boldsymbol{ConFCO}_{1scl}$,

(iii) $\boldsymbol{T_1 ConFCO}$,

(iv) $\overline{T}_0\boldsymbol{ConFCO}$.

(2) *Each of the subcategories $\boldsymbol{ConFCO}_{icl}$, $i = 0,1,2$, are epireflective subcategory of $\boldsymbol{ConFCO}$.*

Proof. **(1)** It follows from Theorems 4.4 and 4.6.

(2) Note that these subcategories are full and isomorphism-closed. We need to show that they are closed under subspaces and products.

Let $(B,L) \in \boldsymbol{ConFCO}_{1cl}$ and L_M be the subspace structures on M induced by the inclusion map $i : M \subset B$ and $[x] \cap [y] \in L_M$ for $x,y \in M$ with $x \neq y$. By 2.1,

$$i([x] \cap [y]) = i([x]) \cap i([y]) = [x] \cap [y] \in K$$

for $x,y \in X$ with $x \neq y$, a contradiction since, by Theorem 4.4, $(B,L) \in \boldsymbol{ConFCO}_{1cl}$. Hence, $[x] \cap [y] \notin L_M$ for all $x,y \in M$ with $x \neq y$ and by Theorem 4.4, $(M,L_M) \in \boldsymbol{ConFCO}_{1cl}$.

Let $(B,L) \in \boldsymbol{ConFCO}_{0cl}$, then the proof similar to above by using the Theorem 4.3 in place of Theorem 4.4.

Let $(B,L) \in \boldsymbol{ConFCO}_{2cl}$ and for all $x,y \in M$ with $x \neq y$ and for any $\alpha, \beta \in L_M$, if $\alpha \subset [x]$ and $\beta \subset [y]$, then $\alpha \cup \beta$ is improper. By 2.1 and Remark 3.2,

$$i(\alpha) = \alpha \subset i([x]) = [x],$$
$$i(\beta) = \beta \subset i([y]) = [y].$$
$$i(\alpha \cup \beta) = i(\alpha) \cup i(\beta) = \alpha \cup \beta \in K$$

for $x,y \in B$ with $x \neq y$, a contradiction since, by Theorem 4.5, $(B,K) \in \boldsymbol{ConFCO}_{2cl}$.

Suppose $(B_i,K_i) \in \boldsymbol{ConFCO}_{1cl}$ for all $i \in I$, and $B = \prod_{i \in I} B_i$. We show that $(B,K) \in \boldsymbol{ConFCO}_{1cl}$, where K is the product structure on B. Suppose there exist $x = (x_1,x_2,...)$, $y = (y_1,y_2,...) \in B$ with $x \neq y$ such that $[x] \cap [y] \in K$. Since $x \neq y$, there exists $j \in J$ such that $x_j \neq y_j$. $[x] \cap [y] \in K$ implies by 2.1,

$$\pi_j([x] \cap [y]) = \pi_j([x]) \cap \pi_j([y]) = [x_j] \cap [y_j] \in K_j$$

for $x_j \neq y_j$ which contradicts to (B_j,K_j) being in $\boldsymbol{ConFCO}_{1cl}$. Hence, $[x] \cap [y] \notin K$ for $x,y \in B$ with $x \neq y$.

The proof for $(B_i,K_i) \in \boldsymbol{ConFCO}_{0cl}$, $i \in I$ similar.

Suppose $(B_i,K_i) \in \boldsymbol{ConFCO}_{2cl}$, $i \in I$, $(B = \prod_{i \in I} B_i, K)$ for any $x,y \in B$ with $x \neq y$ and $\alpha, \beta \in K$ with $\alpha \subset [x]$ and $\beta \subset [y]$. By Theorem 4.5, we show that $\alpha \cup \beta$ is improper. Note that $\pi_i(\alpha), \pi_i(\beta) \in K$ and

$$\pi_i(\alpha) \subset \pi_i([x]) = [x_i],$$

$$\pi_i(\beta) \subset \pi_i([y]) = [y_i].$$

Since $(B_i,K_i)\in \boldsymbol{ConFCO}_{2cl}$ for all $i\in I$, by Theorem 4.5, $\pi_i(\alpha)\cup\pi_i(\beta)$ is improper.

Let

$$\sigma_0=[\prod_{i\in I}U_i : U_i\in\pi_i(\alpha)]$$

and

$$\beta_0=[\prod_{i\in I}V_i : V_i\in\pi_i(\beta)]$$

.
Note that

$$\pi_i\sigma_0=\pi_i\alpha\in K$$

and

$$\pi_i\beta_0=\pi_i\beta\in K$$

for all $i\in I$. By 2.1, $\beta_0,\sigma_0\in K$, $\sigma_0\subset[x]$ and $\beta_0\subset[y]$. Note that by Theorem 4.5,

$$\pi_i(\sigma_0)\cup\pi_i(\beta_0)=\pi_i(\alpha)\cup\pi_i(\beta)$$

is improper. By Remark 3.2, $\pi_i(\alpha\cup\beta)$ is improper and consequently, $\alpha\cup\beta$ is improper and by Theorem 4.5, $(B,K)\in\boldsymbol{ConFCO}_{2cl}$. □

Remark 4.8. **(1)** *By Theorems 4.3-4.5, we have*

$$\boldsymbol{ConFCO}_{2cl}=\boldsymbol{ConFCO}_{2scl}\subset\boldsymbol{ConFCO}_{1cl}=\boldsymbol{ConFCO}_{1scl}\subset\boldsymbol{ConFCO}_{0cl}=\boldsymbol{ConFCO}_{0scl}.$$

(2) *By Theorem 2.9 of referance [6], we have*

$$\boldsymbol{FCO}_{2scl}\subset\boldsymbol{FCO}_{2cl}=\boldsymbol{FCO}_{1cl}=\boldsymbol{FCO}_{1scl}\subset\boldsymbol{FCO}_{0cl}=\boldsymbol{ConFCO}_{0scl},$$

where $\boldsymbol{FCO}$ *is the category of filter convergence spaces.*

(3) *By Lemma 2.11 of referance [6], the subcategories* $\boldsymbol{Born}_{icl}$ *(resp.* $\boldsymbol{Born}_{iscl}, i=0,1,2$) *are the same. Moreover, for* $i=1,2$

$$\boldsymbol{Born}_{icl}\subset\boldsymbol{Born}_{iscl},$$

where $\boldsymbol{Born}$ *is the category of bornological spaces.*

(4) *By Theorem 4.5 of referance [8], for* $i=1,2$

$$\boldsymbol{Prord}_{icl}=\boldsymbol{Prord}_{iscl}\subset\boldsymbol{Prord}_{0cl}=\boldsymbol{Prord}_{0scl},$$

where $\boldsymbol{Prord}$ *is the category of preordered spaces.*
(5)

$$\boldsymbol{Top}_{2cl}=\boldsymbol{Top}_{2scl}\subset\boldsymbol{Top}_{1cl}=\boldsymbol{Top}_{1scl}\subset\boldsymbol{Top}_{0cl}=\boldsymbol{Top}_{0scl}.$$

5. Conclusion

(A) Let $(B,K)\in\boldsymbol{ConFCO}$. The following are equivalent.

(1) $(B,K)\in\boldsymbol{ConFCO}_{1scl}$,

(2) $(B,K)\in\boldsymbol{ConFCO}_{1cl}$,

(3) (B,K) is T_1,

(4) (B,K) is $\overline{T}_0$,

(5) $[x]\cap[y]\notin K$ for each $x,y\in B$ with $x\neq y$.

(B) The categories $\boldsymbol{ConFCO}_{0cl}$ and $\boldsymbol{ConFCO}_{0scl}$ are isomorphic.

(C) The categories $\boldsymbol{ConFCO}_{icl}$, $i=0,1,2$ are epireflective subcategory of $\boldsymbol{ConFCO}$.

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