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# Undetermined Coefficients Method for Sequential Fractional Differential Equations

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## Abstract

The undetermined coefficients method is presented for nonhomogeneous sequential fractional differential equations involving Caputo fractional derivative of order $n\alpha$ where $n-1<n\alpha\leq n$ and $n\in\mathbb{N}$. By employing the proposed method, a particular solution of the considered equation is obtained. Some details about estimating the particular solution required to apply this method are explained. This method is shown to be particularly effective for nonhomogeneous fractional differential equations when the fractional differential equations involve some specific right-hand side functions.

**Keywords**

*Fractional Differential Equation*
*Undetermined Coefficients*
*Caputo Derivative*
*Particular Solution*

## 1. Introduction

Fractional calculus has attracted the attention of researchers in recent decades, as modeling using fractional differential equations is convenient for estimating the evolutionary tendency of the systems affected by past memories.

This paper deals with the following nonhomogeneous sequential fractional differential equation

$$\sum_{i=0}^{n} b_i D^{i\alpha} u(t) = f(t), \tag{1}$$

where $n-1<n\alpha\leq n$, $b_i\in\mathbb{R}$, $n\in\mathbb{N}$ , and $f(t)\in C^{\infty}(0,t)$. $D^{n\alpha}$ is called sequential fractional derivative operator and it is defined as follows

$$D^{n\alpha}u(t)=D^{(n-1)\alpha}({}^{c}D^{\alpha}u(t)),$$

where ${}^{c}D^{\alpha}$ is the Caputo derivatives.

The sequential fractional derivative equation is first investigated in the monograph [1]. Its generalized version is then studied in [2]. Recently, many studies have been considered on the solutions of sequential fractional differential equations. These studies include different types of fractional derivatives, initial values, and boundary values. The uniqueness and existence of the solution of the periodic boundary value and the initial value problem for Riemann-Liouville (R-L) sequential fractional differential equations are considered in [3,4]. Similarly, solutions of impulsive R-L sequential fractional differential equations are studied in [5] and some specific solutions of sequential fractional differential equations with R-L derivatives are investigated in [6]. In [7] and [8], the uniqueness and existence of the solution are proved for sequential fractional differential equations involving the Hadamard derivative and Caputo–Hadamard derivative, respectively. Some existing results are obtained for Caputo-type sequential fractional differential equations with three-point, semi-periodic non-local, and mixed-type boundary conditions [9-11]. The uniqueness and Ulam-stability of solutions for specific sequential fractional differential equations involving Caputo derivative are studied in [12]. Additionally, in [13], the solution of Equation (1) is considered for $n=2$ and $f(t)=0$. We refer the reader to the papers [14-20] for some recent work on this subject.

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The purpose of this article is to examine the method of uncertain coefficients to obtain particular solutions to Equation (1) with some specific functions $f(t)$. The method is a generalization of the well-known method of nonhomogeneous linear ordinary differential equations for sequential fractional differential equations with Caputo derivative. A similar method has been used for nonhomogeneous differential equations involving conformable fractional derivatives in [21], whereas there is no study in the literature for the Caputo derivative.

The layout of this article is as follows. In Section 2, some definitions and applications of fractional calculus are presented. The method for the solution to the related problem is proposed in Section 3, and an illustrative example is given to present the application of the method in Section 4. Finally, some conclusions are described in the last section.

## 2. Preliminaries

**Definition 1:** *The Caputo fractional derivative of order $\alpha$ is defined as [22]:*

$$ {}^{c}D^{\alpha}u(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t (t-s)^{n-\alpha-1}\left(\frac{d^n u(s)}{ds^n}\right)ds, & n-1<\alpha<n, \\ \dfrac{d^n u(s)}{ds^n}, & \alpha = n. \end{cases} $$

For $0<\alpha<1$, the Caputo fractional derivative is formed as:

$$ {}^{c}D^{\alpha}u(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(s)}{(t-s)^{\alpha}}ds, & 0<\alpha<1, \\ u'(t), & \alpha = 1. \end{cases} $$

**Definition 2:** *The sequential Caputo fractional derivative operator of order $n-1<n\alpha\le n$ is defined as [13]:*

$$ D^{n\alpha}u(t) = D^{(n-1)\alpha}({}^{c}D^{\alpha}u(t)). $$

**Definition 3:** *The Mittag-Leffler function is defined as:*

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, $$

*where $Re(\alpha)>0$ [23].*

**Definition 4:** *The Mittag-Leffler function with 2-parameters is defined as:*

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, $$

*where $\alpha,\beta\in\mathbb{C}$ and $Re(\alpha)>0$ [23].*

**Definition 5:** *The Mittag-Leffler function with 3-parameters is defined as:*

$$ E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+k)z^k}{k!\Gamma(\alpha k+\beta)}, $$

*where $\alpha,\beta$ and $\gamma\in\mathbb{C}$ and $Re(\alpha)>0$ [23].*

Since $(\gamma)_k := \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)} = \gamma(\gamma+1)\cdots(\gamma+k-1)$ for $\gamma\in\mathbb{N}$, Mittag-Leffler with 3-parameters is rewritten as

$$ E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty}\frac{(\gamma)_k z^k}{k!\Gamma(\alpha k+\beta)}, $$

and holds the following property

$$ E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\alpha\gamma}\left(E_{\alpha,\beta-1}^{\gamma-1}(z) + (1-\beta+\alpha\gamma)E_{\alpha,\beta}^{\gamma-1}(z)\right), $$

which yields to reduce in the third parameter [24]. By repeatedly applying the property, the third parameter can be reduced to one. That is, a relation can be established between the Mittag-Leffler function with 3-parameters and the Mittag-Leffler function with 2-parameters when $\gamma\in\mathbb{N}$. We refer the reader to the paper [25] for more detail on the Mittag-Leffler function with 3-parameters.

**Definition 6:** *The parametrized form of $f(t)\in C^{\infty}(0,t)$ is defined as follows:*

$$ f_{\alpha}(t) = \sum_{k=0}^{\infty}\left.\frac{d^k f(t)}{dt^k}\right|_{t=0}\frac{t^k}{\Gamma(\alpha k+1)}, \tag{2} $$

*where $\alpha>0$.*

**Definition 7:** *The 2-parametrized form of $f(t)\in C^{\infty}(0,t)$ is defined as follows:*

$$ f_{\alpha,\beta}(t) = \sum_{k=0}^{\infty}\left.\frac{d^k f(t)}{dt^k}\right|_{t=0}\frac{t^k}{\Gamma(\alpha k+\beta)}, \tag{3} $$

*where $\alpha,\beta>0$.*

**Remark 1:** *Parametrized and 2-parametrized forms of function $f(t)=e^t$ are Mittag-Leffler function $E_{\alpha}(t)$ and Mittag-Leffler function with two parameters $E_{\alpha,\beta}(t)$,respectively. Moreover, the following notations are used for parametrized and 2-parametrized forms of* *functions $f(t)=sin(t)$ and $f(t)=cos(t)$*

$$\sin_{\alpha}(t)=\sum_{k=0}^{\infty}(-1)^{k}\frac{t^{2k+1}}{\Gamma((2k+1)\alpha+1)},$$
$$\sin_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}(-1)^{k}\frac{t^{2k+1}}{\Gamma((2k+1)\alpha+\beta)},$$
$$\cos_{\alpha}(t)=\sum_{k=0}^{\infty}(-1)^{k}\frac{t^{2k}}{\Gamma(2k\alpha+1)},$$
$$\cos_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}(-1)^{k}\frac{t^{2k}}{\Gamma(2k\alpha+\beta)}.$$

**Theorem 1:** *Let $f(t)\in C^{\infty}(0,x)$, $\frac{\alpha}{\rho}=m\in\mathbb{N}$ and $r\in\mathbb{C}$. If $\frac{d^k}{dt^k}f(t)\Big|_{t=0}=\frac{d^{mk}}{dt^{mk}}f(t)\Big|_{t=0}$ for all $k\in\mathbb{N}$, then*

$$f_{\alpha,\beta}(rt^{\alpha})=\frac{1}{m}\sum_{i=1}^{m}f_{\rho,\beta}(s_it^{\rho}),$$

*where $s_i$ are roots of $s^m=r$.*

***Proof.*** *By definition of the 2-parametrized form of $f(t)$, we have*

$$\sum_{i=1}^{m}f_{\rho,\beta}(s_it^{\rho})=\sum_{k=0}^{\infty}\sum_{i=1}^{m}c_k\frac{s_i^kt^{\rho k}}{\Gamma(\rho k+\beta)}, \tag{4}$$

*where $c_k=\frac{d^kf(t)}{dt^k}\Big|_{t=0}$. On the other hand, the roots $s_i$ of the equation $s^m=r$ satisfy following sum:*

$$\sum_{i=1}^{m}(s_i)^j=\begin{cases}mr^k & \text{if there exist } k\in\mathbb{N} \text{ such that } j=km,\\ 0 & \text{otherwise.}\end{cases} \tag{5}$$

*By plugging equation (5) into equation (4), we have*

$$\sum_{k=0}^{\infty}\sum_{i=1}^{m}c_k\frac{s_i^kt^{\rho k}}{\Gamma(\rho k+\beta)}=\sum_{k=0}^{\infty}c_k\frac{mr^kt^{\alpha k}}{\Gamma(\rho k+\beta)}=mf_{\alpha,\beta}(rt^{\alpha})$$

Theorem 1 shows that the first parameter of a 2-parametrized form of $f(t)$ can be changed under appropriate conditions. Moreover, the following equations are the direct result of theorem 1 for some specific functions $f(t)$.

$$E_{\alpha,\beta}(rt^{\alpha})=\frac{1}{m}\sum_{i=1}^{m}E_{\rho,\beta}(s_it^{\rho}), \tag{6}$$

$$\sin_{\alpha,\beta}(rt^{\alpha})=\begin{cases}\frac{1}{m}\sum_{i=1}^{m}\sin_{\rho,\beta}(s_it^{\rho}) & \text{if } m=4k+1, k\in\mathbb{N},\\ -\frac{1}{m}\sum_{i=1}^{m}\sin_{\rho,\beta}(s_it^{\rho}) & \text{if } m=4k+3, k\in\mathbb{N},\end{cases} \tag{7}$$

$$\sin_{\alpha,\beta}(rt^{\alpha})=$$

$$\frac{1}{m}\sum_{i=1}^{m}\cos_{\rho,\beta}(s_it^{\rho}) \quad \text{if } m=2k+1, k\in\mathbb{N}, \tag{8}$$

*where $\frac{\alpha}{\rho}=m\in\mathbb{N}$.*

## 3. Undetermined Coefficient Method

In this section, we introduce undetermined coefficient method to find a particular solution to the following nonhomogeneous sequential fractional differential equation for specific classes of right-hand side function

$$\sum_{i=0}^{n}b_iD^{i\alpha}u(t)=f(t), \tag{9}$$

where $n-1<n\alpha\le n$, $b_i\in\mathbb{R}$, and $f(t)\in C^{\infty}(0,t)$.

For $f(t)=0$, the solutions of homogeneous equation (9) are in the form of $E_{\alpha}(rt^{\alpha})$ where $r$ is the root of the corresponding characteristic equation

$$P(r)=\sum_{i=0}^{n}b_ir^i. \tag{10}$$

If the characteristic equation (10) has $k$ distinct roots $r_i$, a solution of equation (9) with $f(t)=0$ is as follows:

$$u_{h_1}=\sum_{i=1}^{k}c_iE_{\alpha}(r_it^{\alpha}),$$

where $c_i\in\mathbb{R}$. Moreover, in [26], the following solutions of the equation (9) with $f(t)=0$ are obtained if the characteristic equation (10) has $k$ coincident roots $r_0$

$$u_{h_2}=c_0E_{\alpha}(r_0t^{\alpha})+\sum_{i=1}^{k-1}c_i\frac{t^{i\alpha}}{i\alpha}E_{\alpha,i\alpha}^{i}(r_0t^{\alpha}).$$

**Theorem 2:** *Let $f(t)=\sum_{i=0}^{m}c_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}$ in equation (9) where $c_i\in\mathbb{R}$. If the corresponding characteristic equation has no root at $r=0$, then there are real constants $a_0,a_1,\cdots,a_m$ such that*

$$u_p=\sum_{i=0}^{m}a_i\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}, \tag{11}$$

*is a particular solution of equation (9).*

***Proof.*** *By plugging equation (11) into equation (9), we have the following linear algebraic system*

$$\begin{bmatrix}b_0 & b_1 & b_2 & \cdots & b_m\\ 0 & b_0 & b_1 & \cdots & b_{m-1}\\ 0 & 0 & b_0 & \cdots & b_{m-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & b_0\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\\ \vdots\\ a_m\end{bmatrix}=\begin{bmatrix}c_0\\ c_1\\ c_2\\ \vdots\\ c_m\end{bmatrix}. \tag{12}$$

*Here $b_0\neq 0$ since the characteristic equation has no* *root at $r = 0$. Thus, the linear algebraic system has a unique solution.*

**Remark 2:** *If the characteristic equation has $k$ coincident root at $r = 0$, the particular solution (11) is formed as follows*

$$u_p = \sum_{i=k}^{m+k} a_i \frac{t^{i\alpha}}{\Gamma(i\alpha+1)},$$

*and the linear algebraic system (12) is reobtained as follows*

$$\begin{bmatrix} b_k & b_{k+1} & b_{k+2} & \cdots & b_m & 0 & 0 & \cdots & 0 \\ 0 & b_k & b_{k+1} & \cdots & b_{m-1} & b_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_k & b_{k+1} & b_{k+2} & \cdots & b_m & 0 \\ 0 & \cdots & 0 & 0 & b_k & b_{k+1} & \cdots & b_{m-1} & b_m \\ 0 & \cdots & 0 & 0 & 0 & b_k & \cdots & b_{m-2} & b_{m-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & b_k & b_{k+1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_k \end{bmatrix} \begin{bmatrix} a_k \\ a_{k+1} \\ a_{k+2} \\ \vdots \\ a_{m+k} \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}.$$

**Remark 3:** *Let $q_0 < q_1 < \cdots < q_m$ be positive real numbers and $GCD(\alpha, q_0, q_1, \cdots, q_m) = \beta$. In Theorem 2, the right-hand side function is taken as $f(t) = P(t^\alpha)$, where $P(t)$ is a polynomial. However, more generally, if the right-hand side function is of the form $f(x) = \sum_{i=0}^{m} c_i t^{q_i}$, the particular solution of the equation is considered as follows*

$$u_p = \sum_{i=0}^{k} a_i \frac{t^{i\beta}}{\Gamma(i\beta+1)},$$

*where $q_m = k\beta$, $k \in \mathbb{N}$. In this case, the characteristic equation is established by using $E_\beta(rt^\beta)$*

**Theorem 3:** *Let $f(t) = E_q(ct^q)$ in equation (9) and $GCD(\alpha, q) = \beta$. If the characteristic equation formed using $E_\beta(rt^\beta)$ has no root at $r = s_i$ where $s_i$ are roots of $s^m = c$ for $\frac{q}{\beta} = m \in \mathbb{N}$, then there are real constants $a_0, a_1, \cdots, a_m$ such that*

$$u_p = \sum_{i=1}^{m} a_i E_\beta(s_i t^\beta), \tag{13}$$

*is a particular solution of equation (9).*

***Proof.*** *From equation (6), we have*

$$f(t) = E_q(ct^q) = \frac{1}{m}\sum_{i=1}^{m} E_\beta(s_i t^\beta).$$

*On the other hand, by plugging equation (13) into equation (9), we obtain the following equality for $i = 1,2,\cdots,m$,*

$$a_i\left(\sum_{k=0}^{n} b_k (s_i)^k\right) \sum_{i=1}^{m} E_\beta(s_i t^\beta) = \frac{1}{m}\sum_{i=1}^{m} E_\beta(s_i t^\beta),$$

*which gives*

$$a_i = \frac{1}{m\sum_{k=0}^{n} b_k (s_i)^k}.$$

*Since the characteristic equation has no root at $r = s_i$, $\sum_{k=0}^{n} b_k (s_i)^k \neq 0$.*

**Remark 4** *Let $s$ be a root of $s^m = c$ in Theorem 3. If the characteristic equation has $k$ coincident root at $r = s$, a particular solution (13) is formed as follows:*

$$u_p = \frac{t^{k\beta}}{k\beta} E^k_{\beta,k\beta}(st^\beta).$$

**Theorem 4** *Let $f(t) = sin_q(ct^q) + cos_q(ct^q)$ in equation (9), $GCD(\alpha, q) = \beta$ and $\frac{q}{\beta} = m$ be odd number. If the corresponding characteristic equation has no root at $r = s_i$ where $s_i$ are roots of $s^m = c$, then there are real constants $a_0, a_1, \cdots, a_m$ and $d_0, d_1, \cdots, d_m$ such that*

$$u_p = \sum_{i=1}^{m} a_i \sin_\beta(ct^\beta) + d_i \cos_\beta(ct^\beta), \tag{14}$$

*is a particular solution of equation (9).*

By the principle of superposition, if the right-hand side function $f(t)$ is a linear combination of the functions in Theorem 1-4, a particular solution of equation (9) is a linear combination of the particular solutions defined in the related theorems. Additionally, the general solution is $u_g = u_h + u_p$ where $u_h$ is the solution of homogeneous equation (9) with $f(t) = 0$, and $u_p$ is the particular solution of equation (9).

## 4. Illustrative Example

**Example:** Let us consider the general solution of the following equation

$$D^{\frac{6}{4}}u(t) + D^{\frac{3}{4}}u(t) - 2u(t) = e^t + t. \tag{15}$$

The characteristic equation $r^2 + r - 2 = 0$ of equation (15) is formed using $E_{\frac{3}{4}}(rt^{\frac{3}{4}})$, so $r = 1$ and $r = -2$ are 1-fold roots. Hence, the homogeneous solution is obtained as follows

$$u_h = c_1 E_{\frac{3}{4}}(t^{\frac{3}{4}}) + c_2 E_{\frac{3}{4}}(-2t^{\frac{3}{4}}), \tag{16}$$

where $c_1, c_2 \in \mathbb{R}$.

Since $GCD(\frac{3}{4},1)=\frac{1}{4}$, the characteristic equation is formed using $E_{\frac{1}{4}}(\lambda t^{\frac{1}{4}})$ as follows

$$\lambda^6+\lambda^3-2=0, \quad (17)$$

and the roots of the equation are

$$\lambda_1=1, \lambda_{2,3}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}, \lambda_4=-\sqrt[3]{2}, \lambda_{5,6}=2^{-\frac{2}{3}}(1\pm i\sqrt{3})$$

In order to determine the form of the particular solution, we use the following equality from equation (6)

$$\begin{aligned} e^t &= \frac{1}{4}\left[E_{\frac{1}{4}}(t^{\frac{1}{4}})+E_{\frac{1}{4}}(-t^{\frac{1}{4}})+E_{\frac{1}{4}}(it^{\frac{1}{4}})+E_{\frac{1}{4}}(-it^{\frac{1}{4}})\right] \\ &= \frac{1}{4}\left[E_{\frac{1}{4}}(t^{\frac{1}{4}})+E_{\frac{1}{4}}(-t^{\frac{1}{4}})+2\cos_{\frac{1}{4}}(t^{\frac{1}{4}})\right]. \end{aligned}$$

Therefore, the particular solution is considered as follows

$$\begin{aligned} u_p &= a_0+a_1\frac{t^{\frac{1}{4}}}{\Gamma(\frac{5}{4})}+a_2\frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}+a_3\frac{t^{\frac{3}{4}}}{\Gamma(\frac{7}{4})}+a_4t \\ &\quad +a_5\frac{t^{\frac{1}{4}}}{\frac{1}{4}}E_{\frac{1}{4},\frac{1}{4}}(t^{\frac{1}{4}})+a_6E_{\frac{1}{4}}(-t^{\frac{1}{4}}) \\ &\quad +a_7\cos_{\frac{1}{4}}(t^{\frac{1}{4}})+a_8\sin_{\frac{1}{4}}(t^{\frac{1}{4}}). \end{aligned}$$

The particular solution contains the term $\frac{t^{\frac{1}{4}}}{\frac{1}{4}}E_{\frac{1}{4},\frac{1}{4}}(t^{\frac{1}{4}})$ because the characteristic equation has a 1-fold root of $\lambda=1$. Substituting $u_p$ in equation (15), we have

$a_0=a_2=a_3=0,\ a_1=\frac{1}{4},\ a_4=-\frac{1}{2},\ a_5=\frac{1}{36},\ a_6=-\frac{1}{8},\ a_7=-\frac{3}{5},\ a_8=-\frac{1}{5}.$

Hence, the general solution of the equation (15) is obtained as follows

$$\begin{aligned} u_g &= c_1E_{\frac{3}{4}}(t^{\frac{3}{4}})+c_2E_{\frac{3}{4}}(-2t^{\frac{3}{4}})+\frac{t^{\frac{1}{4}}}{4\Gamma(\frac{5}{4})}-\frac{t}{2}+\frac{t^{\frac{1}{4}}}{9}E_{\frac{1}{4},\frac{1}{4}}(t^{\frac{1}{4}}) \\ &\quad -\frac{1}{8}E_{\frac{1}{4}}(-t^{\frac{1}{4}}) \\ &\quad -\frac{1}{5}\left[3\cos_{\frac{1}{4}}(t^{\frac{1}{4}})+\sin_{\frac{1}{4}}(t^{\frac{1}{4}})\right]. \end{aligned}$$

It is clear from equation (6) that the homogeneous solution (16) is equal to the homogeneous solution obtained from the roots of the characteristic equation (17).

## 5. Conclusions

The particular solution to Equation (1) is constructed when Equation (1) involves the right-hand side functions $f(t)=\sum_{k=0}^{m}c_k\frac{t^k}{k!}$, $f(t)=e^t$, $f(t)=\sin(t)$, and $f(t)=\cos(t)$ or their parametrized forms or linear combinations of them. To obtain the particular solution, the method of uncertain coefficients is presented for Caputo sequential fractional derivative equation of order $n\alpha$ where $n-1<n\alpha\leq n$ and $n\in\mathbb{N}$. This method is based on the appropriate particular solution estimation. The necessary details for proper estimation have been obtained. It is shown that Equation (1) is transformed into a linear algebraic equation with the benefit of proper solution estimation. Therefore, the method discussed is particularly effective for nonhomegeneous fractional differential equations.

## Declaration of Ethical Standards

The author of this article declares that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

## Conflict of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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