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WEIGHTED OSTROWSKI AND ČEBYŠEV TYPE INEQUALITIES WITH APPLICATIONS

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Abstract. Weighted Ostrowski and Čebyšev type inequalities on time scales for single and double integrals have been derived which unify the corresponding continuous and discrete versions and some applications for quantum calculus are also given.

1. Introduction

In 1937, Ostrowski gave a useful formula to estimate the absolute value of derivation of a differentiable function by its integral mean, the so called Ostrowski's inequality [11]

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\ ds\right| \le \sup_{a\le t\le b}|f'(t)|(b-a)\left[\frac{1}{4}+\frac{\left(t-\frac{a+b}{2}\right)^2}{(b-a)^2}\right], \tag{1.1}$$

by means of the Montgomery's identity [9, p.565].

In 1980, Pečarić [13] gave the weighted generalization of the Montgomery identity as:

$$f(x)=\int_a^b w(t)\ f(t)\ dt+\int_a^b Q_w(x,t)\ f'(t)\ dt, \tag{1.2}$$

where the weighted Peano kernel Q_w is defined by:

$$Q_w(x,t)=\begin{cases} W(t), & t\in[a,x] \\ W(t)-1, & t\in(x,b], \end{cases}$$

with $W(t)=\int_a^t w(x)\ dx$ for $t\in[a,b]$.

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K. Boukerrioua et. al [2], further generalized (1.2), while on the other hand in 1882, P. L. Čebyšev [5] proved the following inequality, the so called Čebyšev inequality, for two absolutely continuous functions $f, g : [a, b] \to \mathbb{R}$

$$|T(f,g)| \leq \frac{(b-a)^2}{12} \; \|f'\|_{\infty;[a,b]} \;\; \|g'\|_{\infty;[a,b]}, \tag{1.3}$$

where

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\; g(x)\; dx - \left(\frac{1}{b-a}\int_a^b f(x)\; dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\; dx\right).$$

During the past few years, many researchers have given considerable attention to the inequalities (1.1) and (1.3) and various generalizations, extensions, variants of these have appeared in literature [3, 4, 6, 7, 12, 14, 15, 16, 17, 18], and the references therein. The main aim of this paper is to prove some new results in general time scales, generalizing some results in literature, which unify discrete, continuous and many other cases. As a consequence some new Ostrowski and Čebyšev type inequalities have been proved with some new applications. In the whole paper $\mathbb{T}_i$, $1 \leq i \leq 2$, is considered as a time scale.

2. Ostrowski type inequalities for single integral

We may begin with the following lemma:

Lemma 2.1. *Let $a, b \in \mathbb{T}_1$; $c, d \in \mathbb{R}_+ \cup \{0\}$ and let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If f^Δ is an integrable on $[a, b]$ and $w : [a, b] \to \mathbb{R}$ is a weight function such that $\int_a^b w(t)\; \Delta t = d$. Let $\phi : [c, d] \to \mathbb{R}$ be a continuously differentiable function such that $\phi(0) = 0$ and $\phi(d) \neq 0$, then*

$$f(x) = \frac{1}{\phi(d)}\int_a^b [(\phi\circ W)(t)]^\Delta \, f^\sigma(t)\Delta t + \frac{1}{\phi(d)}\int_a^b K_{w,\phi}(x,t) f^\Delta(t)\Delta t, \tag{2.1}$$

where, $K_{w,\phi}$ is the generalized weighted Peano kernel defined by:

$$K_{w,\phi}(x,t) = \begin{cases} \phi(W(t)), & t \in [a,x] \\ \\ \phi(W(t)) - \phi(d), & t \in (x,b], \end{cases}$$

with $W : [a, b] \to [c, d]$ defined by $W(t) = \int_a^t w(x)\; \Delta x$ for $t \in [a, b]$.

Proof. By integration

$$\begin{aligned}
\int_a^b K_{w,\phi}(x,t) f^\Delta(t)\Delta t &= \int_a^x K_{w,\phi}(x,t) f^\Delta(t)\Delta t + \int_x^b K_{w,\phi}(x,t) f^\Delta(t)\Delta t. \\
&= \int_a^x \phi(W(t)) f^\Delta(t)\Delta t + \int_x^b [\phi(W(t)) - \phi(d)] f^\Delta(t)\Delta t. \\
&= \int_a^b \phi(W(t)) f^\Delta(t)\Delta t - \phi(d)[f(b) - f(x)]. \\
&= \phi(W(b)) f(b) - \phi(W(a)) f(a) - \int_a^b [(\phi\circ W)(t)]^\Delta \, f^\sigma(t)\Delta t \\
&\quad + f(x)\phi(d) - \phi(d) f(b),
\end{aligned}$$

which is equivalent to (2.1) □

Remark 2.1. For $\mathbb{T}_1 = \mathbb{R}$ and $d = 1$, Lemma 2.1 coincides [2, Theorem 2.1]

Corollary 2.1. *(Discrete case) Let* $\mathbb{T}_1 = \mathbb{Z}$, $a = 0, b = n, c = 0, d = m, x = i, t = k, s = l$ *and* $f(x) = x_i$*, then* (2.1) *reduces to*

$$x_i = \frac{1}{\phi(m)} \sum_{k=1}^{n} f(k) \Delta\phi \left(\sum_{p=1}^{k-1} w(p) \right) + \frac{1}{\phi(m)} \sum_{k=1}^{n} \Delta f(k-1) K_{w,\phi}(i, k-1),$$

where

$$K_{w,\phi}(i,k) = \begin{cases} \phi\left(\sum_{p=0}^{k-1} w(p)\right), & 1 \le k \le i \\ \\ \phi\left(\sum_{p=0}^{k-1} w(p)\right) - \phi(m), & i+1 \le k \le n. \end{cases} \tag{2.2}$$

Corollary 2.2. (*Quantum calculus case*) *Let* $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$ *with* $q_1 > 1$*. Suppose* $a = q_1^i, b = q_1^j, d = q_2^l$ *for some* $i < j$, $i < r$; $l, r \in \mathbb{N}_0$ *and* $q_2 > 1$*, then* (2.1) *reduces to*

$$f\left(q_1^m\right) = \frac{1}{\phi\left(q_2^l\right)} \sum_{r=i}^{j-1} \frac{f(q_1^{r+1}) \ \Delta\phi\left(\sum_{u=i}^{r-1} w\left(q_1^u\right)\right)}{q_1^r(q_1-1)} - \frac{1}{\phi\left(q_2^l\right)} \sum_{r=i}^{j-1} K_{w,\phi}\left(q_1^m, q_1^r\right) \frac{\Delta f\left(q_1^r\right)}{(q_1-1)q_1^r},$$

where Δ *is the forward difference operator defined by:*

$$\Delta\phi \left(\sum_{u=i}^{r-1} w\left(q_1^u\right) \right) = \phi \left(\sum_{u=i}^{r} w\left(q_1^u\right) \right) - \phi \left(\sum_{u=i}^{r-1} w\left(q_1^u\right) \right).$$

Remark 2.2. By setting $\phi(x) = x$; $d = 1$ and $W(t) = \frac{t-a}{b-a}$, Lemma 2.1 reduces to [8, Lemma A].

A generalization of Ostrowski's inequality on time scale may be considered as follows:

Theorem 2.1. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable function on* (a,b) *such that* f^{Δ} *is bounded on* (a,b)*, that is,* $\left\|f^{\Delta}\right\|_{\infty;[a,b]} := \sup_{t\in(a,b)} \left|f^{\Delta}(t)\right| < \infty$ *and* $\|\phi \circ W\| = \int_a^b |\phi(W(t))| \Delta t$*. If* f^{Δ} *is integrable on* $[a,b]$*, then for* $x \in [a,b]$

$$\left| f(x) - \frac{1}{\phi(d)} \int_a^b \left[(\phi \circ W)(t)\right]^{\Delta} f^{\sigma}(t) \Delta t \right| \le \frac{\left\|f^{\Delta}\right\|_{\infty;[a,b]}}{|\phi(d)|} \left[\|\phi \circ W\| + |\phi(d)|(b-x)\right], \tag{2.3}$$

Proof. By properties of modulus and Lemma 2.1

$$\left|f(x)-\frac{1}{\phi(d)}\int_a^b[(\phi\circ W)(t)]^{\Delta}f^{\sigma}(t)\,\Delta t\right|=\left|\frac{1}{\phi(d)}\int_a^b K_{W,\phi}(x,t)\,f^{\Delta}(t)\,\Delta t\right|$$
$$\leq\frac{1}{|\phi(d)|}\int_a^b|K_{W,\phi}(x,t)|\left|f^{\Delta}(t)\right|\Delta t$$
$$\leq\frac{1}{|\phi(d)|}\left\{\sup_{t\in(a,b)}\left|f^{\Delta}(t)\right|\right\}\left[\int_a^x|\phi(W(t))|\,\Delta t+\int_x^b|\phi(W(t))-\phi(d)|\,\Delta t\right]$$
$$\leq\frac{\left\|f^{\Delta}\right\|_{\infty;[a,b]}}{|\phi(d)|}\left[\int_a^x|\phi(W(t))|\,\Delta t+\int_x^b|\phi(W(t))|\,\Delta t+\int_x^b|\phi(d)|\,\Delta t\right]$$
$$=\frac{\left\|f^{\Delta}\right\|_{\infty;[a,b]}}{|\phi(d)|}\left[\int_a^b|\phi(W(t))|\,\Delta t+|\phi(d)|(b-x)\right]$$
$$=\frac{\left\|f^{\Delta}\right\|_{\infty;[a,b]}}{|\phi(d)|}\left[\|\phi\circ W\|+|\phi(d)|(b-x)\right],$$

which is as required. □

Corollary 2.3. *(Continuous case) Let $\mathbb{T}_1=\mathbb{R}$, then (2.3) becomes*

$$\left|f(x)-\frac{1}{\phi\left(\int_a^b w(t)\,dt\right)}\int_a^b w(t)\,\phi'\left(\int_a^t w(x)\,dx\right)f(t)\,dt\right|\tag{2.4}$$
$$\leq\frac{\|f'\|_{\infty;[a,b]}}{\left|\phi\left(\int_a^b w(t)\,dt\right)\right|}\left[\int_a^b\left|\phi\left(\int_a^t w(x)\,dx\right)\right|dt+\left|\phi\left(\int_a^b w(t)\,dt\right)\right|(b-x)\right].$$

For instance, let $w(t)=\frac{1}{b-a}$ and $\phi(t)=t$, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\|f'\|_{\infty;[a,b]}\left[\frac{b-a}{2}+(b-x)\right].\tag{2.5}$$

Corollary 2.4. *(Discrete case) Let $\mathbb{T}_1=\mathbb{Z}$, $a=0,b=n,c=0,d=m,x=i,t=k,s=l$ and $f(x)=x_i$, then (2.3) reduces to*

$$\left|x_i-\frac{1}{\phi(m)}\sum_{k=1}^{n}f(k)\Delta\phi\left(\sum_{p=1}^{k-1}w(p)\right)\right|\tag{2.6}$$
$$\leq\frac{1}{|\phi(m)|}\left[\sum_{k=1}^{n}\phi\left(\sum_{p=1}^{k-1}w(p)\right)+|\phi(m)|(n-i)\right]\max_{1\leq i\leq n-1}|\Delta x_i|.$$

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that f^{Δ} is integrable on $[a,b]$. If $q=\frac{p}{p-1}$ for $p>1$ and $\left\|f^{\Delta}\right\|_{q;[a,b]}=\left(\int_a^b\left|f^{\Delta}(t)\right|^q\Delta t\right)^{1/q}$,* *then for* $x \in [a,b]$

$$\left| f(x) - \frac{1}{\phi(d)} \int_a^b [(\phi \circ W)(t)]^{\Delta} f^{\sigma}(t) \Delta t \right| \leq \frac{\left\| f^{\Delta} \right\|_{q;[a,b]}}{|\phi(d)|} \times \left[\|\phi \circ W\|_{p;[a,x]} + \|\phi \circ W\|_{p;[x,b]} + |\phi(d)| (b-x)^{1/p} \right]. \tag{2.7}$$

Proof. By Hölder's inequality and Lemma 2.1

$$\begin{aligned}
&\left| f(x) - \frac{1}{\phi(d)} \int_a^b [(\phi \circ W)(t)]^{\Delta} f^{\sigma}(t) \Delta t \right| \\
&= \left| \frac{1}{\phi(d)} \int_a^b K_{W,\phi}(x,t) f^{\Delta}(t) \Delta t \right| \\
&\leq \frac{1}{|\phi(d)|} \int_a^b |K_{W,\phi}(x,t)| \left| f^{\Delta}(t) \right| \Delta t \\
&\leq \frac{1}{|\phi(d)|} \left(\int_a^b |K_{W,\phi}(x,t)|^p \Delta t \right)^{1/p} \left(\int_a^b \left| f^{\Delta}(t) \right|^q \Delta t \right)^{1/q}
\end{aligned}$$

But,

$$\begin{aligned}
&\left(\int_a^b |K_{W,\phi}(x,t)|^p \Delta t \right)^{1/p} \\
&= \left(\int_a^x |\phi(W(t))|^p \Delta t + \int_x^b |\phi(W(t)) - \phi(d)|^p \Delta t \right)^{1/p} \\
&\leq \left(\int_a^x |\phi(W(t))|^p \Delta t \right)^{1/p} + \left(\int_x^b |\phi(W(t)) - \phi(d)|^p \Delta t \right)^{1/p} \\
&\leq \left(\int_a^x |\phi(W(t))|^p \Delta t \right)^{1/p} + \left(\int_x^b |\phi(W(t))|^p \Delta t + \int_x^b |\phi(d)|^p \Delta t \right)^{1/p} \\
&\leq \left(\int_a^x |\phi(W(t))|^p \Delta t \right)^{1/p} + \left(\int_x^b |\phi(W(t))|^p \Delta t \right)^{1/p} + \left(\int_x^b |\phi(d)|^p \Delta t \right)^{1/p} \\
&= \|\phi \circ W\|_{p;[a,x]} + \|\phi \circ W\|_{p;[x,b]} + |\phi(d)| (b-x)^{1/p}
\end{aligned}$$

Combining the above obtained inequalities we get the required result. □

Theorem 2.3. *Let* $f, g : [a,b] \to \mathbb{R}$ *be differentiable functions such that* f^{Δ}, g^{Δ} *are integrable functions on* $[a,b]$*. If* ϕ, w *are as in Lemma 2.1, then*

$$|S(f,g;\phi,W)| \leq \frac{H^2(x)}{\phi^2(d)} \parallel f^{\Delta} \parallel_{\infty;[a,b]} \parallel g^{\Delta} \parallel_{\infty;[a,b]} \parallel \phi' \parallel^2_{\infty;[a,b]},$$

where

$$S(f,g;\phi,W)=f(x)g(x)-\frac{1}{\phi(d)}\left\{f(x)\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}g^{\sigma}(t)\Delta t\right.$$
$$\left.+\ g(x)\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}f^{\sigma}(t)\Delta t\right\}+\frac{1}{\phi^2(d)}\left[\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}f^{\sigma}(t)\Delta t\right]$$
$$\times\left[\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}g^{\sigma}(t)\Delta t\right], \tag{2.8}$$

and $H(x)=\| w \|_{\infty;[a,b]}(b-a)(1-a)+|d|(b-x)$.

Proof. By Lemma 2.1 for functions f and g

$$f(x)-\frac{1}{\phi(d)}\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}f^{\sigma}(t)\Delta t=\frac{1}{\phi(d)}\int_a^b K_{w,\phi}(x,t)f^{\Delta}(t)\Delta t. \tag{2.9}$$

$$g(x)-\frac{1}{\phi(d)}\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta}g^{\sigma}(t)\Delta t=\frac{1}{\phi(d)}\int_a^b K_{w,\phi}(x,t)g^{\Delta}(t)\Delta t. \tag{2.10}$$

Multiplying equations (2.9) and (2.10) simultaneously and using properties of modulus:

$$|S(f,g;\phi,W)|=\frac{1}{\phi^2(d)}\left|\left\{\int_a^b K_{w,\phi}(x,t)f^{\Delta}(t)\Delta t\right\}\left\{\int_a^b K_{w,\phi}(x,t)g^{\Delta}(t)\Delta t\right\}\right|$$
$$\le\frac{1}{\phi^2(d)}\max_{t\in[a,b]}\left|f^{\Delta}(t)\right|\max_{t\in[a,b]}\left|g^{\Delta}(t)\right|\left[\int_a^b\left|K_{w,\phi}(x,t)\right|\Delta t\right]^2$$
$$=\frac{\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}}{\phi^2(d)}\left[\int_a^x|\phi(W(t))|\,\Delta t+\int_x^b|\phi(W(t))-\phi(d)|\,\Delta t\right]^2$$

By Mean Value Theorem, there exist $\eta_t,\xi_t\in[c,d]$ such that

$$\phi(W(t))=\phi'(\xi_t)W(t)\quad\text{and}\quad\phi(W(t))-\phi(d)=\phi'(\eta_t)(W(t)-d).$$

$$|S(f,g;\phi,W)|$$
$$\le\frac{\|\phi'\|^2_{\infty;[a,b]}\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}}{\phi^2(d)}\left\{\int_a^x|W(t)|\Delta t+\int_x^b|W(t)|\Delta t+|d|\int_x^b\Delta t\right\}^2$$
$$\le\frac{\|\phi'\|^2_{\infty;[a,b]}\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}}{\phi^2(d)}\left\{\int_a^b\left(\int_a^t|w(x)|\Delta x\right)\Delta t+|d|(b-x)\right\}^2$$
$$\le\frac{\|\phi'\|^2_{\infty;[a,b]}\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}}{\phi^2(d)}\left[\|w\|_{\infty;[a,b]}(b-a)(1-a)+|d|(b-x)\right]^2.$$

This completes the proof of the theorem. □

Applications of theorem 2.3 to discrete and continuous cases give the following results.

Corollary 2.5. (*continuous case*) *Let $\mathbb{T}_1 = \mathbb{R}$. In this case delta integral is the usual Riemann integral from calculus, then*

$$\left| f(x)g(x) - \frac{1}{\phi(d)} \left\{ f(x) \int_a^b g(t)\ w(t)\ \phi'\left(W_1(t)\right)\ dt + g(x) \int_a^b f(t)\ w(t)\phi'\left(W_1(t)\right) dt \right\} \right.$$
$$\left. + \frac{1}{\phi^2(d)} \left[\int_a^b f(t)\ w(t)\ \phi'\left(W_1(t)\right)\ dt \right] \left[\int_a^b g(t)\ w(t)\ \phi'\left(W_1(t)\right)\ dt \right] \right|$$
$$\leq \frac{\| f' \|_{\infty;[a,b]} \| g' \|_{\infty;[a,b]} \| \phi' \|^2_{\infty;[a,b]}}{\phi^2(d)}\ H^2(x),$$

where

$$W_1(u) = \int_a^u w(l)\ dl. \tag{2.11}$$

Corollary 2.6. (*Discrete case*) *Let $\mathbb{T}_1 = \mathbb{Z}$, $a = 0, b = n, d = m, x = i$, $f(x) = x_i$ and $g(x) = y_i$, then*

$$\left| x_i y_i - \frac{1}{\phi(m)} \left\{ x_i \sum_{k=1}^{n} y_k\ \Delta\phi \left(\sum_{p=1}^{k-1} w(p) \right) + y_i \sum_{k=1}^{n} x_k\ \Delta\phi \left(\sum_{p=1}^{k-1} w(p) \right) \right\} \right.$$
$$\left. + \frac{1}{\phi^2(m)} \times \left[\sum_{k=1}^{n} y_k\ \Delta\phi \left(\sum_{p=1}^{k-1} w(p) \right) \right] \left[\sum_{k=1}^{n} x_k\ \Delta\phi \left(\sum_{p=1}^{k-1} w(p) \right) \right] \right|$$
$$\leq \frac{1}{\phi^2(m)} \max_{1\leq i\leq n-1} |\Delta x_i| \max_{1\leq i\leq n-1} |\Delta y_i| \left[\max_{1\leq i\leq n-1} |\phi'(i)| \right]^2$$
$$\times \left[n \max_{1\leq i\leq n-1} |w(i)| + |m|(n-i) \right]^2.$$

Corollary 2.7. (*Quantum calculus case*) *Let $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$ with $q_1 > 1$. Suppose $a = q_1^i, b = q_1^j, c = q_2^k, d = q_2^l$ for some $i < j$; $i < r$; $k < l$; $l, r \in \mathbb{N}_0$ and $q_2 > 1$, then*

$$\left| f\left(q_1^m\right) g\left(q_1^m\right) - \frac{1}{q_1^r (q_1 - 1) \phi\left(q_2^l\right)} \sum_{r=i}^{j-1} \left(f\left(q_1^m\right) g\left(q_1^{r+1}\right) + g\left(q_1^m\right) f\left(q_1^{r+1}\right) \right) \right.$$
$$\times \Delta\phi \left(\sum_{u=i}^{r-1} w\left(q_1^u\right) \right) - \frac{1}{q_1^{2r} (q_1 - 1)^2 \phi^2\left(q_2^l\right)} \left[\sum_{r=i}^{j-1} g\left(q_1^{r+1}\right) \Delta\phi \left(\sum_{u=i}^{r-1} w\left(q_1^u\right) \right) \right]$$
$$\left. \times \left[\sum_{r=i}^{j-1} f\left(q_1^{r+1}\right) \Delta\phi \left(\sum_{u=i}^{r-1} w\left(q_1^u\right) \right) \right] \right|$$
$$\leq \frac{1}{\phi^2\left(q_2^l\right)(q_1 - 1)^2} \sup_{i<r<j-1} \left| \frac{\Delta f\left(q_1^r\right)}{q_1^r} \right| \sup_{i<r<j-1} \left| \frac{\Delta g\left(q_1^r\right)}{q_1^r} \right| \left[\sup_{k<s<l-1} \left|\phi'\left(q_2^s\right)\right| \right]^2$$
$$\times \left[\max_{i<r<j-1} \left| w\left(q_1^r\right) \right| \left(\sum_{r=i}^{j-1} q_1^r \right) \left(1 - q_1^i\right) + q_2^l \sum_{p=m}^{j-1} q_1^p \right]^2.$$

3. Ostrowski type inequality for double integrals

Lemma 3.1. *Let $a, b \in \mathbb{T}_1$; $c, d \in \mathbb{T}_2$ and let $f : [a, b] \times [c, d] \to \mathbb{R}$ be such that the partial derivatives $\frac{\partial f(t,s)}{\Delta_1 t}$, $\frac{\partial f(t,s)}{\Delta_2 s}$ and $\frac{\partial^2 f(t,s)}{\Delta_2 s\ \Delta_1 t}$ exist and are continuous on $[a, b] \times [c, d]$, then*

(3.1)

$$f(x,y) = \frac{1}{\phi^2(d)}\left[\int_a^b\int_c^d K_{w,\phi}(x,t)K_{w,\phi}(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}\ \Delta_2 s\Delta_1 t + \int_a^b\int_c^d K_{w,\phi}(x,t)\right.$$
$$\times\left[(\phi\circ W)(s)\right]^{\Delta}\frac{\partial f(t,\sigma_2(s))}{\Delta_1 t}\Delta_2 s\Delta_1 t + \int_a^b\int_c^d\left[(\phi\circ W)(t)\right]^{\Delta}K_{w,\phi}(y,s)\frac{\partial f(\sigma_1(t),s)}{\Delta_2 s}$$
$$\left.\times\ \Delta_2 s\Delta_1 t + \int_a^b\int_c^d\left[(\phi\circ W)(t)\right]^{\Delta}\left[(\phi\circ W)(s)\right]^{\Delta} f(\sigma_1(t),\sigma_2(s))\Delta_2 s\Delta_1 t\right]$$

Proof. By Lemma 2.1 for partial delta map $f(., y)$ we have

$$f(x,y) = \frac{1}{\phi(d)}\int_a^b K_{w,\phi}(x,t)\frac{\partial f(t,y)}{\Delta_1 t}\Delta_1 t + \frac{1}{\phi(d)}\int_a^b\left[(\phi\circ W)(t)\right]^{\Delta} f(\sigma_1(t),y)\Delta_1 t, \tag{3.2}$$

for all $(x, y) \in [a, b]\times[c, d]$. Also application of the same lemma for the partial delta map $f(\sigma_1(t), .)$ yields:

$$f(\sigma_1(t),y) = \frac{1}{\phi(d)}\int_c^d K_{w,\phi}(y,s)\frac{\partial f(\sigma_1(t),s)}{\Delta_2 s}\Delta_2 s + \frac{1}{\phi(d)}\int_c^d\left[(\phi\circ W)(s)\right]^{\Delta} f(\sigma_1(t),\sigma_2(s))\Delta_2 s. \tag{3.3}$$

Similarly for the partial delta map $\frac{\partial f(t,.)}{\Delta_1 t}$, Lemma 2.1 provides:

$$\frac{\partial f(t,y)}{\Delta_1 t} = \frac{1}{\phi(d)}\int_c^d K_{w,\phi}(y,s)\frac{\partial^2 f(t,s)}{\Delta_2 s\Delta_1 t}\Delta_2 s + \frac{1}{\phi(d)}\int_c^d\left[(\phi\circ W)(s)\right]^{\Delta}\frac{\partial f(t,\sigma_2(s))}{\Delta_1 t}\Delta_2 s. \tag{3.4}$$

From equations (3.2)-(3.4) we obtain (3.1). □

Application of Lemma 3.1 to different time scales gives some new results.

Corollary 3.1. (*Continuous case*) *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$. In this case delta integral is the Riemann integral from calculus, then*

$$f(x,y) = \frac{1}{\phi^2(d)}\left[\int_a^b\int_c^d K_{w,\phi}(x,t)K_{w,\phi}(y,s)\frac{\partial^2 f(t,s)}{\partial s\ \partial t}\ ds\ dt + \int_a^b\int_c^d K_{w,\phi}(x,t)\right.$$
$$\times\ w(s)\phi'\left(W_1(s)\right)\frac{\partial f(t,s)}{\partial t}\ ds\ dt + \int_a^b\int_c^d K_{w,\phi}(y,s)\ w(t)\phi'\left(W_1(t)\right)\frac{\partial f(t,s)}{\partial s}\ ds\ dt$$
$$\left. + \int_a^b\int_c^d w(t)\ w(s)\phi'\left(W_1(t)\right)\ \phi'\left(W_1(s)\right) f(t,s)\ ds\ dt\right]$$

Corollary 3.2. (*Discrete case*) *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, $a = 0, b = n, d = m, x = i, y = j, t = k, s = l$ and $f(p,q) = x_p y_q$, then*

$$f(i,j) = \frac{1}{\phi^2(m)} \left[\sum_{k=1}^{n} \sum_{l=1}^{m} K_{w,\phi}(i,k)\ K_{w,\phi}(j,l)\ \Delta x_k\ \Delta y_l + \sum_{k=1}^{n} \sum_{l=1}^{m} K_{w,\phi}(i,k)\ w(l) \right.$$
$$\times \Delta\phi \left(\sum_{q=1}^{l-1} w(q) \right) \Delta x_k + \sum_{k=1}^{n} \sum_{l=1}^{m} K_{w,\phi}(j,l) \Delta\phi \left(\sum_{p=1}^{k-1} w(p) \right) \Delta y_l$$
$$\left. + \sum_{k=1}^{n} \sum_{l=1}^{m} w(k)\ w(l)\ \Delta\phi \left(\sum_{q=1}^{l-1} w(q) \right) \Delta\phi \left(\sum_{p=1}^{k-1} w(p) \right) \right],$$

where, $K_{w,\phi}$ is given by (2.2).

Corollary 3.3. (*Quantum calculus case*) *Let $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$ and $\mathbb{T}_2 = q_2^{\mathbb{N}_0}$ with $q_1 > 1$ and $q_2 > 1$. Suppose $a = q_1^i, b = q_1^j, c = q_2^k, d = q_2^l$ for some $i < j$; $k < l$, then*

$$f\left(q_1^m, q_2^n\right) = \frac{1}{\phi^2\left(q_2^l\right)} \left[\sum_{r=i}^{j-1} \sum_{s=k}^{l-1} K_{w,\phi}\left(q_1^m, q_1^r\right) K_{w,\phi}\left(q_2^n, q_2^s\right) \left[\Delta_1 f\left(q_1^r, q_2^{s+1}\right) - \Delta_1 f\left(q_1^r, q_2^s\right)\right] \right.$$
$$+ \sum_{r=i}^{j-1} \sum_{s=k}^{l-1} K_{w,\phi}\left(q_1^m, q_1^r\right) \Delta\phi \left(\sum_{v=j}^{s-1} w\left(q_2^v\right) \right) \frac{\Delta_1 f\left(q_1^r, q_2^{s+1}\right)}{q_1^r q_2^s (q_1 - 1)(q_2 - 1)} + \sum_{r=i}^{j-1} \sum_{s=k}^{l-1} K_{w,\phi}\left(q_2^n, q_2^s\right)$$
$$\times \Delta\phi \left(\sum_{u=i}^{r-1} w\left(q_1^u\right) \right) \frac{\Delta_2 f\left(q_1^{r+1}, q_2^s\right)}{q_1^r q_2^s (q_1 - 1)(q_2 - 1)} + \sum_{r=i}^{j-1} \sum_{s=k}^{l-1} \Delta\phi \left(\sum_{v=j}^{s-1} w\left(q_2^v\right) \right) \Delta\phi \left(\sum_{u=i}^{r-1} w\left(q_1^u\right) \right)$$
$$\left. \times \frac{f\left(q_1^{r+1}, q_2^{s+1}\right)}{q_1^r q_2^s (q_1 - 1)(q_2 - 1)} \right],$$

where, Δ_1 is the forward difference operator with respect to first component and Δ_2 is with respect to second component.

Theorem 3.1. *Let the conditions of Lemma 3.1 be satisfied, then*

$$\left| f(x,y) - \frac{1}{\phi^2(d)} \int_a^b \int_c^d \left[(\phi \circ W)(t)\right]^\Delta \left[(\phi \circ W)(s)\right]^\Delta f(\sigma_1(t), \sigma_2(s)) \Delta_2 s \Delta_1 t \right| \tag{3.5}$$
$$\leq (M_1 + M_2 + M_3) \|w\|_{\infty;[a,b]} \|w\|_{\infty;[c,d]} \left[\int_a^b \Psi(t) \Delta_1 t \right] \left[\int_c^d \Psi(s) \Delta_2 s \right]$$
$$+ (M_1 + M_3)|\phi(d)|(d-y)\|w\|_{\infty;[a,b]} \left[\int_a^b \Psi(t) \Delta_1 t \right] + (M_2 + M_3)|\phi(d)|(b-x)$$
$$\times \|w\|_{\infty;[c,d]} \left[\int_c^d \Psi(s) \Delta_2 s \right] + M_3 \phi^2(d)(b-x)(d-y),$$

for all $(x,y) \in [a,b] \times [c,d]$, where

$$M_1 = \sup_{s \in [c,d]} \left| \frac{\partial f(t,s)}{\Delta_2 s} \right|, \quad M_2 = \sup_{t \in [a,b]} \left| \frac{\partial f(t,s)}{\Delta_1 t} \right| \quad \text{and} \quad M_3 = \sup_{(t,s) \in [a,b] \times [c,d]} \left| \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \right|$$

and

$$\Psi(u) = \int_0^1 \phi'(W(u) + h\mu(u)\ w(u))\ dh.$$

Proof. By properties of modulus and Lemma 3.1

$$\left| f(x,y) - \frac{1}{\phi^2(d)} \int_a^b \int_c^d \left[(\phi \circ W)(t)\right]^{\Delta} \left[(\phi \circ W)(s)\right]^{\Delta} f(\sigma_1(t), \sigma_2(s)) \Delta_2 s \Delta_1 t \right|$$

$$= \frac{1}{\phi^2(d)} \left| \int_a^b \int_c^d K_{w,\phi}(x,t) K_{w,\phi}(y,s) \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \; \Delta_2 s \Delta_1 t + \int_a^b \int_c^d K_{w,\phi}(x,t) \left[(\phi \circ W)(s) \right]^{\Delta} \right.$$

$$\left. \times \frac{\partial f(t, \sigma_2(s))}{\Delta_1 t} \Delta_2 s \Delta_1 t + \int_a^b \int_c^d \left[(\phi \circ W)(t) \right]^{\Delta} K_{w,\phi}(y,s) \frac{\partial f(\sigma_1(t), s)}{\Delta_2 s} \Delta_2 s \Delta_1 t \right|$$

$$\leq \frac{1}{\phi^2(d)} \left\{ \int_a^b \int_c^d |K_{w,\phi}(x,t)| \; |K_{w,\phi}(y,s)| \; \left| \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \right| \Delta_2 s \Delta_1 t + \int_a^b \int_c^d |K_{w,\phi}(x,t)| \right.$$

$$\times \left| \left[(\phi \circ W)(s) \right]^{\Delta} \right| \; \left| \frac{\partial f(t, \sigma_2(s))}{\Delta_1 t} \right| \Delta_2 s \Delta_1 t + \int_a^b \int_c^d \left| \left[(\phi \circ W)(t) \right]^{\Delta} \right|$$

$$\left. \times \; |K_{w,\phi}(y,s)| \; \left| \frac{\partial f(\sigma_1(t), s)}{\Delta_2 s} \right| \Delta_2 s \Delta_1 t \right\}$$

$$\leq \frac{1}{\phi^2(d)} \left\{ \sup_{(t,s) \in [a,b] \times [c,d]} \left| \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \right| \int_a^b \int_c^d |K_{w,\phi}(x,t)| \; |K_{w,\phi}(y,s)| \, \Delta_2 s \Delta_1 t \right.$$

$$+ \sup_{t \in [a,b]} \left| \frac{\partial f(t,s)}{\Delta_1 t} \right| \int_a^b \int_c^d |K_{w,\phi}(x,t)| \left| \left[(\phi \circ W)(s) \right]^{\Delta} \right| \Delta_2 s \Delta_1 t$$

$$\left. + \sup_{s \in [c,d]} \left| \frac{\partial f(t,s)}{\Delta_2 s} \right| \int_a^b \int_c^d \left| \left[(\phi \circ W)(t) \right]^{\Delta} \right| |K_{w,\phi}(y,s)| \, \Delta_2 s \Delta_1 t \right\}$$

$$= \frac{1}{\phi^2(d)} \left[M_3 \left\{ \int_a^b |K_{w,\phi}(x,t)| \;\; \Delta_1 t \right\} \left\{ \int_c^d \;\; |K_{w,\phi}(y,s)| \, \Delta_2 s \right\} + M_2 \left\{ \int_a^b |K_{w,\phi}(x,t)| \, \Delta_1 t \right\} \right.$$

$$\left. \times \left\{ \int_c^d \left| \left[(\phi \circ W)(s) \right]^{\Delta} \right| \Delta_2 s \right\} + M_1 \left\{ \int_a^b \left| \left[(\phi \circ W)(t) \right]^{\Delta} \right| \Delta_1 t \right\} \left\{ \int_c^d |K_{w,\phi}(y,s)| \, \Delta_2 s \right\} \right]$$

$$\leq \frac{1}{\phi^2(d)} \left[M_3 \left\{ \int_a^b |w(t)| \; \left| \int_0^1 \phi'(W(t) + h \; \mu(t) \; w(t)) dh \right| \Delta_1 t + |\phi(d)| (b - x) \right\} \right.$$

$$\times \left\{ \int_c^d |w(s)| \; \left| \int_0^1 \phi'(W(s) + h \; \mu(s) \; w(s)) dh \right| \Delta_2 s + |\phi(d)| (d - y) \right\}$$

$$+ \; M_2 \left\{ \int_a^b |w(t)| \; \left| \int_0^1 \phi'(W(t) + h \; \mu(t) \; w(t)) dh \right| \Delta_1 t + |\phi(d)| (b - x) \right\}$$

$$\times \left\{ \int_c^d |w(s)| \; \left| \int_0^1 \phi'(W(s) + h \; \mu(s) \; w(s)) dh \right| \Delta_2 s \right\}$$

$$+ \; M_1 \left\{ \int_a^b |w(t)| \; \left| \int_0^1 \phi'(W(t) + h \; \mu(t) \; w(t)) dh \right| \Delta_1 t \right\}$$

$$\left. \times \left\{ \int_c^d |w(s)| \; \left| \int_0^1 \phi'(W(s) + h \; \mu(s) \; w(s)) dh \right| \Delta_2 s + |\phi(d)| (d - y) \right\} \right]$$

$$\leq \frac{1}{\phi^2(d)} \left[M_3 \left\{ \sup_{t \in [a,b]} |w(t)| \int_a^b \; \left| \int_0^1 \phi'(W(t) + h \; \mu(t) \; w(t)) dh \right| \Delta_1 t + |\phi(d)| (b - x) \right\} \right.$$

$$\times \left\{ \sup_{s \in [c,d]} |w(s)| \int_c^d \left| \int_0^1 \phi'(W(s) + h \; \mu(s) \; w(s)) dh \right| \Delta_2 s + |\phi(d)| (d - y) \right\}$$

$$
\begin{aligned}
&+ M_2 \left\{ \sup_{t\in[a,b]} |w(t)| \int_a^b \left| \int_0^1 \phi'(W(t)+h\ \mu(t)\ w(t))dh \right| \Delta_1 t + |\phi(d)|(b-x) \right\} \\
&\times \left\{ \int_c^d |w(s)| \ \left| \int_0^1 \phi'(W(s)+h\ \mu(s)\ w(s))dh \right| \Delta_2 s \right\} \\
&+ M_1 \left\{ \sup_{t\in[a,b]} |w(t)| \int_a^b \left| \int_0^1 \phi'(W(t)+h\ \mu(t)\ w(t))dh \right| \Delta_1 t \right\} \\
&\times \left\{ \sup_{s\in[c,d]} |w(s)| \int_c^d \left| \int_0^1 \phi'(W(s)+h\ \mu(s)\ w(s))dh \right| \Delta_2 s + |\phi(d)|(d-y) \right\} \Bigg] \\
= \frac{1}{\phi^2(d)} &\Bigg[M_3 \left\{ \|w\|_{\infty;[a,b]} \int_a^b |\Psi(t)|\Delta_1 t + |\phi(d)|(b-x) \right\} \\
&\times \left\{ \|w\|_{\infty;[c,d]} \int_c^d |\Psi(s)|\Delta_2 s + |\phi(d)|(d-y) \right\} \\
&+ M_2 \left\{ \|w\|_{\infty;[a,b]} \int_a^b |\Psi(t)|\Delta_1 t + |\phi(d)|(b-x) \right\} \left\{ \|w\|_{\infty;[c,d]} \int_c^d |\Psi(s)|\Delta_2 s \right\} \\
&+ M_1 \left\{ \|w\|_{\infty;[a,b]} \int_a^b |\Psi(t)|\Delta_1 t \right\} \left\{ \|w\|_{\infty;[c,d]} \int_c^d |\Psi(s)|\Delta_2 s + |\phi(d)|(d-y) \right\} \Bigg],
\end{aligned}
$$

which is equivalent to (3.5). □

The followings are the discrete and continuous cases of the Theorem 3.1.

Corollary 3.4. (*continuous case*) *Let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$. In this case delta integral is the Riemann integral from calculus, then*

$$
\begin{aligned}
&\left| f(x,y) - \frac{1}{\phi^2(d)} \int_a^b \int_c^d w(t)\ w(s)\phi'(W_1(t))\phi'(W_1(s))\ f(t,s)\ ds\ dt \right| \\
&\le (M_1+M_2+M_3)\|w\|_{\infty;[a,b]}\|w\|_{\infty;[c,d]} \left[\int_a^b \phi'(W_1(t))\ dt \right] \left[\int_c^d \phi'(W_1(s))\ ds \right] \\
&\quad + (M_1+M_3)|\phi(d)|(d-y)\|w\|_{\infty;[a,b]} \left[\int_a^b \phi'(W_1(t))\ dt \right] + (M_2+M_3)|\phi(d)|(b-x) \\
&\quad \times \|w\|_{\infty;[c,d]} \left[\int_c^d \phi'(W_1(s))\ ds \right] + M_3\ \phi^2(d)(b-x)(d-y),
\end{aligned}
$$

where, $W_1(u)$ is given by (2.11).

Corollary 3.5. *(Discrete case) Let* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$, $a = 0 = c, b = n, d = m, x = i, y = j, t = k, s = l$ *and* $f(p,q) = x_p\ y_q$, *then*

$$\begin{aligned}
&\left| x_i\ y_j - \frac{1}{\phi^2(m)} \sum_{k=1}^{n} \sum_{l=1}^{m} \Delta\phi\left(\sum_{q=1}^{l-1}\ w(q)\right)\ \Delta\phi\left(\sum_{p=1}^{k-1}\ w(p)\right) \times x_k\ y_l \right| \\
&\le (M_1 + M_2 + M_3) \\
&\quad \times \max_{1\le i\le n-1} |w(i)| \times \max_{1\le j\le m-1} |w(j)| \sum_{k=1}^{n}\sum_{l=1}^{m} \frac{1}{w(k)\ w(l)} \Delta\phi\left(\sum_{p=1}^{k-1}\ w(p)\right) \Delta\phi\left(\sum_{q=1}^{l-1}\ w(q)\right) \\
&\quad + (M_1 + M_3)|\phi(m)|(m-j) \max_{1\le i\le n-1} |w(i)| \sum_{k=1}^{n} \frac{1}{w(k)} \Delta\phi\left(\sum_{p=1}^{k-1}\ w(p)\right) + (M_2 + M_3)|\phi(m)| \\
&\quad \times (n-i) \max_{1\le j\le m-1} |w(j)| \sum_{l=1}^{m} \frac{1}{w(l)} \Delta\phi\left(\sum_{q=1}^{l-1}\ w(q)\right) + M_3\phi^2(m)(n-i)(m-j),
\end{aligned}$$

where

$$M_1 = \max_{1\le l\le m-1} |\Delta y_l|\,,\ M_2 = \max_{1\le k\le n-1} |\Delta x_k|\ \text{ and } M_3 = \max_{1\le l\le m-1; 1\le k\le n-1} |\Delta x_k \Delta y_l|$$

4. Čebyšev type inequalities

Theorem 4.1. *Let the conditions of Theorem 2.3 be satisfied, then*

$$|T(f,g;\phi,W)| \le \frac{\| f^{\Delta} \|_\infty \| g^{\Delta} \|_\infty \| w \|_\infty}{\phi^2(d)} \int_a^b |\Psi(x)|\ G^2(x)\ \Delta x, \tag{4.1}$$

where

$$T(f,g;\phi,W) = \int_a^b S(f,g;\phi,W)\ [(\phi\circ W)(x)]^{\Delta}\ \Delta x,$$

$$\text{and }\ G(x) = \|w\|_{\infty;[a,b]} \int_a^b |\Psi(t)|\Delta_1 t + |\phi(d)|(b-x).$$

Proof. By Lemma 2.1, the following identities hold for all $x \in [a,b]$

$$f(x) - \frac{1}{\phi(d)} \int_a^b [(\phi\circ W)(t)]^{\Delta}\ f^{\sigma}(t)\Delta t = \frac{1}{\phi(d)} \int_a^b K_{w,\phi}(x,t) f^{\Delta}(t)\Delta t. \tag{4.2}$$

$$g(x) - \frac{1}{\phi(d)} \int_a^b [(\phi\circ W)(t)]^{\Delta}\ g^{\sigma}(t)\Delta t = \frac{1}{\phi(d)} \int_a^b K_{w,\phi}(x,t) g^{\Delta}(t)\Delta t. \tag{4.3}$$

Multiplying both sides of equations (4.2) and (4.3) we get (2.8), then multiplying by $[(\phi\circ W)(x)]^{\Delta}$ and integrating over $x\in[a,b]$

$$\begin{aligned}
&\int_a^b f(x)g(x)\,[(\phi\circ W)(x)]^{\Delta}\,\Delta x\\
&\quad-\frac{1}{\phi(d)}\left\{\int_a^b f(x)[(\phi\circ W)(x)]^{\Delta}\Delta x\right\}\left\{\int_a^b[(\phi\circ W)(t)]^{\Delta}\ \times g^{\sigma}(t)\Delta t\right\}\\
&\quad-\frac{1}{\phi(d)}\left\{\int_a^b g(x)[(\phi\circ W)(x)]^{\Delta}\Delta x\right\}\left\{\int_a^b[(\phi\circ W)(t)]^{\Delta}f^{\sigma}(t)\Delta t\right\}\\
&\quad+\frac{1}{\phi^2(d)}\left[\int_a^b[(\phi\circ W)(t)]^{\Delta}f^{\sigma}(t)\Delta t\right]\left[\int_a^b[(\phi\circ W)(t)]^{\Delta}g^{\sigma}(t)\Delta t\right]\left[\int_a^b[(\phi\circ W)(x)]^{\Delta}\Delta x\right]\\
&=\frac{1}{\phi^2(d)}\int_a^b[(\phi\circ W)(x)]^{\Delta}\left[\int_a^b K_{w,\phi}(x,t)g^{\Delta}(t)\Delta t\right]\left[\int_a^b K_{w,\phi}(x,t)f^{\Delta}(t)\Delta t\right]\Delta x,
\end{aligned}$$

that is,

$$\begin{aligned}
T(f,g;\phi,W)&=\int_a^b S(f,g;\phi,W)[(\phi\circ W)(x)]^{\Delta}\Delta x\\
&=\frac{1}{\phi^2(d)}\int_a^b[(\phi\circ W)(x)]^{\Delta}\left[\int_a^b K_{w,\phi}(x,t)g^{\Delta}(t)\Delta t\right]\left[\int_a^b K_{w,\phi}(x,t)f^{\Delta}(t)\Delta t\right]\Delta x.
\end{aligned}$$

By using properties of modulus

$$\begin{aligned}
&|T(f,g;\phi,W)|\\
&\le\frac{\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}}{\phi^2(d)}\int_a^b\left|[(\phi\circ W)(x)]^{\Delta}\right|\left[\int_a^b|K_{w,\phi}(x,t)|\,\Delta t\right]^2\Delta x\\
&\le\frac{\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}}{\phi^2(d)}\\
&\qquad\times\int_a^b|\Psi(x)|\,|w(x)|\left[\|w\|_{\infty;[a,b]}\int_a^b|\Psi(t)|\ \Delta t+|\phi(d)|(b-x)\right]^2\Delta x\\
&\le\frac{\|f^{\Delta}\|_{\infty;[a,b]}\|g^{\Delta}\|_{\infty;[a,b]}\|w\|_{\infty;[a,b]}}{\phi^2(d)}\\
&\qquad\times\int_a^b|\Psi(x)|\left[\|w\|_{\infty;[a,b]}\int_a^b|\Psi(t)|\ \Delta t+|\phi(d)|(b-x)\right]^2\Delta x
\end{aligned}$$

□

Corollary 4.1. (*continuous case*) *Let $\mathbb{T}_1 = \mathbb{R}$. In this case delta integral is the Riemann integral from calculus, then*

$$\begin{aligned}&\Bigg| \int_a^b f(x)\ g(x)\ w(x)\ \phi'(W(x))\ dx - \frac{1}{\phi(d)} \left(\int_a^b f(x)\ w(x)\ \phi'(W(x))\ dx \right) \\ &\quad \times \left(\int_a^b\ w(t)\ \phi'(W(t))\ g(t)\ dt \right) - \frac{1}{\phi(d)} \left(\int_a^b g(x)\ w(x)\ \phi'(W(x))\ dx \right) \\ &\quad \times \left(\int_a^b\ w(t)\ \phi'(W(t))\ f(t)\ dt \right) + \frac{1}{\phi^2(d)} \left(\int_a^b g(t)\ w(t)\ \phi'(W(t))\ dt \right) \\ &\quad \times \left(\int_a^b\ w(t)\ \phi'(W(t))\ f(t)\ dt \right) \left(\int_a^b w(x)\ \phi'(W(x))\ dx \right) \Bigg| \\ &\leq \frac{\| f' \|_{\infty;[a,b]} \| g' \|_{\infty;[a,b]} \| w \|_{\infty;[a,b]}}{\phi^2(d)} \int_a^b |\phi'(W(x))|\ G^2(x)\ \Delta x,\end{aligned}$$

where

$$G(x) = \|w\|_{\infty;[a,b]} \int_a^b |\phi'(W(t))|\ dt + |\phi(d)|(b-x).$$

Remark 4.1. For Theorems 3.1 and 4.1 the applications for quantum calculus can also be given.

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