HERMITE–HADAMARD TYPE INEQUALITIES VIA DIFFERENTIABLE $h_\varphi-$ PREINVEX FUNCTIONS FOR FRACTIONAL INTEGRALS

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Abstract. In this paper, we consider a new class of convex functions which is called $h_\varphi-$preinvex functions. We prove several Hermite–Hadamard type inequalities for differentiable $h_\varphi-$preinvex functions via Fractional Integrals. Some special cases are also discussed. Our results extend and improve the corresponding ones in the literature.

1. Introduction

The following inequality is well-known in the literature as Hermite-Hadamard inequality. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then, the following inequality holds;

$$(1.1) \qquad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Recently, Hermite-Hadamard type inequality has been the subject of intensive research. Hermite–Hadamard inequality can be considered as a necessary and sufficient condition for a function to be convex. It provides estimates of the mean value of the continuous convex function.

In recent years, several extensions and generalizations have been proposed for classical convexity (see $[1-24]$). A significant generalization of classical convex functions is that of $\varphi-$convex functions introduced by Noor [10]. Noor [10] have investigated the basic properties of $\varphi-$convex functions and showed that $\varphi-$convex functions are nonconvex functions. Noor [9] extended Hermite–Hadamard type inequalities for $\varphi-$convex functions.

Motivated by the ongoing research on generalizations and extensions of classical convexity, Varosanec [23] introduced the class of $h-$convex functions. Noor et al. [13] introduced another generalization of classical convexity which is called $h_\varphi-$convex functions. Finally, Noor et al. Several Hermite–Hadamard type inequalities are proved for $h_\varphi-$preinvex functions [20].

Weir and Mond [24] investigated the class of preinvex functions. It is well-known that the preinvex functions and invex sets may not be convex functions and convex sets. For recent investigation on preinvexity (see $[1, 4, 5 - 8, 17, 22]$).

Motivated and inspired by the recent research in this field, we consider a new class of convex functions, which is called h_φ−preinvex functions. This class includes several known and new classes of convex functions such as φ−convex functions [10], preinvex functions [21], h−convex functions [23] as special cases. We derive several new Hermite–Hadamard type fractional integral inequalities for h_φ−preinvex functions and their variant forms. Results obtained in this paper continue to hold for these special cases. Our results represent significantly generalized of the previous results. The interested readers are encouraged to find novel and innovative applications of h_φ−preinvex functions and fractional integrals.

2. Preliminaries

Let $\mathbb{R}^n$ be the finite dimensional Euclidian space. Also let $0 \leq \varphi \leq \dfrac{\pi}{2}$ be a continuous function.

Definition 2.1. ([12]) The set $K_{\varphi n}$ in $\mathbb{R}^n$ is said to be φ−invex at u with respect to $\varphi(.)$, if there exists a bifunction $\eta(.,.) : K_{\varphi n} \times K_{\varphi n} \to \mathbb{R}$, such that

$$(2.1) \qquad u + te^{i\varphi}\eta(v, u) \in K_{\varphi n}, \qquad \forall u, v \in K_{\varphi n}, \ t \in [0, 1].$$

The φ−invex set $K_{\varphi n}$ is also called φn-connected set. Note that the convex set with $\varphi = 0$ and $\eta(v, u) = v - u$ is a φ−invex set, but the converse is not true.

Definition 2.2. ([10]) Let K_φ be a set in $\mathbb{R}^n$. Then the set K_φ is said to be φ−convex with respect to φ, if and only if

$$u + te^{i\varphi}(v, u) \in K_\varphi, \qquad \forall u, v \in K_\varphi, \ t \in [0, 1].$$

For $\varphi = 0$, the set K_φ reduces to the classical convex set K. That is,

$$u + t(v, u) \in K, \qquad \forall u, v \in K, \ t \in [0, 1].$$

Definition 2.3. ([24]) A set K_n is said to be invex set with respect to bifunction $\eta(.,.)$, if

$$u + t\eta(v, u) \in K_n, \qquad \forall u, v \in K_n, \ t \in [0, 1].$$

The invex set K_n is also called η−connected set.

Definition 2.4. ([20])Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function. A function f on the set $K_{\varphi n}$ is said to be h_φ−preinvex function with respect to φ and bifunction η, if

$$(2.2) \qquad f(u + te^{i\varphi}\eta(v, u)) \leq h(1 - t)f(u) + h(t)f(v),$$

$\forall u, v \in K_{\varphi n}, \ t \in [0, 1]$.

Remark 2.5. One can deduce several known concepts from Definition 4 as:

1. For $h(t) = t$ Definition 4 reduces to the definition for φ−preinvex functions (see [12]).
2. For $\varphi = 0$ Definition 4 reduces to the definition for h−preinvex functions (see [17]).
3. For $\eta(v, u) = v - u$ Definition 4 reduces to the definition for h_φ−convex functions (see [13]).

4. For $\varphi = 0$ and $\eta(v, u) = v - u$ Definition 4 reduces to the definition for $h-$convex functions (see [23]).

Now we discuss some special cases of Definition 4.

I. For $h(t) = t^s$ where $s \in (0, 1]$ in (2.2) we have the definition for $s_\varphi-$preinvex functions.

Definition 2.6. A function f on the set $K_{\varphi n}$ is said to be $s_\varphi-$preinvex function with respect to φ and η, if

$$(2.3) \qquad f(u + te^{i\varphi}\eta(v, u)) \le (1 - t)^s f(u) + t^s f(v),$$

$\forall u, v \in K_{\varphi n},\ t \in [0, 1]$.

II. For $h(t) = 1$ in (2.2) we have the definition for $P_\varphi-$preinvex functions.

Definition 2.7. A function f on the set $K_{\varphi n}$ is said to be $s_\varphi-$preinvex function with respect to φ and η, if

$$(2.4) \qquad f(u + te^{i\varphi}\eta(v, u)) \le f(u) + f(v),$$

$\forall u, v \in K_{\varphi n},\ t \in [0, 1]$.

Definition 2.8. ([25]) Let $f \in L^1[a, b]$. The Riemann-Liouville fractional integral $J_{a+}^\alpha f(x)$ and $J_{b-}^\alpha f(x)$ of order $\alpha > 0$ are defined by

$$(2.5) \qquad J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t)\, dt, \quad x > a$$

and

$$(2.6) \qquad J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t)\, dt \quad x < b$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha - 1} du$ is Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

3. Main results

In this section, we will discuss our main results for $h_\varphi-$preinvex functions and fractional integrals.

Using the technique of [1] and [20], we prove the following Lemma which plays a key role in our study.

Lemma 3.1. *Let $I \subseteq \mathbb{R}$ be a open invex set with respect to bifunction $\eta : I \times I \to \mathbb{R}$ where $\eta(b, a) > 0$. If $f' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$ and $\alpha \ge 0$, then*

$$\left[f(a) + f(a + e^{i\varphi}\eta(b, a)) \right] - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b, a)]^\alpha} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right\}$$

$$(3.1)$$

$$= e^{i\varphi}\eta(b, a) \int_0^1 [t^\alpha - (1 - t)^\alpha] f'(a + te^{i\varphi}\eta(b, a)) dt.$$

Proof. Let,

$$\int\limits_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'(a + te^{i\varphi}\eta(b,a))dt = \left.\frac{((1-t)^\alpha - t^\alpha)\, f(a + te^{i\varphi}\eta(b,a))}{e^{i\varphi}\eta(b,a)}\right|_0^1$$

$$+ \frac{\alpha}{e^{i\varphi}\eta(b,a)} \int\limits_0^1 \left[(1-t)^{\alpha-1} + t^{\alpha-1}\right] f(a + te^{i\varphi}\eta(b,a))dt$$

$$= -\frac{\left[f(a) + f(a + e^{i\varphi}\eta(b,a))\right]}{e^{i\varphi}\eta(b,a)} + \frac{\alpha}{e^{i\varphi}\eta(b,a)} \int\limits_0^1 \left[(1-t)^{\alpha-1} + t^{\alpha-1}\right] f(a + te^{i\varphi}\eta(b,a))dt$$

$$= -\frac{1}{e^{i\varphi}\eta(b,a)} \left[\left[f(a) + f(a + e^{i\varphi}\eta(b,a))\right]\right.$$

$$\left. - \alpha \int\limits_a^{a + e^{i\varphi}\eta(b,a)} \left[\left(1 - \frac{u-a}{e^{i\varphi}\eta(b,a)}\right)^{\alpha-1} + \left(\frac{u-a}{e^{i\varphi}\eta(b,a)}\right)^{\alpha-1}\right] f(u)\frac{du}{e^{i\varphi}\eta(b,a)}\right]$$

$$= -\frac{\left[f(a) + f(a + e^{i\varphi}\eta(b,a))\right]}{e^{i\varphi}\eta(b,a)}$$

$$+ \frac{\alpha}{e^{i\varphi}\eta(b,a)} \int\limits_a^{a + e^{i\varphi}\eta(b,a)} \left[\left(a + e^{i\varphi}\eta(b,a) - u\right)^{\alpha-1} + \left(\frac{u-a}{e^{i\varphi}\eta(b,a)}\right)^{\alpha-1}\right] f(u)\, du$$

$$= -\frac{\left[f(a) + f(a + e^{i\varphi}\eta(b,a))\right]}{e^{i\varphi}\eta(b,a)} + \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^{\alpha+1}} \left[J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x)\right].$$

This complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.2. *Let $I \subseteq \mathbb{R}$ be a open invex set with respect to bifunction $\eta : I \times I \to \mathbb{R}$. Suppose $f : I \to \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. Let $\left|f'\right|$ be $h_\varphi-$preinvex function. Then, for $\eta(b,a) > 0$ and $\alpha > 0$,*

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x)\right] \right|$$

(3.2)

$$\leq \quad e^{i\varphi}\eta(b,a) \left[\left|f'(a)\right| + \left|f'(b)\right|\right] \int\limits_0^1 |(1-t)^\alpha - t^\alpha|\, h(t)dt.$$

Proof. Using Lemma 2 and $h_\varphi-$preinvexity of $\left|f'\right|$, we have

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x)\right] \right|$$

$$\leq e^{i\varphi}\eta(b,a) \int\limits_0^1 |(1-t)^\alpha - t^\alpha| \left|f'(a + te^{i\varphi}\eta(b,a))\right| dt$$

$$\leq e^{i\varphi}\eta(b,a) \int\limits_0^1 |(1-t)^\alpha - t^\alpha| \left(h(1-t)\left|f'(a)\right| + h(t)\left|f'(b)\right|\right) dt$$

$$= e^{i\varphi}\eta(b,a)\left[\int_0^1 |(1-t)^\alpha - t^\alpha|\, h\,(1-t)\left|f^{'}(a)\right|dt\right.$$

$$\left. + \int_0^1 |(1-t)^\alpha - t^\alpha|\, h\,(t)\left|f^{'}(a)\right|dt\right]$$

$$= e^{i\varphi}\eta(b,a)\left[\int_0^1 |t^\alpha - (1-t)^\alpha|\, h\,(t)\left|f^{'}(a)\right|dt\right.$$

$$\left. + \int_0^1 |(1-t)^\alpha - t^\alpha|\, h\,(t)\left|f^{'}(b)\right|dt\right]$$

$$= e^{i\varphi}\eta(b,a)\left[\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right]\int_0^1 |(1-t)^\alpha - t^\alpha|\, h(t)dt.$$

This completes the proof. $\qquad\qquad\square$

Now we have some special cases.

I. If $h(t) = t$, then we have the result for $\varphi-$preinvexity.

Corollary 1. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f^{'} \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f^{'}\right|$ is $\varphi-$preinvex on I, then, for $\eta(b,a) > 0$,

$$\left|f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha}\left[J^\alpha_{a^+}f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-}f(x)\right]\right|$$

(3.3)

$$\leq\ e^{i\varphi}\eta(b,a)\left[\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right]\frac{1}{\alpha+2}\left[1 - \frac{1}{2^{\alpha+2}}\right].$$

II. If $h(t) = t^s$, then we have the result for $s_\varphi-$preinvexity

Corollary 2. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f^{'} \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f^{'}\right|$ is $s_\varphi-$preinvex on I, then, for $\eta(b,a) > 0$ and $s \in (0,1]$,

$$\left|f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha}\left[J^\alpha_{a^+}f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-}f(x)\right]\right|$$

(3.4)
$$\leq e^{i\varphi}\eta(b,a)\left[\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right]\left[B_{\frac{1}{2}}\left(\alpha+1, s+1\right)\right.$$

$$\left.\left(-B_{\frac{1}{2}}\left(s+1, \alpha+1\right) + \frac{1}{\alpha+s+1}\left[1 - \frac{1}{2^{s+\alpha}}\right]\right)\right].$$

III. If $h(t) = 1$, then we have the result for $P_\varphi-$preinvexity.

Corollary 3. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f^{'} \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f^{'}\right|$ is $P_\varphi-$preinvex on I, then, for $\eta(b,a) > 0$,

(3.5)
$$\left|f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha}\left[J^\alpha_{a^+}f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-}f(x)\right]\right|$$

$$\leq e^{i\varphi}\eta(b,a)\left[\left|f^{'}(a)\right| + \left|f^{'}(b)\right|\right]\frac{2}{\alpha+1}\left[1 - \frac{1}{2^\alpha}\right].$$

Theorem 3.3. *Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f^{'} \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. Let*

$\left|f'\right|^p$ be $h_\varphi-$preinvex on I, $\frac{1}{p} + \frac{1}{q} = 1$ and $\eta(b,a) > 0$. Then

(3.6)
$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J^\alpha_{a^+} f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-} f(x) \right] \right|$$
$$\leq \left(\frac{2}{\alpha p + 1} \right)^{\frac{1}{p}} e^{i\varphi}\eta(b,a) \left[1 - \frac{1}{2^{\alpha p}} \right]^{\frac{1}{p}} \left\{ \left[\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}} \right] \int_0^1 h(t)dt \right\}^{\frac{p-1}{p}}.$$

Proof. Using Lemma 2, we have

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J^\alpha_{a^+} f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-} f(x) \right] \right|$$
$$= \left| e^{i\varphi}\eta(b,a) \int_0^1 \left[(1-t)^\alpha - t^\alpha \right] f'(a + te^{i\varphi}\eta(b,a))dt \right|$$
$$\leq e^{i\varphi}\eta(b,a) \int_0^1 \left| (1-t)^\alpha - t^\alpha \right| \left| f'(a + te^{i\varphi}\eta(b,a)) \right| dt.$$

From Holders inequality, we have

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J^\alpha_{a^+} f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-} f(x) \right] \right|$$
$$\leq e^{i\varphi}\eta(b,a) \left(\int_0^1 \left| (1-t)^\alpha - t^\alpha \right|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f'(a + te^{i\varphi}\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}}$$
$$= e^{i\varphi}\eta(b,a) \left(\int_0^{\frac{1}{2}} \left((1-t)^\alpha - t^\alpha \right)^p dt + \int_{\frac{1}{2}}^1 \left(t^\alpha - (1-t)^\alpha \right)^p dt \right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left| f'(a + te^{i\varphi}\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}}.$$

Here $h_\varphi-$preinvexity of $\left|f'\right|^p$, we have following inequality

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J^\alpha_{a^+} f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-} f(x) \right] \right|$$
$$\leq \frac{2^{\frac{1}{p}}}{(\alpha p + 1)^{\frac{1}{p}}} e^{i\varphi}\eta(b,a) \left[1 - \frac{1}{2^{\alpha p}} \right]^{\frac{1}{p}} \left(\int_0^1 \left[h(1-t) \left| f'(a) \right|^q + h(t) \left| f'(b) \right|^q \right] dt \right)^{\frac{1}{q}}$$
$$= \left(\frac{2}{\alpha p + 1} \right)^{\frac{1}{p}} e^{i\varphi}\eta(b,a) \left[1 - \frac{1}{2^{\alpha p}} \right]^{\frac{1}{p}} \left\{ \left[\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}} \right] \int_0^1 h(t)dt \right\}^{\frac{p-1}{p}}.$$

This completes the proof. $\qquad\square$

Now we have some special cases for (3.6).
I. If $h(t) = t$, then we have the result for $\varphi-$preinvexity.

Corollary 4. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. Let

$\left|f'\right|^p$ be $\varphi-$preinvex on I, $\frac{1}{p} + \frac{1}{q} = 1$ and $\eta(b,a) > 0$. Then

$$(3.7) \quad \begin{aligned} &\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right] \right| \\ &\leq \left(\frac{2}{\alpha p + 1} \right)^{\frac{1}{p}} e^{i\varphi}\eta(b,a) \left[1 - \frac{1}{2^{\alpha p}} \right]^{\frac{1}{p}} \left\{ \frac{\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}}{2} \right\}^{\frac{p-1}{p}}. \end{aligned}$$

II. If $h(t) = t^s$, then we have the result for $s_\varphi-$preinvexity

Corollary 5. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. Let $\left|f'\right|^p$ be $s_\varphi-$preinvex on I, $\frac{1}{p} + \frac{1}{q} = 1$, $\eta(b,a) > 0$ and $s \in (0,1]$. Then

$$(3.8) \quad \begin{aligned} &\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right] \right| \\ &\leq \left(\frac{2}{\alpha p + 1} \right)^{\frac{1}{p}} e^{i\varphi}\eta(b,a) \left[1 - \frac{1}{2^{\alpha p}} \right]^{\frac{1}{p}} \left\{ \frac{\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}}{s+1} \right\}^{\frac{p-1}{p}}. \end{aligned}$$

III. If $h(t) = 1$, then we have the result for $P_\varphi-$preinvexity.

Corollary 6. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. Let $\left|f'\right|^p$ be $P_\varphi-$preinvex on I, $\frac{1}{p} + \frac{1}{q} = 1$ and $\eta(b,a) > 0$. Then

$$(3.9) \quad \begin{aligned} &\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right] \right| \\ &\leq \left(\frac{2}{\alpha p + 1} \right)^{\frac{1}{p}} e^{i\varphi}\eta(b,a) \left[1 - \frac{1}{2^{\alpha p}} \right]^{\frac{1}{p}} \left\{ \left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}}. \end{aligned}$$

Theorem 3.4. *Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. Let $\left|f'\right|^q$, $q > 1$ is $h_\varphi-$preinvex on I, then, for $\eta(b,a) > 0$,*

$$(3.10) \quad \begin{aligned} &\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right] \right| \\ &\leq \left(\frac{2}{\alpha + 1} \right)^{\frac{1}{p}} \left[1 - \frac{1}{2^\alpha} \right]^{\frac{1}{p}} e^{i\varphi}\eta(b,a) \\ &\times \left\{ \left[\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}} \right] \int_0^1 |(1-t)^\alpha - t^\alpha| h(t) dt \right\}^{\frac{p-1}{p}}. \end{aligned}$$

Proof. Using Lemma 2, we have

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^{\alpha}} \left[J_{a+}^{\alpha} f(x) + J_{(a+e^{i\varphi}\eta(b,a))-}^{\alpha} f(x) \right] \right|$$

$$= \left| e^{i\varphi}\eta(b,a) \int_{0}^{1} \left[(1-t)^{\alpha} - t^{\alpha} \right] f'(a + te^{i\varphi}\eta(b,a)) dt \right|$$

$$\leq e^{i\varphi}\eta(b,a) \int_{0}^{1} |(1-t)^{\alpha} - t^{\alpha}| \left| f'(a + te^{i\varphi}\eta(b,a)) \right| dt$$

Using power-mean inequality, we have

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^{\alpha}} \left[J_{a+}^{\alpha} f(x) + J_{(a+e^{i\varphi}\eta(b,a))-}^{\alpha} f(x) \right] \right|$$

$$\leq e^{i\varphi}\eta(b,a) \left(\int_{0}^{1} |(1-t)^{\alpha} - t^{\alpha}| dt \right)^{\frac{1}{p}} \left(\int_{0}^{1} \left| f'(a + te^{i\varphi}\eta(b,a)) \right|^{q} dt \right)^{\frac{1}{q}}$$

$$= e^{i\varphi}\eta(b,a) \left(\int_{0}^{\frac{1}{2}} \left((1-t)^{\alpha} - t^{\alpha} \right) dt + \int_{\frac{1}{2}}^{1} \left(t^{\alpha} - (1-t)^{\alpha} \right) dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_{0}^{1} \left| f'(a + te^{i\varphi}\eta(b,a)) \right|^{q} dt \right)^{\frac{1}{q}}$$

now using h_{φ}−preinvexity of $\left| f' \right|^{q}$, we have

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^{\alpha}} \left[J_{a+}^{\alpha} f(x) + J_{(a+e^{i\varphi}\eta(b,a))-}^{\alpha} f(x) \right] \right|$$

$$\leq \frac{2^{\frac{1}{p}}}{(\alpha+1)^{\frac{1}{p}}} e^{i\varphi}\eta(b,a) \left[1 - \frac{1}{2^{\alpha}} \right]^{\frac{1}{p}} \left(\int_{0}^{1} \left[h(1-t) \left| f'(a) \right|^{q} + h(t) \left| f'(b) \right|^{q} \right] dt \right)^{\frac{1}{q}}$$

$$= \left(\frac{2}{\alpha+1} \right)^{\frac{1}{p}} e^{i\varphi}\eta(b,a) \left[1 - \frac{1}{2^{\alpha}} \right]^{\frac{1}{p}}$$

$$\times \left\{ \left[\left| f'(a) \right|^{\frac{p}{p-1}} + \left| f'(b) \right|^{\frac{p}{p-1}} \right] \int_{0}^{1} |(1-t)^{\alpha} - t^{\alpha}| h(t) dt \right\}^{\frac{p-1}{p}}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we have some special cases for (3.10).

I. If $h(t) = t$, then we have the result for φ−preinvexity.

Corollary 7. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left| f' \right|^{q}$, $q > 1$ is φ−preinvex on I, then, for $\eta(b,a) > 0$,

(3.11)

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^{\alpha}} \left[J_{a+}^{\alpha} f(x) + J_{(a+e^{i\varphi}\eta(b,a))-}^{\alpha} f(x) \right] \right|$$

$$\leq \left(\frac{2}{\alpha+1} \right)^{\frac{1}{p}} \left[1 - \frac{1}{2^{\alpha}} \right]^{\frac{1}{p}} e^{i\varphi}\eta(b,a)$$

$$\times \left\{ \left[\left| f'(a) \right|^{\frac{p}{p-1}} + \left| f'(b) \right|^{\frac{p}{p-1}} \right] \frac{1}{\alpha+2} \left[1 - \frac{1}{2^{\alpha+2}} \right] \right\}^{\frac{p-1}{p}}.$$

II. If $h(t) = t^{s}$, then we have the result for s_{φ}−preinvexity

Corollary 8. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f^{'} \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f^{'}\right|^q$, $q > 1$ is s_φ−preinvex on I, then, for $\eta(b,a) > 0$ and $s \in (0, 1]$,

(3.12)

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right] \right|$$

$$\leq \left(\frac{2^{\alpha+1} - 2}{2^\alpha (\alpha + 1)} \right) e^{i\varphi}\eta(b,a) \left\{ \left[\left|f^{'}(a)\right|^{\frac{p}{p-1}} + \left|f^{'}(b)\right|^{\frac{p}{p-1}} \right] \right.$$

$$\left. x \left[B_{\frac{1}{2}}(\alpha + 1, s + 1) - B_{\frac{1}{2}}(s + 1, \alpha + 1) + \frac{1}{\alpha + s + 1} \left[1 - \frac{1}{2^{s+\alpha}} \right] \right] \right\}^{\frac{p-1}{p}}.$$

III. If $h(t) = 1$, then we have the result for P_φ−preinvexity.

Corollary 9. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ is a differentiable function such that $f^{'} \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f^{'}\right|^q$, $q > 1$ is P_φ−preinvex on I, then, for $\eta(b,a) > 0$

(3.13)

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left[J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right] \right|$$

$$\leq \left(\frac{2^{\alpha+1} - 2}{2^\alpha (\alpha + 1)} \right) e^{i\varphi}\eta(b,a) \left\{ \left|f^{'}(a)\right|^{\frac{p}{p-1}} + \left|f^{'}(b)\right|^{\frac{p}{p-1}} \right\}^{\frac{p-1}{p}}.$$

Using the technique of [1], we prove the following result which helps us in proving our next results.

Lemma 3.5. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$ where $\eta(b,a) > 0$. If $f^{''} \in L_1[a, a + e^{i\varphi}\eta(b,a)]$, then

$$f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right\}$$

(3.14)

$$= \left[e^{i\varphi}\eta(b,a) \right]^2 \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right] f^{''}(a + te^{i\varphi}\eta(b,a)) dt.$$

Proof. Let

$$\int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right] f^{''}(a + te^{i\varphi}\eta(b,a)) dt$$

$$= \left. \frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \frac{f^{'}(a + te^{i\varphi}\eta(b,a))}{e^{i\varphi}\eta(b,a)} \right|_0^1$$

$$- \frac{1}{e^{i\varphi}\eta(b,a)} \int_0^1 \left[(1-t)^\alpha - t^\alpha \right] f^{'}(a + te^{i\varphi}\eta(b,a)) dt$$

$$= - \frac{1}{e^{i\varphi}\eta(b,a)} \int_0^1 \left[(1-t)^\alpha - t^\alpha \right] f^{'}(a + te^{i\varphi}\eta(b,a)) dt$$

$$= \frac{1}{(e^{i\varphi}\eta(b,a))^2} \left[\left(f(a) + f(a + e^{i\varphi}\eta(b,a)) \right) \right.$$

$$\left. - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right\} \right].$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Theorem 3.6. *Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f''\right|$, is h_φ-preinvex on I, then, for $\eta(b,a) > 0$,*

(3.15)
$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right\} \right|$$
$$\leq \left[e^{i\varphi}\eta(b,a) \right]^2 \left\{ \left(\left|f''(a)\right| + \left|f''(b)\right| \right) \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] h(t)dt \right\}.$$

Proof. Using Lemma 3, we have

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right\} \right|$$
$$= \left| \left[e^{i\varphi}\eta(b,a) \right]^2 \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] f''(a + te^{i\varphi}\eta(b,a))dt \right|$$
$$\leq \left[e^{i\varphi}\eta(b,a) \right]^2 \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] \left|f''(a + te^{i\varphi}\eta(b,a))\right| dt$$
$$\leq \left[e^{i\varphi}\eta(b,a) \right]^2 \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] \left(h(1-t)\left|f''(a)\right| + h(t)\left|f''(b)\right| \right) dt$$
$$= \left[e^{i\varphi}\eta(b,a) \right]^2 \left(\left|f''(a)\right| + \left|f''(b)\right| \right) \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] h(t)dt$$

This completes the proof. □

Now, we discuss some special cases for (3.15).

I. If $h(t) = t$, then we have the result for φ-preinvexity.

Corollary 10. *Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f''\right|$ is φ-preinvex on I, then, for $\eta(b,a) > 0$,*

(3.16)
$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right\} \right|$$
$$\leq \left[e^{i\varphi}\eta(b,a) \right]^2 \left\{ \frac{\alpha}{2(\alpha+1)(\alpha+2)} \left(\left|f''(a)\right| + \left|f''(b)\right| \right) \right\}.$$

II. If $h(t) = t^s$, then we have the result for s_φ-preinvexity

Corollary 11. *Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f''\right|$ is s_φ-preinvex on I, then, for $\eta(b,a) > 0$ and $s \in (0,1]$,*

(3.17)
$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right\} \right|$$
$$\leq \left[e^{i\varphi}\eta(b,a) \right]^2 \left\{ \left(\left|f''(a)\right| + \left|f''(b)\right| \right) \left(\frac{1}{(\alpha+s+2)(s+1)} - \frac{B_{\frac{1}{2}}(s+1,\alpha)}{\alpha+1} \right) \right\}.$$

III. If $h(t) = 1$, then we have the result for P_φ-preinvexity.

Corollary 12. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f''\right|$ is $P_\varphi-$preinvex on I, then, for $\eta(b,a) > 0$,

(3.18)
$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))-}^\alpha f(x) \right\} \right|$$
$$\leq \left[e^{i\varphi}\eta(b,a) \right]^2 \left\{ \left(\left| f''(a) \right| + \left| f''(b) \right| \right) \frac{\alpha}{(\alpha + 1)(\alpha + 2)} \right\}.$$

Theorem 3.7. *Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f''\right|$ is $P_\varphi-$preinvex on I, then, for $\eta(b,a) > 0$,*

(3.19)
$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))-}^\alpha f(x) \right\} \right|$$
$$\leq \left[e^{i\varphi}\eta(b,a) \right]^2 \left(1 - \frac{1}{2^\alpha} \right) \left[\left| f''(a) \right|^q + \left| f''(b) \right|^q \right]^{\frac{1}{q}} \left(\int_0^1 h(t)dt \right)^{\frac{1}{q}}$$

Proof. Using Lemma 3, we have

$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))-}^\alpha f(x) \right\} \right|$$
$$= \left| \left[e^{i\varphi}\eta(b,a) \right]^2 \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right] f''(a + te^{i\varphi}\eta(b,a))dt \right|$$
$$\leq \left[e^{i\varphi}\eta(b,a) \right]^2 \left(\int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \right]^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f''(a + te^{i\varphi}\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}}$$
$$\leq \frac{\left[e^{i\varphi}\eta(b,a) \right]^2}{\alpha + 1} \left(1 - \frac{1}{2^\alpha} \right) \left(\int_0^1 \left\{ h(1-t) \left| f''(a) \right|^q + h(t) \left| f''(b) \right|^q \right\} dt \right)^{\frac{1}{q}}$$
$$= \frac{\left[e^{i\varphi}\eta(b,a) \right]^2}{\alpha + 1} \left(1 - \frac{1}{2^\alpha} \right) \left(\left\{ \left| f''(a) \right|^q + \left| f''(b) \right|^q \right\} \int_0^1 [h(t)] dt \right)^{\frac{1}{q}}$$

This completes the proof. □

We have some special cases for (3.19).
I. If $h(t) = t$, then we have the result for $\varphi-$preinvexity.

Corollary 13. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f''\right|$ is $\varphi-$preinvex on I, then, for $\eta(b,a) > 0$,

(3.20)
$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha + 1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))-}^\alpha f(x) \right\} \right|$$
$$\leq \frac{\left[e^{i\varphi}\eta(b,a) \right]^2}{\alpha + 1} \left(1 - \frac{1}{2^\alpha} \right) \left(\frac{1}{2} \right)^{\frac{1}{q}} \left(\left| f''(a) \right|^q + \left| f''(b) \right|^q \right)^{\frac{1}{q}}$$

II. If $h(t) = t^s$, then we have the result for $s_\varphi-$preinvexity

Corollary 14. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If

$\left|f''\right|$ is s_φ−preinvex on I, then, for $\eta(b,a) > 0$ and $s \in (0,1]$,
(3.21)
$$\left|f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha}\left\{J^\alpha_{a^+}f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-}f(x)\right\}\right|$$
$$\leq \frac{\left[e^{i\varphi}\eta(b,a)\right]^2}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\left|f''(a)\right|^q + \left|f''(b)\right|^q\right)^{\frac{1}{q}}$$

III. If $h(t) = 1$, then we have the result for P_φ−preinvexity.

Corollary 15. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f''\right|$ is P_φ−preinvex on I, then, for $\eta(b,a) > 0$,
(3.22)
$$\left|f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha}\left\{J^\alpha_{a^+}f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-}f(x)\right\}\right|$$
$$\leq \frac{\left[e^{i\varphi}\eta(b,a)\right]^2}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\left(\left|f''(a)\right|^q + \left|f''(b)\right|^q\right)^{\frac{1}{q}}$$

Theorem 3.8. *Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f''\right|$ is P_φ−preinvex on I, then, for $\eta(b,a) > 0$,*
(3.23)
$$\left|f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha}\left\{J^\alpha_{a^+}f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-}f(x)\right\}\right|$$
$$\leq \left[e^{i\varphi}\eta(b,a)\right]^2\left(\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{p}}$$
$$\times \left(\left\{\left|f''(a)\right|^q + \left|f''(b)\right|^q\right\}\int_0^1\left[\frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1}\right]h(t)dt\right)^{\frac{1}{q}}$$

Proof. Using Lemma 3 and well-known power-mean inequality, we have

$$
\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J^\alpha_{a+} f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-} f(x) \right\} \right|
$$

$$
= \left| [e^{i\varphi}\eta(b,a)]^2 \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] f''(a + te^{i\varphi}\eta(b,a)) dt \right|
$$

$$
\leq [e^{i\varphi}\eta(b,a)]^2 \left(\int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] dt \right)^{\frac{1}{p}}
$$

$$
\times \left(\int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] \left| f''(a + te^{i\varphi}\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}}
$$

$$
\leq [e^{i\varphi}\eta(b,a)]^2 \left(\int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] dt \right)^{\frac{1}{p}}
$$

$$
\times \left(\int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] \left\{ h(1-t) \left| f''(a) \right|^q + h(t) \left| f''(b) \right|^q \right\} dt \right)^{\frac{1}{q}}
$$

$$
\leq [e^{i\varphi}\eta(b,a)]^2 \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right)^{\frac{1}{p}} \left\{ \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] \right.
$$

$$
\times \left. \left(h(1-t) \left| f''(a) \right|^q + h(t) \left| f''(b) \right|^q \right) dt \right\}^{\frac{1}{q}}
$$

$$
= [e^{i\varphi}\eta(b,a)]^2 \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right)^{\frac{1}{p}} \left\{ \left\{ \left| f''(a) \right|^q + \left| f''(b) \right|^q \right\} \right.
$$

$$
\times \left. \int_0^1 \left[\frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right] h(t) dt \right\}^{\frac{1}{q}}
$$

This completes the proof. □

We have some special cases for (3.23).
I. If $h(t) = t$, then we have the result for $\varphi-$preinvexity.

Corollary 16. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Suppose that $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left| f'' \right|$ is $\varphi-$preinvex on I, then, for every $\eta(b,a) > 0$,
(3.24)

$$
\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J^\alpha_{a+} f(x) + J^\alpha_{(a+e^{i\varphi}\eta(b,a))^-} f(x) \right\} \right|
$$

$$
\leq [e^{i\varphi}\eta(b,a)]^2 \left(\frac{1}{2} \right)^{\frac{1}{q}} \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \left(\left\{ \left| f''(a) \right|^q + \left| f''(b) \right|^q \right\} \right)^{\frac{1}{q}}
$$

II. If $h(t) = t^s$, then we have the result for $s_\varphi-$preinvexity

Corollary 17. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f'' \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If

$\left|f^{''}\right|$ is s_φ−preinvex on I, then, for every $\eta(b,a) > 0$ and $s \in (0,1]$,

(3.25)
$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right\} \right|$$
$$\leq \left[e^{i\varphi}\eta(b,a)\right]^2 \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right)^{\frac{1}{p}}$$
$$\times \left(\left\{ \left|f^{''}(a)\right|^q + \left|f^{''}(b)\right|^q \right\} \left(\frac{1}{(\alpha+s+2)(s+1)} - \frac{B_{\frac{1}{2}}(s+1,\alpha)}{\alpha+1} \right) \right)^{\frac{1}{q}}$$

III. If $h(t) = 1$, then we have the result for P_φ−preinvexity.

Corollary 18. Let $I \subseteq \mathbb{R}$ be a open invex set with respect to $\eta : I \times I \to \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a twice differentiable function such that $f^{''} \in L_1[a, a + e^{i\varphi}\eta(b,a)]$. If $\left|f^{''}\right|$ is P_φ−preinvex on I, then, for every $\eta(b,a) > 0$,

(3.26)
$$\left| f(a) + f(a + e^{i\varphi}\eta(b,a)) - \frac{\Gamma(\alpha+1)}{[e^{i\varphi}\eta(b,a)]^\alpha} \left\{ J_{a^+}^\alpha f(x) + J_{(a+e^{i\varphi}\eta(b,a))^-}^\alpha f(x) \right\} \right|$$
$$\leq \left[e^{i\varphi}\eta(b,a)\right]^2 \left(\frac{\alpha}{(\alpha+1)(\alpha+2)} \right) \left(\left|f^{''}(a)\right|^q + \left|f^{''}(b)\right|^q \right)^{\frac{1}{q}}$$

Conclusion 1. If we take $\alpha = 1$ in $(3.3) - (3.5)$, $(3.7) - (3.9)$, $(3.11) - (3.13)$, $(3.16) - (3.18)$, $(3.20) - (3.22)$, $(3.24) - (3.26)$, we obtain Crollary $3.3 - 3.26$ in [20].

Conclusion 2. If we take $\varphi = 0$ in (3.1), we obtain Lemma 2.3 in [4].

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