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SOME ESTIMATES FOR THE GENERALIZED FOURIER-DUNKL TRANSFORM IN THE SPACE $L^2_{\alpha,n}$

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Abstract. Some estimates are proved for the generalized Fourier-Dunkl transform in the space $L^2_{\alpha,n}$ on certain classes of functions characterized by the generalized continuity modulus.

1. Introduction

In [5], Abilov et al. proved two useful estimates for the Fourier transform in the space of square integrable functions on certain classes of functions characterized by the generalized continuity modulus, using a translation operator.
In this paper, we consider a first-order singular differential-difference operator Λ on $\mathbb{R}$ which generalizes the Dunkl operator Λ_α, we prove some estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the generalized Fourier-Dunkl transform associated to Λ in $L^2_{\alpha,n}$ analogs of the statements proved in [5]. For this purpose, we use a generalized translation operator.
In section 2, we give some definitions and preliminaries concerning the generalized Fourier-Dunkl transform. The some estimates are proved in section 3.

2. Preliminaries

In this section, we develop some results from harmonic analysis related to the differential-difference operator Λ. Further details can be found in [1] and [6]. In all what follows assume where $\alpha > -1/2$ and n a non-negative integer.
Consider the first-order singular differential-difference operator on $\mathbb{R}$ defined by

$$\Lambda f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x} - 2n\frac{f(-x)}{x}.$$

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For $n = 0$, we regain the differential-difference operator

$$\Lambda_\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x},$$

which is referred to as the Dunkl operator of index $\alpha + 1/2$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. Such operators have been introduced by Dunkl (see [3], [4]) in connection with a generalization of the classical theory of spherical harmonics. Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, ...$$

Let $L^p_{\alpha,n}$, $1 \leq p < \infty$, be the class of measurable functions f on $\mathbb{R}$ for which

$$\|f\|_{p,\alpha,n} = \|M^{-1} f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx\right)^{1/p}.$$

If $p = 2$, then we have $L^2_{\alpha,n} = L^2(\mathbb{R}, |x|^{2\alpha+1})$.
The one-dimensional Dunkl kernel is defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz), z \in \mathbb{C}, \tag{2.1}$$

where

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m! \Gamma(m+\alpha+1)}, z \in \mathbb{C}, \tag{2.2}$$

is the normalized spherical Bessel function of index α. It is well-known that the functions $e_\alpha(\lambda .)$, $\lambda \in \mathbb{C}$, are solutions of the differential-difference equation

$$\Lambda_\alpha u = \lambda u, u(0) = 1.$$

In the terms of $j_\alpha(x)$, we have (see [2])

$$1 - j_\alpha(x) = O(1), x \geq 1, \tag{2.3}$$

$$1 - j_\alpha(x) = O(x^2), 0 \leq x \leq 1, \tag{2.4}$$

$$\sqrt{hx} J_\alpha(hx) = O(1), hx \geq 0, \tag{2.5}$$

where $J_\alpha(x)$ is Bessel function of the first kind, which is related to $j_\alpha(x)$ by the formula

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1)}{x^\alpha} J_\alpha(x), x \in \mathbb{R}^+. \tag{2.6}$$

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, put

$$\varphi_\lambda(x) = x^{2n} e_{\alpha+2n}(i\lambda x),$$

where $e_{\alpha+2n}$ is the Dunkl kernel of index $\alpha + 2n$ given by (1).

Proposition 2.1. *(i) φ_λ satisfies the differential equation*

$$\Lambda \varphi_\lambda = i\lambda \varphi_\lambda.$$

(ii) For all $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$

$$|\varphi_\lambda(x)| \leq |x|^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Dunkl transform we call the integral transform

$$\mathcal{F}_\Lambda f(\lambda) = \int_{\mathbb{R}} f(x)\varphi_{-\lambda}(x)|x|^{2\alpha+1}dx, \lambda \in \mathbb{R}, f \in L^1_{\alpha,n}.$$

Let $f \in L^1_{\alpha,n}$ such that $\mathcal{F}_\Lambda(f) \in L^1_{\alpha+2n} = L^1(\mathbb{R}, |x|^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Dunkl transform is given by the formula

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda f(\lambda)\varphi_\lambda(x)d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n}|\lambda|^{2\alpha+4n+1}d\lambda, \quad a_\alpha = \frac{1}{2^{2\alpha+2}(\Gamma(\alpha+1))^2}.$$

Proposition 2.2. *(i) For every $f \in L^2_{\alpha,n}$,*

$$\mathcal{F}_\Lambda(\Lambda f)(\lambda) = i\lambda\mathcal{F}_\Lambda(f)(\lambda).$$

(ii) For every $f \in L^1_{\alpha,n} \cap L^2_{\alpha,n}$ we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2|x|^{2\alpha+1}dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

(iii) The generalized Fourier-Dunkl transform $\mathcal{F}_\Lambda$ extends uniquely to an isometric isomorphism from $L^2_{\alpha,n}$ onto $L^2(\mathbb{R}, \mu_{\alpha+2n})$.

The generalized translation operators τ^x, $x \in \mathbb{R}$, tied to Λ are defined by

$$\begin{aligned}\tau^x f(y) &= \frac{(xy)^{2n}}{2}\int_{-1}^{1} \frac{f(\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n}\left(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt \\ &+ \frac{(xy)^{2n}}{2}\int_{-1}^{1} \frac{f(-\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n}\left(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right)A(t)dt,\end{aligned}$$

where

$$A(t) = \frac{\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+1/2)}(1+t)(1-t^2)^{\alpha+2n-1/2}.$$

Proposition 2.3. *Let $x \in \mathbb{R}$ and $f \in L^2_{\alpha,n}$. Then $\tau^x f \in L^2_{\alpha,n}$ and*

$$\|\tau^x f\|_{2,\alpha,n} \leq 2x^{2n}\|f\|_{2,\alpha,n}.$$

Furthermore,

$$\mathcal{F}_\Lambda(\tau^x f)(\lambda) = x^{2n}e_{\alpha+2n}(i\lambda x)\mathcal{F}_\Lambda(f)(\lambda). \tag{2.7}$$

The generalized modulus of continuity of function $f \in L^2_{\alpha,n}$ is defined as

$$w(f,\delta)_{2,\alpha,n} = \sup_{0<h\leq\delta} \|\tau^h f(x) + \tau^{-h}f(x) - 2h^{2n}f(x)\|_{2,\alpha,n}, \delta > 0.$$

3. Main Results

The goal of this work is to prove some estimates for the integral

$$J_N^2(f) = \int_{|\lambda|\geq N} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

in certain classes of functions in $L^2_{\alpha,n}$.

Lemma 3.1. *For $f \in L^2_{\alpha,n}$, we have,*

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|^2_{2,\alpha,n} = 4h^{4n} \int_{\mathbb{R}} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

where $r = 0, 1, 2, ...$

Proof. By using the formulas (2.1), (2.2) and (2.7), we conclude that

$$\mathcal{F}_\Lambda(\tau^h f + \tau^{-h} f - 2h^{2n} f)(\lambda) = 2h^{2n}(j_{\alpha+2n}(\lambda h) - 1)\mathcal{F}_\Lambda f(\lambda). \tag{3.1}$$

Now by formula (3.1) and Plancherel equality, we have the result. □

Theorem 3.1. *Given $f \in L^2_{\alpha,n}$. Then there exist a constant $C > 0$ such that, for all $N > 0$,*

$$J_N(f) = O(N^{2n}\omega(f, CN^{-1})_{2,\alpha,n}).$$

Proof. Firstly, we have

$$J_N^2(f) \leq \int_{|\lambda|\geq N} |j| d\mu + \int_{|\lambda|\geq N} |1 - j| d\mu, \tag{3.2}$$

with $j = j_p(\lambda h)$, $p = \alpha + 2n$ and $d\mu = |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)$. The parameter $h > 0$ will be chosen in an instant.
In view of formulas (2.5) and (2.6), there exist a constant $C_1 > 0$ such that

$$|j| \leq C_1(|\lambda|h)^{-p-\frac{1}{2}}.$$

Then

$$\int_{|\lambda|\geq N} |j| d\mu \leq C_1(hN)^{-p-\frac{1}{2}} J_N^2(f).$$

Choose a constant C_2 such that the number $C_3 = 1 - C_1 C_2^{-p-\frac{1}{2}}$ is positif.
Setting $h = C_2/N$ in the inequality (3.2), we have

$$C_3 J_N^2(f) \leq \int_{|\lambda|\geq N} |1 - j| d\mu. \tag{3.3}$$

By Hölder inequality the second term in (3.3) satisfies

$$\begin{aligned} \int_{|\lambda|\geq N} |1 - j| d\mu &= \int_{|\lambda|\geq N} |1 - j|.1.d\mu \\ &\leq \left(\int_{|\lambda|\geq N} |1 - j|^2 d\mu\right)^{1/2} \left(\int_{|\lambda|\geq N} d\mu\right)^{1/2} \\ &\leq \left(\int_{|\lambda|\geq N} |1 - j|^2 d\mu\right)^{1/2} J_N(f). \end{aligned}$$

From Lemma 3.1, we conclude that

$$\int_{|\lambda|\geq N} |1-j|^2 d\mu \leq h^{-4n}\|\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)\|^2_{2,\alpha,n}.$$

Therefore

$$\int_{|\lambda|\geq N} |1-j| d\mu \leq h^{-2n}\|\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)\|_{2,\alpha,n}J_N(f).$$

For $h = C_2/N$, we obtain

$$C_3 J_N^2(f) \leq C_2^{-2n} N^{2n} w(f, C_2/N)_{2,\alpha,n} J_N(f).$$

Consequently

$$C_2^{2n} C_3 J_N(f) \leq N^{2n} w(f, C_2/N)_{2,\alpha,n}.$$

for all $N > 0$. The theorem is proved with $C = C_2$. □

Theorem 3.2. *Let $f \in L^2_{\alpha,n}$. Then, for all $N > 0$,*

$$\omega(f, N^{-1})_{2,\alpha,n} = O\left(N^{-2(n+1)}\left(\sum_{l=0}^{N-1}(l+1)^3 J_l^2(f)\right)^{\frac{1}{2}}\right).$$

Proof. From Lemma 3.1, we have

$$\|\tau^h f(x)+\tau^{-h}f(x)-2h^{2n}f(x)\|^2_{2,\alpha,n} = 4h^{4n}\int_{\mathbb{R}} |j_{\alpha+2n}(\lambda h)-1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

This integral is divided into two

$$\int_{\mathbb{R}} = \int_{|\lambda|\leq N} + \int_{|\lambda|\geq N} = I_1 + I_2,$$

where $N = [h^{-1}]$. We estimate them separately.
From (2.3), we have the estimate

$$I_2 \leq C_4 \int_{|\lambda|\geq N} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = C_4 J_N^2(f).$$

Now, we estimate I_1. From formula (2.4), we have

$$\begin{aligned} I_1 &\leq C_5 h^4 \int_{|\lambda|\leq N} \lambda^4 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = C_5 h^4 \sum_{l=0}^{N-1}\int_{l\leq|\lambda|\leq l+1} \lambda^4 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\ &= C_5 h^4 \sum_{l=0}^{N-1} a_l \left(J_l^2(f) - J_{l+1}^2(f)\right), \end{aligned}$$

with $a_l = (l+1)^4$.
For all integers $m \geq 1$, the Abel transformation shows

$$\begin{aligned} \sum_{l=0}^{m} a_l \left(J_l^2(f) - J_{l+1}^2(f)\right) &= a_0 J_0^2(f) + \sum_{l=1}^{m} (a_l - a_{l-1}) J_l^2(f) - a_m J_{m+1}^2(f) \\ &\leq a_0 J_0^2(f) + \sum_{l=1}^{m} (a_l - a_{l-1}) J_l^2(f), \end{aligned}$$

because $a_m J_{m+1}^2(f) \geq 0$.
Hence

$$I_1 \leq C_5 h^4 \left(J_0^2(f) + \sum_{l=1}^{N-1} \left((l+1)^4 - l^4 \right) J_l^2(f) - N^4 J_N^2(f) \right).$$

Moreover by the finite increments theorem, we have $(l+1)^4 - l^4 \leq 4(l+1)^3$. Then

$$I_1 \leq C_5 N^{-4} \left(J_0^2(f) + 4 \sum_{l=1}^{N-1} (l+1)^3 J_l^2(f) - N^4 J_N^2(f) \right),$$

since $N \leq \frac{1}{h}$. Combining the estimates for I_1 and I_2 gives

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha,n}^2 = O\left(N^{-4-4n} \sum_{l=0}^{N-1} (l+1)^3 J_l^2(f) \right),$$

which implies

$$\omega(f, N^{-1})_{2,\alpha,n} = O\left(N^{-2(n+1)} \left(\sum_{l=0}^{N-1} (l+1)^3 J_l^2(f) \right)^{\frac{1}{2}} \right),$$

and this ends the proof. □

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