Konuralp Journal of Mathematics
Volume 1 No. 2 pp. 50–56 (2013) ©KJM

GENERALIZED RELATIVE ORDER OF FUNCTIONS ANALYTIC IN THE UNIT DISC

RATAN KUMAR DUTTA

Abstract. In this paper we consider generalized relative order of a function analytic in the unit disc with respect to an entire function and prove several theorems.

1. Introduction, Definitions and Notation

Let $f(z)$ be analytic in the unit disc $U : \{z : |z| < 1\}$ and

$$T_f(r) = T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} log^+|f(re^{i\theta})|d\theta$$

is Nevanlinna characteristic function of $f(z)$. If

$$T(r, f) = (1 - r)^{-\mu} \text{ for all } r \text{ in } 0 < r_0(\mu) < r < 1,$$

then the greatest lower bound of all such numbers μ is called Nevanlinna order [5] (Juneja and Kapoor 1985) of f. Thus the Nevanlinna order $\rho(f)$ of f is given by

$$\rho(f) = limsup_{r\to 1}\frac{log\, T(r, f)}{-log(1 - r)}.$$

In [1] Banerjee and Dutta introduce the idea of relative order of a function analytic in the unit disc with respect to an entire function.

Definition 1.1. [1] If f be analytic in U and g be entire, then the relative order of f with respect to g, denoted by $\rho_g(f)$ is defined by

$$\rho_g(f) = inf\{\mu > 0 : T_f(r) < T_g\left[\left(\frac{1}{1-r}\right)^{\mu}\right] \text{ for all } 0 < r_0(\mu) < r < 1\}.$$

Note 1.1. When $g(z) = exp\ z$, then Definition 1.1 coincides with the definition of Nevanlinna order of f.

Date: April 14, 2013 and, in revised form, July 23, 2013.
1991 *Mathematics Subject Classification.* 30D15.
Key words and phrases. Analytic function, entire function, generalized relative order, unit disc.

Notation 1.2. [3] $log^{[0]}x = x$, $exp^{[0]}x = x$ and for positive integer m, $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.

In [2] Datta and Jerin introduce the idea of generalized relative order.

Definition 1.2. [2] Let $T_f(r) = T(r, f)$ denote the Nevanlinna's characteristic function of f. The relative generalized Nevanlinna order $\rho_g^p(f)$ of an analytic function f in U with respect to another entire function g are defined in the following way:

$$\rho_g^p(f) = limsup_{r\to 1} \frac{log^{[p]} T_g^{-1} T_f(r)}{-log(1-r)} \ .$$

Definition 1.3. [1] An entire function g is said to have the property (A), if for any $\sigma > 1, \lambda > 0$ and for all $r, 0 < r < 1$ sufficiently close to 1

$$\left[G\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)\right]^2 < G\left(\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)^{\sigma}\right)$$

where $G(r) = max_{|z|=r}|g(z)|$.

The function $g(z) = exp\ z$ has the property (A) where as $g(z) = z$ has not.

In this paper we consider the definition of generalized relative order of a function analytic in the unit disc U with respect to an entire function and obtain the sum and product theorems. Also we show that the relative order of a function analytic in U with respect to an entire and to the derivative of the entire are same. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions as those are available in [4], [5], [6] and [7]. Throughout we shall assume that f, f_1, f_2 etc, to be function analytic in U and g, g_1, g_2 etc, are non constant entire.

2. Known Lemmas

Lemma 2.1. [1] *Let g be an entire function which has the property (A). Then for any positive integer n and for all $\sigma > 1, \lambda > 0$,*

$$\left[G\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)\right]^n < G\left(\left(\left(\frac{1}{1-r}\right)^{\lambda}\right)^{\sigma}\right)$$

holds for all $r, 0 < r < 1$, sufficiently close to 1.

Lemma 2.1 follows from Definition 1.3.

Lemma 2.2. [1] *If g is entire then*

$$T_g\left(\frac{1}{1-r}\right) \le logG\left(\frac{1}{1-r}\right) \le 3T_g\left(\frac{2}{1-r}\right)$$

for all r, $0 < r < 1$, sufficiently close to 1.

3. Preliminary Theorem

Theorem 3.1. *Let f be analytic in U of generalized relative order $\rho_g^p(f)$ where g is entire. Let $\epsilon > 0$ be arbitrary. Then*

$$T_f(r) = O\left(logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_g^p(f)+\epsilon}\right)\right)$$

holds for all r, $0 < r < 1$, sufficiently close to 1.
Conversely, if for an analytic f in U and entire g having the property (A),

$$T_f(r) = O\left(logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)\right)$$

holds for all r, $0 < r < 1$, sufficiently close to 1, and

$$T_f(r) = O\left(logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k-\epsilon}\right)\right)$$

does not hold for all r, $0 < r < 1$, sufficiently close to 1, then $k = \rho_g^p(f)$.

Proof. From the definition of generalized relative order, we have

$$\begin{aligned}
T_f(r) &< T_g\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_g^p(f)+\epsilon}\right) \text{ for } 0 < r_0 < r < 1, \text{ say} \\
\text{or, } T_f(r) &< logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_g^p(f)+\epsilon}\right) \text{ for } 0 < r_0 < r < 1, \quad \text{by Lemma 2.2.} \\
\text{So, } T_f(r) &= O\left(logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_g^p(f)+\epsilon}\right)\right).
\end{aligned}$$

Conversely, if

$$T_f(r) = O\left(logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)\right)$$

holds for all r, $0 < r < 1$, sufficiently close to 1, then

$$\begin{aligned}
T_f(r) &< [\alpha] logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right), \ \alpha > 1 \\
&= \frac{1}{3} log\left[G\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)\right]^{[3\alpha]} \\
&\le \frac{1}{3} logG\left(\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)^{\sigma}\right) \text{ by Lemma 2.1, for any } \sigma > 1 \\
&\le T_g\left(2\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)^{\sigma}\right), \text{ by Lemma 2.2.}
\end{aligned}$$

$$\therefore \quad logT_g^{-1}T_f(r) \leq \log 2 + \log\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k+\epsilon}\right)^{\sigma}$$
$$\leq \sigma \exp^{[p-2]}\left(\frac{1}{1-r}\right)^{k+\epsilon} + O(1).$$
$$\therefore \quad log^{[2]}T_g^{-1}T_f(r) \leq \exp^{[p-3]}\left(\frac{1}{1-r}\right)^{k+\epsilon} + O(1).$$

So

$$limsup_{r\to 1-}\frac{log^{[p]}T_g^{-1}T_f(r)}{-log(1-r)} \leq k+\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$limsup_{r\to 1-}\frac{log^{[p]}T_g^{-1}T_f(r)}{-log(1-r)} \leq k. \tag{3.1}$$

Again there exists a sequence $\{r_n\}$ of values of r tending to 1_- for which

$$T_f(r) \geq logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k-\epsilon}\right)$$
$$\geq T_g\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{k-\epsilon}\right), \text{ by Lemma 2.2}$$

and so

$$\frac{log^{[p]}T_g^{-1}T_f(r)}{-log(1-r)} \geq k-\epsilon \tag{3.2}$$

for $r = r_n \to 1_-$.
Since $\epsilon > 0$ is arbitrary, combining (3.1) and (3.2), we obtain $k = \rho_g^p(f)$.
This proves the theorem. □

4. Sum and Product Theorems

Theorem 4.1. *Let f_1 and f_2 be analytic in the unit disc U having generalized relative orders $\rho_g^p(f_1)$ and $\rho_g^p(f_2)$ respectively, where g is entire having the property (A). Then*

$$\begin{aligned} (a)\quad \rho_g^p(f_1 \pm f_2) &\leq max\{\rho_g^p(f_1), \rho_g^p(f_2)\} \text{ and} \\ (b)\quad \rho_g^p(f_1.f_2) &\leq max\{\rho_g^p(f_1), \rho_g^p(f_2)\}. \end{aligned}$$

The same inequality holds for the quotient. The equality holds in (b) if $\rho_g^p(f_1) \neq \rho_g^p(f_2)$.

Proof. We may suppose that $\rho_g^p(f_1)$ and $\rho_g^p(f_2)$ both are finite, because if one of them or both are infinite, the inequalities are evident. Let $\rho_1 = \rho_g^p(f_1)$ and $\rho_2 = \rho_g^p(f_2)$ and $\rho_1 \leq \rho_2$. For arbitrary $\epsilon > 0$ and for all $r,\ 0 < r < 1$, sufficiently close to 1, we have

$$T_{f_1}(r) < T_g\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_1+\epsilon}\right) \leq logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_1+\epsilon}\right)$$

and

$$T_{f_2}(r) < T_g\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right) \leq logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right), \quad \text{using Lemma 2.2.}$$

Now for all $r,\ 0<r<1$, sufficiently close to 1,

$$\begin{aligned}
T_{f_1\pm f_2}(r) &\leq T_{f_1}(r)+T_{f_2}(r)+O(1)\\
&\leq logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_1+\epsilon}\right)+logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)+O(1)\\
&\leq 3logG\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)\\
&= \frac{1}{3}\ log\left[G\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)\right]^9\\
&\leq \frac{1}{3}\ logG\left(\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)^{\sigma}\right) \text{ by Lemma 2.1, for any } \sigma>1\\
&\leq T_g\left(2\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)^{\sigma}\right) \text{ by Lemma 2.2.}
\end{aligned}$$

$$\begin{aligned}
\therefore \quad logT_g^{-1}T_{f_1\pm f_2}(r) &\leq log2+log\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}\right)^{\sigma}\\
&\leq \sigma\exp^{[p-2]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}+O(1).
\end{aligned}$$

$$\therefore \quad log^{[2]}T_g^{-1}T_{f_1\pm f_2}(r) \leq \exp^{[p-3]}\left(\frac{1}{1-r}\right)^{\rho_2+\epsilon}+O(1).$$

$$\begin{aligned}
\therefore \quad \rho_g^p(f_1\pm f_2) &= limsup_{r\to 1-}\frac{log^{[p]}T_g^{-1}T_{(f_1\pm f_2)}(r)}{-log(1-r)}\\
&\leq \rho_2+\epsilon.
\end{aligned}$$

Since $\epsilon>0$ is arbitrary,

$$\rho_g^p(f_1\pm f_2)\leq\rho_2\leq max\{\rho_g^p(f_1),\rho_g^p(f_2)\}$$

which proves (a).

For (b), since,

$$T_{f_1.f_2}(r)\leq T_{f_1}(r)+T_{f_2}(r),$$

we obtain similarly as above

$$\rho_g^p(f_1.f_2)\leq max\ \{\rho_g^p(f_1),\rho_g^p(f_2)\}.$$

Let $f=f_1.f_2$ and $\rho_g^p(f_1)<\rho_g^p(f_2)$. Then applying (b), we have $\rho_g^p(f)\leq\rho_g^p(f_2)$.
Again since $f_2=f/f_1$, applying the first part of (b), we have

$$\rho_g^p(f_2)\leq max\ \{\rho_g^p(f),\rho_g^p(f_1)\}.$$

Since $\rho_g^p(f_1) < \rho_g^p(f_2)$, we have

$$\rho_g^p(f) = \rho_g^p(f_2) = max\{\rho_g^p(f_1), \rho_g^p(f_2)\}$$

when $\rho_g^p(f_1) \neq \rho_g^p(f_2)$.
This prves the theorem. □

5. Relative order with respect to the derivative of an entire function

Theorem 5.1. *If f is analytic in the unit disc U and g be transcendental entire having the property (A), then $\rho_g^p(f) = \rho_{g'}^p(f)$ where $g^{'}$ denotes the first derivative of g.*

To prove the theorem we require the following lemmas.

Lemma 5.1. [1] *If g be a transcendental entire, then for all $r,\ 0 < r < 1$, sufficiently close to 1 and for any $\lambda > 0$*

$$T_{g'}\left(\left(\frac{1}{1-r}\right)^{\lambda}\right) \leq 2\,T_g\left(2\left(\frac{1}{1-r}\right)^{\lambda}\right) + O\left(T_g\left(2\left(\frac{1}{1-r}\right)^{\lambda}\right)\right)$$

Lemma 5.2. [1] *Let g be a transcendental entire function, then for all $r,\ 0 < r < 1$, sufficiently close to 1 and $\lambda > 0$*

$$T_g\left(\left(\frac{1}{1-r}\right)^{\lambda}\right) < C\,\left[T_{g'}\left(2\left(\frac{1}{1-r}\right)^{\lambda}\right) + log\left(\frac{1}{1-r}\right)^{\lambda}\right]$$

where C is a constant which is only dependent on $g(0)$.

Proof of the Theorem 5.1:

Proof. From Lemma 5.1 and Lemma 5.2 we obtain for $r,\ 0 < r < 1$, sufficiently close to 1

$$T_{g'}\left(\left(\frac{1}{1-r}\right)^{\lambda}\right) < [K]\,T_g\left(2\left(\frac{1}{1-r}\right)^{\lambda}\right) \tag{5.1}$$

and

$$T_g\left(\left(\frac{1}{1-r}\right)^{\lambda}\right) < [K^{'}]\,T_{g'}\left(2\left(\frac{1}{1-r}\right)^{\lambda}\right) \tag{5.2}$$

where $K,\ K^{'} > 0$ and $\lambda > 0$ be any number.
From the definition of $\rho_{g'}^p(f)$, we get for arbitrary $\epsilon > 0$,

$$T_f(r) < T_{g'}\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho_{g'}(f)+\epsilon}\right)$$

for all $r,\ 0 < r < 1$, sufficiently close to 1.
From (5.1) and by Lemma 2.1 and Lemma 2.2, for all $r,\ 0 < r < 1$, sufficiently close to 1

$$\begin{aligned}
T_f(r) &< [K]T_g\left(2\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho^p_{g'}(f)+\epsilon}\right)\\
&\leq [K]logG\left(2\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho^p_{g'}(f)+\epsilon}\right)\\
&= \frac{1}{3}log\left[G\left(2\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho^p_{g'}(f)+\epsilon}\right)\right]^{3[k]}\\
&\leq \frac{1}{3}log\left(G\left(2\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho^p_{g'}(f)+\epsilon}\right)^{\sigma}\right) \text{ for any } \sigma > 1\\
&\leq T_g\left(2^{\sigma+1}\left(\exp^{[p-1]}\left(\frac{1}{1-r}\right)^{\rho^p_{g'}(f)+\epsilon}\right)^{\sigma}\right).
\end{aligned}$$

$$\therefore \quad \rho^p_g(f) = limsup_{r\to 1-}\ \frac{log^{[p]}T_g^{-1}T_f(r)}{-log(1-r)} \leq \rho^p_{g'}(f)+\epsilon.$$

Since $\epsilon > 0$ is arbitrary, so $\rho^p_g(f) \leq \rho^p_{g'}(f)$.
Using (5.2) we obtain similarly $\rho^p_{g'}(f) \leq \rho^p_g(f)$. So, $\rho^p_g(f) = \rho^p_{g'}(f)$.
This proves the theorem. □

Acknowledgement: Author is thankful to the referee for his/her suggestions towards the improvement of the paper.

References

[1] D. Banerjee and R. K. Dutta, Relative order of functions analytic in the unit disc, Bull. Cal. Math. Soc., Vol. 101, No. 1, (2009), pp 95-104.
[2] S. K. Datta and E. Jerin: Further Results on the Generalised Growth Properties of Functions Analytic in a Unit Disc, International Journal of Contemporary Mathematical Sciences, Vol. 5, No. 23 (2010), pp. 1137-1143.
[3] D. Sato, On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc., Vol. 69, (1963), pp. 411-414.
[4] W. K. Hayman, *Meromorphic functions*, The Clarendon Press, Oxford, 1964.
[5] O. P. Juneja and G. P. Kapoor, *Analytic functions -growth aspects*, Pitman advanced publishing program, 1985.
[6] G. Valiron, *Lectures on the general theory of Integral functions*, Chelsea Publishing Company, 1949.
[7] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers and Science Press, Beijing, 2003.

Department of Mathematics, Patulia High School, Patulia, Kolkata-119, West Bengal, India
E-mail address: ratan_3128@yahoo.com