Journal of Universal Mathematics
Vol.6 No.2 pp.221-226 (2023)
ISSN-2618-5660
DOI: 10.33773/jum.1329709

ON THE GROUP OF POINTWISE INNER AUTOMORPHISMS

ELA AYDIN

0000-0003-4867-0583

Abstract. Let $L_{m,c}$ stand for the free metabelian nilpotent Lie algebra of class c of rank m over a field K of characteristic zero. Automorphisms of the form $\varphi(x_i) = e^{adu_i}(x_i)$ are called pointwise inner, where e^{adu_i}, is the inner automorphism induced by the element $u_i \in L_{m,c}$ for each $i = 1, \dots, m$. In the present study, we investigate the group structure of the group $\mathrm{PInn}(L_{m,c})$ of pointwise inner automorphisms of $L_{m,c}$ for low nilpotency classes.

1. Introduction

Pointwise inner automorphisms of the free metabelian nilpotent Lie algebra $L_{m,c}$ forms a group shown by the author [3], recently. A generating set for the group $\mathrm{PInn}(L_{m,c})$ was provided, as well, in the same study: Each automorphism φ in $\mathrm{PInn}(L_{m,c})$ is of the form

$$\varphi(x_i) = e^{\mathrm{ad}(u_i)}(x_i) = (u_1, \dots, u_m)$$

for some $u_i \in L_{m,c}$, $i = 1, \dots, m$. Let us define the set

$$I_i = \{\varphi_u = (0, \dots, 0, u, 0, \dots, 0) \mid u \in L_{m,c}\}, \quad i = 1, \dots, m,$$

consisting of m-tuples where each coordinate except for i-th position is necessarily filled by zero.

Theorem 1.1. [3] *The set I_i is a group for every $i = 1, \dots, m$.*

Theorem 1.2. [3] *The set $\mathrm{PInn}(L_{m,c})$ of pointwise inner automorphisms of the free metabelian nilpotent Lie algebra $L_{m,c}$ forms a group generated by the set $I_1 \cup \cdots \cup I_m$.*

In the current study, that can be considered as a continuation of the previous one, we show that the group $\mathrm{PInn}(L_{m,2})$ is abelian, and $\mathrm{PInn}(L_{m,3})$ is abelian-by-nilpotent of class 2. Furthermore, we give multiplication rules for compositions of two pointwise inner automorphisms in these groups.

Date: **Received:** 2023-07-19; **Accepted:** 2023-07-29.
2000 *Mathematics Subject Classification.* 17B01; 17B30.
Key words and phrases. Lie algebras, Metabelian, Nilpotent, Pointwise inner.

2. Preliminaries

The free metabelian nilpotent Lie algebra $L_{m,c}$ over a field K of characteristic zero is the free algebra of rank n in the variety of the Lie algebras satisfying the identities

$$[[x, y], [z, t]] = 0, \qquad \text{and} \qquad [y_1, y_2, \dots, y_{c+1}] = 0$$

for all $x, y, z, t, y_1, y_2, \dots, y_{c+1} \in L_{m,c}$. For more information on the Lie algebra $L_{m,c}$ we refer to the books [1, 2]. In this paper, we use the left normed commutators as below.

$$[u_1, \dots, u_{n-1}, u_n] = [[u_1, \dots, u_{n-1}], u_n], \quad n = 3, 4, \dots$$

For each $v \in L_{m,c}$, the linear operator $\mathrm{ad}v : L_{m,c} \to L_{m,c}$ defined by

$$\mathrm{ad}v(u) = [u, v], \quad u \in L_{m,c},$$

is a derivation of $L_{m,c}$ which is nilpotent and $\mathrm{ad}^c v = (\mathrm{ad}v)^c = 0$ because $L_{m,c}^{c+1} = 0$, and thus the linear operator

$$e^{\mathrm{ad}(v)} = 1 + \frac{\mathrm{ad}v}{1!} + \frac{\mathrm{ad}^2 v}{2!} + \cdots + \frac{\mathrm{ad}^{c-1} v}{(c-1)!}$$

is well defined and is an automorphism of $L_{m,c}$. The set of all automorphisms are of the form $e^{\mathrm{ad}(v)}$, $v \in L_{m,c}$, is called the inner automorphism group of $L_{m,c}$ and is denoted by $\mathrm{Inn}(L_{m,c})$. The group $\mathrm{PInn}(L_{m,c})$ of pointwise inner automorphisms can be considered as a generalization of $\mathrm{Inn}(L_{m,c})$.

Our goal is to describe the group structure of the group $\mathrm{PInn}(L_{m,c})$ of pointwise inner automorphisms of the Lie algebra $L_{m,c}$.

3. Main Results

Theorem 3.1. *Let the nilpotency class $c = 2$. Then the group $\mathrm{PInn}(L_{m,2})$ of pointwise inner automorphisms of the free metabelian Lie algebra $L_{m,2}$ is abelian, and the composition of two pointwise inner automorphisms is given by*

$$\varphi_u \varphi_v = \varphi_{u+v}.$$

Proof. Without loss of generality, we verify the formula $\varphi_u \varphi_v = \varphi_{u+v}$ with its action on x_1 only. In this case each element in $L_{m,2}$ is of the form

$$\sum_i c_i x_i + \sum_{i<j} c_{ij} [x_i, x_j]$$

for some $i, j \in K$. Let $\varphi_u = (u_1, \dots, u_m)$ for some $u_i \in L_{m,c}$, and let

$$u_i = u_{i1} + u_{i2}$$

such that u_{i1} and is the linear part and u_{i2} is of the homogeneous degree 2. Because $\varphi(x_i) = e^{\mathrm{ad}(u_i)}(x_i)$, we have the followings.

$$\varphi_u(x_1) = x_1 + [x_1, u_{11} + u_{12}] = x_1 + [x_1, u_{11}].$$

Similarly,

$$\varphi_v(x_1) = x_1 + [x_1, v_{11}].$$

Thus,

$$\begin{aligned}\varphi_u\varphi_v(x_1) &= \varphi_u(x_1) + [\varphi_u(x_1), \varphi_u(v_{11})] \\ &= x_1 + [x_1, u_{11}] + [x_1 + \overline{\varphi_u(x_1)}, v_{11} + \overline{\varphi_u(v_{11})}]\end{aligned}$$

where $\overline{\varphi_u(x_1)}$ is of the homogeneous degree 2, and $\overline{\varphi_u(v_{11})}$ is of the homogeneous degree ≥ 2. Thus $[x_1, \overline{\varphi_u(v_{11})}] = 0$ in $L_{m,2}$. Therefore,

$$\begin{aligned}\varphi_u\varphi_v(x_1) &= x_1 + [x_1, u_{11}] + [x_1 + \overline{\varphi_u(x_1)}, v_{11}] \\ &= x_1 + [x_1, u_{11}] + [x_1, v_{11}] \\ &= x_1 + [x_1, u_{11} + v_{11}] \\ &= \varphi_{u+v}(x_1)\end{aligned}$$

since $[\overline{\varphi_u(x_1)}, v_{11}] = 0$. Finally

$$\varphi_u\varphi_v = \varphi_{u+v} = \varphi_{v+u} = \varphi_v\varphi_u$$

and thus the group $\mathrm{PInn}(L_{m,2})$ is abelian.

□

Theorem 3.2. *Let the nilpotency class $c = 3$. Then the group $\mathrm{PInn}(L_{m,3})$ of pointwise inner automorphisms of the free metabelian Lie algebra $L_{m,3}$ is abelian-by nilpotent of class 2. That is,*

$$[[\varphi_u, \varphi_v], \varphi_w] = 0,$$

where

$$[\varphi_u, \varphi_v] = \varphi_u\varphi_v\varphi_u^{-1}\varphi_v^{-1}$$

Furthermore, the compositon of two pointwise inner automorphisms is given by

$$\varphi_u\varphi_v(x_j) = \varphi_{u+v+\sum_i d_i[x_i,u_{i1}]+\frac{1}{2}[u_{j1},v_{j1}]}(x_j)$$

where u_{j1}, v_{j1} are the linear parts of u_j, v_j in the expression of $\varphi_u = (u_1, \ldots, u_m)$, $\varphi_v = (v_1, \ldots, v_m)$, and

$$d_1x_1 + \cdots + d_mx_m$$

is the linear part of v.

Proof. Assume that $u_1 = u_{11} + u_{12} + u_{13}$, $v = v_{11} + v_{12} + v_{13}$ such that u_{1i} and v_{1i} are of homogeneous degree $i = 1, 2, 3$ and that

$$v_{11} = d_1x_1 + \cdots + d_mx_m$$

for some coefficients $d_1, \ldots, d_m \in K$. Observations give that

$$\begin{aligned}\varphi_u(x_1) &= x_1 + [x_1, u_1] + \frac{1}{2}[x_1, u_1, u_1] \\ &= x_1 + [x_1, u_{11} + u_{12} + u_{13}] + \frac{1}{2}[x_1, u_{11} + u_{12} + u_{13}, u_{11}] \\ &= x_1 + [x_1, u_{11} + u_{12}] + \frac{1}{2}[x_1, u_{11}, u_{11}]\end{aligned}$$

due to the nilpotency $c = 3$. We also have that

$$\varphi_v(x_1) = x_1 + [x_1, v_{11} + v_{12}] + \frac{1}{2}[x_1, v_{11}, v_{11}]$$

by similar calculations. Note that one may express

$$\varphi_u(x_1) = x_1 + A_2 + A_3$$

and

$$\varphi_v(x_1) = x_1 + B_2 + B_3$$

such that

$$A_2 = [x_1, u_{11}] \quad A_3 = [x_1, u_{12}] + \frac{1}{2}[x_1, u_{11}, u_{11}]$$

and

$$B_2 = [x_1, v_{11}] \quad B_3 = [x_1, v_{12}] + \frac{1}{2}[x_1, v_{11}, v_{11}]$$

That is, A_i and B_i are of homogeneous degree $i = 2, 3$. Straightforward computations yiled that

$$\varphi_u(A_3) = \varphi_v(A_3) = A_3$$

and

$$\varphi_u(B_3) = \varphi_v(B_3) = B_3$$

due to that fact that $c = 3$. Now we are ready to make computations on $\varphi_u\varphi_v$ with its action on x_1.

$$\varphi_u\varphi_v(x_1) = \varphi_u(x_1) + \varphi_u(B_2) + B_3$$

Let us consider the second summand above.

$$\begin{aligned}
\varphi_u(B_2) &= \varphi_u([x_1, v_{11}]) \\
&= [\varphi_u(x_1), \varphi_u(v_{11})] \\
&= [x_1 + A_2 + A_3, \varphi_u(v_{11})] \\
&= [x_1, \varphi_u(v_{11})] + [A_2, \varphi_u(v_{11})] \\
&= [x_1, \varphi_u(v_{11})] + [[x_1, u_{11}], \varphi_u(v_{11})] \\
&= [x_1, \varphi_u(v_{11})] + [[x_1, u_{11}], v_{11} + \overline{v_{11}}] \qquad (\deg(\overline{v_{11}}) \geq 2) \\
&= [x_1, \varphi_u(v_{11})] + [[x_1, u_{11}], v_{11}]
\end{aligned}$$

implies that

$$\varphi_u\varphi_v(x_1) = x_1 + A_2 + A_3 + [x_1, \varphi_u(v_{11})] + [[x_1, u_{11}], v_{11}] + B_3$$

where

$$\begin{aligned}
\varphi_u(v_{11}) &= d_1\varphi_u(x_1) + \cdots + d_m\varphi_u(x_m) \\
&= d_1 e^{\mathrm{ad}(u_1)}(x_1) + \cdots + d_m e^{\mathrm{ad}(u_m)}(x_m) \\
&= d_1(x_1 + [x_1, u_1] + \frac{1}{2}[x_1, u_1, u_1]) + \cdots + d_m(x_m + [x_m, u_m] + \frac{1}{2}[x_m, u_m, u_m]) \\
&= \sum_i d_i x_i + \sum_i d_i[x_i, u_i] + \frac{1}{2}\sum_i d_i[x_i, u_i, u_i]) \\
&= v_{11} + \sum_i d_i[x_i, u_{i1} + u_{i2}] + \frac{1}{2}\sum_i d_i[x_i, u_{i1}, u_{i1}])
\end{aligned}$$

Now nilpotency of degree 3 gives

$$\begin{aligned}
[x_1, \varphi_u(v_{11})] &= \left[x_1, v_{11} + \sum_i d_i[x_i, u_{i1} + u_{i2}]\right] \\
&= [x_1, v_{11}] + \left[x_1, \sum_i d_i[x_i, u_{i1}]\right]
\end{aligned}$$

Finally $S = \varphi_{u+v+\sum_i d_i[x_i,u_{i1}]+\frac{1}{2}[u_{11},v_{11}]}(x_1) - \varphi_u\varphi_v(x_1)$ is equal to

$$\begin{aligned}
S =& x_1 + \left[x_1, u_{11}+u_{12}+v_{11}+v_{12}+\sum_i d_i[x_i,u_{i1}] + \frac{1}{2}[u_{11},v_{11}]\right] + \frac{1}{2}[x_1,u_{11}+v_{11},u_{11}+v_{11}] \\
&- x_1 - A_2 - A_3 - [x_1,v_{11}] - [x_1, \sum_i d_i[x_i,u_{i1}]] - [[x_1,u_{11}],v_{11}] - B_3 \\
=& \frac{1}{2}[x_1,[u_{11},v_{11}]] + \frac{1}{2}[x_1,u_{11},v_{11}] + \frac{1}{2}[x_1,v_{11},u_{11}] - [[x_1,u_{11}],v_{11}] \\
=& \frac{1}{2}[x_1,[u_{11},v_{11}]] + \frac{1}{2}[x_1,u_{11},v_{11}] + \left(\frac{1}{2}[x_1,u_{11},v_{11}] + \frac{1}{2}[u_{11},v_{11},x_1]\right) - [[x_1,u_{11}],v_{11}] \\
=& 0
\end{aligned}$$

that verifies the multiplication rule. Now direct computations yiled that $[[\varphi_u,\varphi_v],\varphi_w] = 0$ as a consequence of the multiplication rule showed above given in the expression of the theorem. □

4. Conclusion

In this study, group structures of the groups $\mathrm{PInn}(L_{m,2})$, and $\mathrm{PInn}(L_{m,3})$ were provided via multiplication rules in them. The next step might be extending the nilpotency class $c \geq 4$, and obtain new results.

5. Acknowledgments

The authors would like to thank the reviewers and editors of Journal of Universal Mathematics.

Funding

The authors declared that has not received any financial support for the research, authorship or publication of this study.

The Declaration of Conflict of Interest/ Common Interest

The authors declared no conflict of interest or common interest

The Declaration of Ethics Committee Approval

This study does not be necessary ethical committee permission or any special permission.

The Declaration of Research and Publication Ethics

The authors declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the authors declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

References

[1] Yu.A. Bahturin, Identical Relations in Lie Algebras (Russian), "Nauka", Moscow, 1985. Translation: VNU Science Press, Utrecht, (1987).
[2] V. Drensky, Free Algebras and PI-Algebras, Springer, Singapore, (1999).

[3] E. Aydın, Pointwise inner automoprphisms of relatively free Lie algebras, Journal of Universal Mathematics, Vol.5, No.2, pp.76-80 (2022).

Department of Mathematics, Çukurova University, 01330 Balcali, Adana, Turkey
Email address: eaydin@cu.edu.tr