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# NEW FRACTIONAL DERIVATIVE IN COLOMBEAU ALGEBRA

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Abstract. In this paper we introduce an approach to fractional derivatives involving singularities based on the theory of algebras of generalized functions in the Colombeau algebra $\mathcal{G}$, using new definition of fractional derivative called conformable fractional derivative introduced by the authors Khalil et al. in ([8]).

## 1. Introduction

This paper is extension of fractional derivatives in Colombeau algebra of generalized functions in order to solve problems involving a multiplication of distributions and other nonlinear operations with singularities, provided by Colombeau theory, but including non-integer derivatives and operations among them. There are many fractional order equations with a lack of the solution in classical spaces, especially in the space of distributions involving nonlinear operations and singularities. In this way, many problems with fractional derivatives involving such kind of operation, would have been solved. Another reason for introducing fractional derivatives into Colombeau theory is an extension of the Colombeau theory to derivatives of arbitrary order, i.e. to non-integer ones [2].
In the last decades, fractional, or non-integer, differentiation has played a very important role in various fields such as mechanics, electricity, chemistry, biology, economics, modeling problems, anomalous diffusion and notably control theory and signal and image processing. It has been found that fractional differential equations play a crucial role in modeling anomalous diffusion, time-dependent materials and processes with long-range dependence, allometric scaling laws, as well as power law in complex systems.
The theory of algebra of generalized functions provides extension to derivatives of arbitrary order [2], [6]. It allows us to solve nonlinear partial differential problems with fractional order of temporal or spatial derivatives. These problems sometimes better describe the structure of the problems in nature than ODEs or PDEs do.
The paper is organized as follows, in the first section we give some basic preliminaries such as notations and definitions of the objects we shall work with. We also introduce different spaces of Colombeau algebra of generalized functions and results concerning conformable fractional derivative. In the second section we prove the Fractional derivatives of Colombeau generalized.In the third section we prove the fractional integral of Colombeau generalized.

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## 2. Preliminaries

2.1. **Definition of the colombeau algebra.** We use the following notations

$$\mathcal{A}_q = \{\varphi \in \mathcal{D}(\mathbb{R}^+)/ \int_{\mathbb{R}^+} \varphi(x)dx = 1, \quad \int_{\mathbb{R}^+} x^\alpha \varphi(x)dx = 0 \quad \text{for} \quad 1 \leq |\alpha| \leq q\}$$

$q = 1, 2, \ldots$

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon}) \quad \text{for} \quad \varphi \in \mathcal{D}(\mathbb{R}^+) \tag{2.1}$$

We denote by

$\mathcal{E}(\mathbb{R}^+) = \{u : \mathcal{A}_1 \times \mathbb{R}^+ \to \mathbb{C}/ \quad \text{with} \quad u(\varphi, x) \quad \text{is} \quad \mathcal{C}^\infty \quad \text{to the second variable} \quad x\}$

$$u(x, \varphi_\varepsilon) = u_\varepsilon(x) \quad \forall \varphi \in \mathcal{A}_1$$

$$\begin{aligned}
\mathcal{E}_M(\mathbb{R}^+) &= \{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^+)/\forall K \subset\subset \mathbb{R}^+, \forall m \in \mathbb{N}, \exists N \in \mathbb{N} \quad \text{such that} \\
&\quad \sup_{x\in K} |D^m u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-N}) \quad \text{as} \quad \varepsilon \to 0\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}(\mathbb{R}^+) &= \{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^+)/\forall K \subset\subset \mathbb{R}^+, \forall m \in \mathbb{N}, \forall p \in \mathbb{N} \quad \text{such that} \\
&\quad \sup_{x\in K} |D^m u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{p}) \quad \text{as} \quad \varepsilon \to 0\}
\end{aligned}$$

The Colombeau algebra is defined as a factor set $\mathcal{G}(\mathbb{R}^+) = \mathcal{E}_M(\mathbb{R}^+)/\mathcal{N}(\mathbb{R}^+)$, where the elements of the set $\mathcal{E}_M(\mathbb{R}^+)$ are moderate while the elements of the set $\mathcal{N}(\mathbb{R}^+)$ are negligible.
We introduce $C^k$-Colombeau generalized in the following way.
Denote by

$$\begin{aligned}
\mathcal{E}^k_M(\mathbb{R}^+) &= \{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^+)/\forall K \subset\subset \mathbb{R}^+, \forall m \in \mathbb{N}, \exists N \in \mathbb{N} \quad \text{such that} \\
&\quad \sup_{x\in K} |D^m u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-N}), m \in \{0, ..., k\}, \quad \text{as} \quad \varepsilon \to 0\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}^k(\mathbb{R}^+) &= \{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^+)/\forall K \subset\subset \mathbb{R}^+, \forall m \in \mathbb{N}, \forall p \in \mathbb{N} \quad \text{such that} \\
&\quad \sup_{x\in K} |D^m u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{p}), m \in \{0, ..., k\} \quad \text{as} \quad \varepsilon \to 0\}
\end{aligned}$$

The $C^k$- Colombeau algebra is defined as a factor set $\mathcal{G}^k(\mathbb{R}^+) = \mathcal{E}^k_M(\mathbb{R}^+)/\mathcal{N}^k(\mathbb{R}^+)$

2.2. **New definition of fractional derivative.** We recall some notations, definitions, and results concerning conformable fractional derivative which are used throughout this paper. By $C(I, R)$ we denote the Banach space of all continuous functions from $I$ into $R$ with the norm $\|f\|_\infty = \sup_{t\in I} |f(t)|$

**Definition 2.1.** [8] Given a function $f : [0, \infty) \to R$, then the "conformable fractional derivative" of $f$ of order $\alpha$ is defined by

$$T_\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for all $t > 0$, $\alpha \in (0, 1)$. If $f$ is $\alpha$-differentiable in some $(0, a)$, $a > 0$ and $\lim_{t\to 0^+} T_\alpha(f)(t)$ exists and define by $f^{(\alpha)}(0) = \lim_{t\to 0^+} T_\alpha(f)(t)$ .

**Definition 2.2.** [8] The $\alpha-$fractional integral of a continuous function f starting from $a \geq 0$ of order $\alpha \in (0,1)$ is defined by

$$I_\alpha^a(f)(t) = I_1^a(f)(t) = \int_a^t s^{\alpha-1} f(s)ds$$

**Lemma 2.3.** [8] *Assume that $f[a,\infty) \to R$, such that is continuous and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$T_\alpha^a I_\alpha^a f(t) = f(t)$$

In the right case we have :

**Lemma 2.4.** [9] *Assume that $f(-\infty, b] \to R$, such that is continuous and $0 < \alpha \leq 1$. Then, for all $t < b$ we have*

$$T_\alpha^b I_\alpha^b f(t) = f(t)$$

**Lemma 2.5.** [9] *Let $f$, $h : [a,\infty) \to R$ be functions such that $T_\alpha^a f(t)$ exists for all $t > a$, $f$ is differentiable on $(0,a)$ and*

$$T_\alpha^a f(t) = (t-a)^{1-\alpha} h(t) \tag{2.2}$$

*Then $h(t) = f'(t)$ for all $t > a$ .*

**Lemma 2.6.** [9] *Let $f : (a,b) \to R$ be differentiable and $0 < \alpha \leq 1$. Then for all $t > a$ we have*

$$I_\alpha^a T_\alpha^a f(t) = f(t) - f(a)$$

## 3. Fractional derivatives of Colombeau generalized

Let $(u_\varepsilon)_{\varepsilon>0}$ be a representative of a Colombeau generalized $u \in \mathcal{G}([0,\infty))$. By (2.2), the fractional derivative of $(u_\varepsilon)_{\varepsilon>0}$ is defined by

$$D^\alpha u_\varepsilon(x) = x^{1-\alpha} \frac{d}{dx} u_\varepsilon(x) \tag{3.1}$$

**Lemma 3.1.** *Let $(u_\varepsilon)_{\varepsilon>0}$ be a representative of $u \in \mathcal{G}([0,\infty))$. Then, for every $\alpha > 0$, $\sup\limits_{x\in[0,T]} \Big| D^\alpha u_\varepsilon(x) \Big|$ has a moderate bound.*

*Proof.*

$$\begin{aligned} \sup_{x\in[0,T]} \Big| D^\alpha u_\varepsilon(x) \Big| = \sup_{x\in[0,T]} \Big| x^{1-\alpha} \frac{d}{dx} u_\varepsilon(x) \Big| &\leq T^{1-\alpha} \sup_{x\in[0,T]} \Big| \frac{d}{dx} u_\varepsilon(x) \Big| \\ &\leq T^{1-\alpha} C \varepsilon^{-N} \\ &\leq C_{\alpha,T} \varepsilon^{-N} \end{aligned}$$

□

**Lemma 3.2.** *Let $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$ be two different representative of $u \in \mathcal{G}([0,\infty))$. Then, for every $\alpha > 0$, $\sup\limits_{x\in[0,T]} \Big| D^\alpha u_{1,\varepsilon}(x) - D^\alpha u_{2,\varepsilon}(x) \Big|$ is negligible.*

*Proof.*

$$\begin{aligned}\sup_{x\in[0,T]}\Big|D^\alpha u_{1,\varepsilon}(x)-D^\alpha u_{2,\varepsilon}(x)\Big| &= \sup_{x\in[0,T]}\Big|x^{1-\alpha}\frac{d}{dx}u_{1,\varepsilon}(x)-x^{1-\alpha}\frac{d}{dx}u_{2,\varepsilon}(x)\Big|\\ &= \sup_{x\in[0,T]}\Big|x^{1-\alpha}(\frac{d}{dx}u_{1,\varepsilon}(x)-\frac{d}{dx}u_{2,\varepsilon}(x))\Big|\\ &\leq T^{1-\alpha}\sup_{x\in[0,T]}\Big|\frac{d}{dx}u_{1,\varepsilon}(x)-\frac{d}{dx}u_{2,\varepsilon}(x)\Big|\end{aligned}$$

Since $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$ represent the same Colombeau generalized $u$ we have that
$\sup_{x\in[0,T]}\Big|\frac{d}{dx}u_{1,\varepsilon}(x)-\frac{d}{dx}u_{2,\varepsilon}(x)\Big|$ is negligible. Therefore, $\sup_{x\in[0,T]}\Big|D^\alpha u_{1,\varepsilon}(x)-D^\alpha u_{2,\varepsilon}(x)\Big|$ is negligible, too. □

After proving the previous two lemmas we are able to introduce the $\alpha$-fractional derivative of a Colombeau generalized on $[0,\infty)$.

**Definition 3.3.** Let $u\in\mathcal{G}([0,\infty))$ be a Colombeau generalized on $[0,\infty)$ The $\alpha$-fractional derivative of $u$, in notation $D^\alpha u(x)=[D^\alpha u_\varepsilon(x)]$ is an element of $\mathcal{G}^0([0,\infty))$ satisfying (3.1).

*Remark* 3.4. For $0<\alpha<1$ the first-order derivative of

$$\frac{d}{dx}D^\alpha u_\varepsilon(x)=(1-\alpha)x^{-\alpha}\frac{d}{dx}u_\varepsilon(x)+x^{1-\alpha}\frac{d^2}{dx^2}u_\varepsilon(x)$$

and it does not reach its upper limit . In general, the $k$-th order derivative $\frac{d^k}{dx^k}D^\alpha u_\varepsilon(x)$ it does not reach its upper limit on $[0,\infty)$

The new fractional derivative of a Colombeau generalized $u\in\mathcal{G}([0,\infty))$ If one wants this to be of $\mathcal{G}([0,\infty))$, then the regularization of the fractional derivative has to be done.

**Definition 3.5.** Let $(u_\varepsilon)_{\varepsilon>0}$ be a representative of a Colombeau generalized $u\in\mathcal{G}([0,\infty))$. The regularized of new fractional derivative of $(u_\varepsilon)_{\varepsilon>0}$, is defined by :

$$\tilde{D}^\alpha u_\varepsilon(x)=\begin{cases}(D^\alpha u_\varepsilon*\varphi_\varepsilon)(x), 0<\alpha<1\\ u_\varepsilon^{'}(x)=\frac{d}{dx}u_\varepsilon(x)\end{cases}\tag{3.2}$$

where $D^\alpha u_\varepsilon(x)$ is given by (3.1) and $\varphi_\varepsilon(x)$ is given by (2.1).
The convolution in (3.2) is $D^\alpha u_\varepsilon*\varphi_\varepsilon(x)=\int_0^\infty D^\alpha u_\varepsilon(s)\varphi_\varepsilon(x-s)ds$

**Lemma 3.6.** *Let $(u_\varepsilon)_{\varepsilon>0}$ be a representative of $u\in\mathcal{G}([0,\infty))$.*
*Then, for every $\alpha>0$ and every $k\in\{0,1,2,\ldots\}$, $\sup_{x\in[0,T]}\Big|\frac{d^k}{dx^k}\tilde{D}^\alpha u_\varepsilon(x)\Big|$ has a moderate bound.*

*Proof.* Let $\epsilon \in (0,1)$ For $\alpha \in \mathbb{N}, \tilde{D}^\alpha u_\varepsilon(x)$ is the usual derivative of order $\alpha$ of $u_\varepsilon(x)$ and the assertion immediately follows. In case when $0 < \alpha < 1$, we have

$$\begin{aligned}
\sup_{x\in[0,T]}\left|\tilde{D}^\alpha u_\varepsilon(x)\right| &= \sup_{x\in[0,T]}\left|(D^\alpha u_\varepsilon * \varphi_\varepsilon)(x)\right| \\
&\leq \sup_{x\in[0,T]}\left|\int_0^\infty D^\alpha u_\varepsilon(s)\varphi_\varepsilon(x-s)ds\right| \\
&\leq \sup_{s\in K}\left|D^\alpha u_\varepsilon(s)\right| \sup_{x\in[0,T]}\left|\int_K \varphi_\varepsilon(x-s)ds\right| \\
&\leq C\sup_{s\in K}\left|D^\alpha u_\varepsilon(s)\right|
\end{aligned}$$

for some constant $C > 0$
Since, according to Lemma 3.1, $\sup_{s\in K}\left|D^\alpha u_\varepsilon(s)\right|$ has a moderate bound, for every $\alpha > 0$, it follows that, $\sup_{x\in[0,T]}\left|\tilde{D}^\alpha u_\varepsilon(x)\right|$ has a moderate bound, too.
For arbitrary order derivative, we have

$$\begin{aligned}
\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}\tilde{D}^\alpha u_\varepsilon(x)\right| &\leq \sup_{s\in K}\left|D^\alpha u_\varepsilon(s)\right| \sup_{x\in[0,T]}\left|\int_K \frac{d^k}{dx^k}\varphi_\varepsilon(x-s)ds\right| \\
&\leq \frac{C}{\epsilon^k}\sup_{s\in K}\left|D^\alpha u_\varepsilon(s)\right|
\end{aligned}$$

$k \in \mathbb{N}$ for some constant $C > 0$
according to Lemma 3.1, $\sup_{s\in K}\left|D^\alpha u_\varepsilon(s)\right|$ has a moderate bound.

Therefore $\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}\tilde{D}^\alpha u_\varepsilon(x)\right|$ has a moderate bound,too. □

**Lemma 3.7.** *Let $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$ be two different representative of $u \in \mathcal{G}([0,\infty))$.*
*Then, for every $\alpha > 0$, and every $k \in \{0,1,2,\ldots\}$, $\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}(\tilde{D}^\alpha u_{1,\varepsilon}(x) - \tilde{D}^\alpha u_{2,\varepsilon}(x))\right|$ is negligible.*

*Proof.*

$$\begin{aligned}
\sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}(\tilde{D}^\alpha u_{1,\varepsilon}(x) - \tilde{D}^\alpha u_{2,\varepsilon}(x))\right| &= \sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}((D^\alpha u_{1,\varepsilon} * \varphi_\varepsilon)(x) - (D^\alpha u_{2,\varepsilon} * \varphi_\varepsilon)(x))\right| \\
&= \sup_{x\in[0,T]}\left|\frac{d^k}{dx^k}((D^\alpha u_{1,\varepsilon} - D^\alpha u_{2,\varepsilon}) * \varphi_\varepsilon)(x)\right| \\
&= \sup_{x\in[0,T]}\left|((D^\alpha u_{1,\varepsilon} - D^\alpha u_{2,\varepsilon}) * \frac{d^k}{dx^k}\varphi_\varepsilon)(x)\right| \\
&\leq \sup_{s\in K}\left|(D^\alpha u_{1,\varepsilon} - D^\alpha u_{2,\varepsilon})(s)\right| \sup_{x\in[0,T]}\left|\int_K \frac{d^k}{dx^k}\varphi_\varepsilon(x-s)ds\right| \\
&\leq C\sup_{s\in K}\left|(D^\alpha u_{1,\varepsilon} - D^\alpha u_{2,\varepsilon})(s)\right|
\end{aligned}$$

by Lemma 3.2 $\sup_{s\in K}\Big|(D^\alpha u_{1,\varepsilon}-D^\alpha u_{2,\varepsilon})(s)\Big|$ is negligible then $\sup_{x\in[0,T]}\Big|\frac{d^k}{dx^k}(\tilde{D}^\alpha u_{1,\varepsilon}(x)-\tilde{D}^\alpha u_{2,\varepsilon}(x))\Big|$ is negligible. □

Now we introduce the regularized new fractional derivative of a Colombeau generalized on $[0,\infty)$ in the following way.

**Definition 3.8.** Let $u\in\mathcal{G}([0,\infty))$ be a Colombeau generalized on $[0,\infty)$ The $\alpha$th fractional derivative of $u$, in notation $\tilde{D}^\alpha u(x)=[\tilde{D}^\alpha u_\varepsilon(x)]$ is the element of $\mathcal{G}([0,\infty))$ satisfying (3.2).

## 4. Fractional integral of Colombeau generalized

Let $(u_\varepsilon)_{\varepsilon>0}$ be a representative of a Colombeau generalized $u\in\mathcal{G}([0,\infty))$. The fractional integral of $(u_\varepsilon)_{\varepsilon>0}$ is defined by

$$I^\alpha u_\varepsilon(x)=\int_0^x(s^{\alpha-1}u_\varepsilon(s))ds \tag{4.1}$$

**Lemma 4.1.** *Let $(u_\varepsilon)_{\varepsilon>0}$ be a representative of $u\in\mathcal{G}([0,\infty))$. Then for every $\alpha>0$, $\sup_{x\in[0,T]}\Big|I^\alpha u_\varepsilon(x)\Big|$ has a moderate bound.*

*Proof.*

$$\begin{aligned}
\sup_{x\in[0,T]}\Big|I^\alpha u_\varepsilon(x)\Big| &= \sup_{x\in[0,T]}\Big|\int_0^x(s^{\alpha-1}u_\varepsilon(s))ds\Big|\\
&\le \sup_{s\in[0,T]}\Big|\,u_\varepsilon(s)\Big|\int_0^x(s^{\alpha-1})ds\\
&\le \frac{T^\alpha}{\alpha}\sup_{s\in[0,T]}\Big|\,u_\varepsilon(s)\Big|\\
&\le C_{\alpha,T}\varepsilon^{-N}
\end{aligned}$$

□

**Lemma 4.2.** *Let $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$ be two different representative of $u\in\mathcal{G}([0,\infty))$. Then for every $\alpha>0$, $\sup_{x\in[0,T]}\Big|I^\alpha u_{1,\varepsilon}(x)-I^\alpha u_{2,\varepsilon}(x)\Big|$ is negligible.*

*Proof.*

$$\begin{aligned}
\sup_{x\in[0,T]}\Big|I^\alpha u_{1,\varepsilon}(x)-I^\alpha u_{2,\varepsilon}(x)\Big| &= \sup_{x\in[0,T]}\Big|\int_0^x(s^{\alpha-1}u_{1,\varepsilon}(s))ds-\int_0^x(s^{\alpha-1}u_{2,\varepsilon}(s))ds\Big|\\
&= \sup_{x\in[0,T]}\Big|\int_0^x(s^{\alpha-1}(u_{1,\varepsilon}(s)-u_{2,\varepsilon}(s))ds\Big|\\
&\le \sup_{s\in[0,T]}\Big|\,u_{1,\varepsilon}(s)-u_{2,\varepsilon}(s)\Big|\Big|\int_0^x(s^{\alpha-1})ds\Big|\\
&\le \frac{T^\alpha}{\alpha}\sup_{s\in[0,T]}\Big|\,u_{1,\varepsilon}(s)-u_{2,\varepsilon}(s)\Big|\Big|
\end{aligned}$$

Since $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$represent the same Colombeau generalized $u$ we have that

$$\sup_{x\in[0,T]}\Big|u_{1,\varepsilon}(x)-u_{2,\varepsilon}(x)\Big|$$

is negligible.Therefore,

$$\sup_{x\in[0,T]}\Big|I^{\alpha}u_{1,\varepsilon}(x)-I^{\alpha}u_{2,\varepsilon}(x)\Big|$$

is negligible, too. □

After proving the previous two lemmas we are able to introduce the $\alpha$ fractional integral of a Colombeau generalized on $[0,\infty)$.

**Definition 4.3.** Let $u\in\mathcal{G}([0,\infty))$ be a Colombeau generalized on $[0,\infty)$ The $\alpha$th fractional integral of $u$, in notation $I^{\alpha}u(x)=[I^{\alpha}u_{\varepsilon}(x)]$ is an element of $\mathcal{G}^{0}([0,\infty))$ satisfying (4.1).

*Remark* 4.4. For $0<\alpha<1$ the first-order derivative of $\frac{d}{dx}I^{\alpha}u_{\varepsilon}(x)=x^{\alpha-1}u_{\varepsilon}(x)$ and it does not reach its upper limit . In general, the $k$th order derivative $\frac{d^k}{dx^k}I^{\alpha}u_{\varepsilon}(x)$ it does not reach its upper limit on $[0,\infty)$

The new fractional integral of a Colombeau generalized $u\in\mathcal{G}([0,\infty))$ If one wants this to be of $\mathcal{G}([0,\infty))$, then the regularization of the fractional integral has to be done.

**Definition 4.5.** Let $(u_{\varepsilon})_{\varepsilon>0}$ be a representative of a Colombeau generalized $u\in\mathcal{G}([0,\infty))$. The regularized new fractional integral of $(u_{\varepsilon})_{\varepsilon>0}$, is defined by:

$$\tilde{I}^{\alpha}u_{\varepsilon}(x)=(I^{\alpha}u_{\varepsilon}*\varphi_{\varepsilon})(x),0<\alpha<1 \tag{4.2}$$

where $I^{\alpha}u_{\varepsilon}(x)$ is given by (4.1) and $\varphi_{\varepsilon}(x)$is given by (2.1).

The convolution in (4.2) is $I^{\alpha}u_{\varepsilon}*\varphi_{\varepsilon}(x)=\displaystyle\int_0^{\infty}I^{\alpha}u_{\varepsilon}(s)\varphi_{\varepsilon}(x-s)ds$

**Lemma 4.6.** *Let $(u_{\varepsilon})_{\varepsilon>0}$ be a representative of $u\in\mathcal{G}([0,\infty))$. Then, for every $\alpha>0$ and every $k\in\{0,1,2,...\}$,*

$$\sup_{x\in[0,T]}\Big|\frac{d^k}{dx^k}\tilde{I}^{\alpha}u_{\varepsilon}(x)\Big|$$

*has a moderate bound.*

*Proof.* Let $\epsilon\in(0,1)$ $0<\alpha<1$, we have

$$\begin{aligned}\sup_{x\in[0,T]}\Big|\tilde{I}^{\alpha}u_{\varepsilon}(x)\Big| &= \sup_{x\in[0,T]}\Big|(I^{\alpha}u_{\varepsilon}*\varphi_{\varepsilon})(x)\Big|\\ &\le \sup_{x\in[0,T]}\Big|\int_0^{\infty}I^{\alpha}u_{\varepsilon}(s)\varphi_{\varepsilon}(x-s)ds\Big|\\ &\le \sup_{s\in K}\Big|I^{\alpha}u_{\varepsilon}(s)\Big|\sup_{x\in[0,T]}\Big|\int_K\varphi_{\varepsilon}(x-s)ds\Big|\\ &\le C\sup_{s\in K}\Big|I^{\alpha}u_{\varepsilon}(s)\Big|\end{aligned}$$

for some constant $C > 0$
Since, according to Lemma 3.1, $\sup_{s\in K}\Big|I^{\alpha}u_{\varepsilon}(s)\Big|$ has a moderate bound, for every $\alpha > 0$, it follows that, $\sup_{x\in[0,T]}\Big|\tilde{I}^{\alpha}u_{\varepsilon}(x)\Big|$ has a moderate bound, too.
For arbitrary order derivative, we have

$$\begin{aligned}\sup_{x\in[0,T]}\Big|\frac{d^k}{dx^k}\tilde{I}^{\alpha}u_{\varepsilon}(x)\Big| &\leq \sup_{s\in K}\Big|I^{\alpha}u_{\varepsilon}(s)\Big|\sup_{x\in[0,T]}\Big|\int_K \frac{d^k}{dx^k}\varphi_{\varepsilon}(x-s)ds\Big| \\ &\leq \frac{C}{\epsilon^k}\sup_{s\in K}\Big|I^{\alpha}u_{\varepsilon}(s)\Big|\end{aligned}$$

$k \in \mathbb{N}$ for some constant $C > 0$
according to Lemma 3.1, $\sup_{s\in K}\Big|I^{\alpha}u_{\varepsilon}(s)\Big|$ has a moderate bound.
Therefore $\sup_{x\in[0,T]}\Big|\frac{d^k}{dx^k}\tilde{I}^{\alpha}u_{\varepsilon}(x)\Big|$ has a moderate bound, too. □

**Lemma 4.7.** *Let $(u_{1,\varepsilon})_{\varepsilon>0}$ and $(u_{2,\varepsilon})_{\varepsilon>0}$ be two different representative of $u \in \mathcal{G}([0,\infty))$. Then, for every $\alpha > 0$, and every $k \in \{0,1,2,\ldots\}$,*

$$\sup_{x\in[0,T]}\Big|\frac{d^k}{dx^k}(\tilde{I}^{\alpha}u_{1,\varepsilon}(x) - \tilde{I}^{\alpha}u_{2,\varepsilon}(x))\Big|$$

*is negligible.*

*Proof.*

$$\begin{aligned}\sup_{x\in[0,T]}\Big|\frac{d^k}{dx^k}(\tilde{I}^{\alpha}u_{1,\varepsilon}(x) - \tilde{I}^{\alpha}u_{2,\varepsilon}(x))\Big| &= \sup_{x\in[0,T]}\Big|\frac{d^k}{dx^k}((I^{\alpha}u_{1,\varepsilon} * \varphi_{\varepsilon})(x) - (I^{\alpha}u_{2,\varepsilon} * \varphi_{\varepsilon})(x))\Big| \\ &= \sup_{x\in[0,T]}\Big|\frac{d^k}{dx^k}((I^{\alpha}u_{1,\varepsilon} - I^{\alpha}u_{2,\varepsilon}) * \varphi_{\varepsilon})(x)\Big| \\ &= sup_{x\in[0,T]}\Big|((I^{\alpha}u_{1,\varepsilon} - I^{\alpha}u_{2,\varepsilon}) * \frac{d^k}{dx^k}\varphi_{\varepsilon})(x)\Big| \\ &\leq \sup_{s\in K}\Big|(I^{\alpha}u_{1,\varepsilon} - I^{\alpha}u_{2,\varepsilon})(s)\Big|\sup_{x\in[0,T]}\Big|\int_K \frac{d^k}{dx^k}\varphi_{\varepsilon}(x-s)ds\Big| \\ &\leq C\sup_{s\in K}\Big|(I^{\alpha}u_{1,\varepsilon} - I^{\alpha}u_{2,\varepsilon})(s)\Big|\end{aligned}$$

By to lemma 4.2, $\sup_{s\in K}\Big|(I^{\alpha}u_{1,\varepsilon} - I^{\alpha}u_{2,\varepsilon})(s)\Big|$ is negligible .
Then, $\sup_{x\in[0,T]}\Big|\frac{d^k}{dx^k}(\tilde{I}^{\alpha}u_{1,\varepsilon}(x) - \tilde{I}^{\alpha}u_{2,\varepsilon}(x))\Big|$ is negligible. □

Now we introduce the regularized new fractional integral of a Colombeau generalized on $[0,\infty)$ in the following way.

**Definition 4.8.** Let $u \in \mathcal{G}([0,\infty))$ be a Colombeau generalized on $[0,\infty)$ The $\alpha$th fractional integral of $u$, in notation

$$\tilde{I}^{\alpha}u(x) = [\tilde{I}^{\alpha}u_{\varepsilon}(x)]$$

is the element of $\mathcal{G}([0,\infty))$ satisfying (4.1).

**Proposition 1.** Let $u \in \mathcal{G}([0,\infty))$, then

$$D^\alpha I^\alpha u = u$$

*Proof.* Let $(u_\varepsilon)_{\varepsilon>0}$ be a representative of a Colombeau generalized of $u \in \mathcal{G}([0,\infty))$

$$\begin{aligned}
D^\alpha I^\alpha (u_\varepsilon)(x) &= (x^{1-\alpha})\frac{d}{dx} I^\alpha (u_\varepsilon(x)) \\
&= (x^{1-\alpha})\frac{d}{dx}\int_0^x (s^{\alpha-1}u_\varepsilon(s))ds \\
&= (x^{1-\alpha})(x^{\alpha-1}u_\varepsilon(x)) \\
&= u_\varepsilon(x)
\end{aligned}$$

□

**Proposition 2.** Let $u \in \mathcal{G}([0,\infty))$, then

$$I^\alpha D^\alpha u = u_\varepsilon(x) - u_\varepsilon(0) \quad \text{and} \quad I^\alpha D^\alpha u \in \mathcal{G}([0,\infty))$$

*Proof.* Let $(u_\varepsilon)_{\varepsilon>0}$ be a representative of a Colombeau generalized of $u \in \mathcal{G}([0,\infty))$

$$\begin{aligned}
I^\alpha D^\alpha (u_\varepsilon)(x) &= \int_0^x (s^{\alpha-1}) D^\alpha (u_\varepsilon(s))ds \\
&= \int_0^x (s^{\alpha-1}(s^{1-\alpha})\frac{d}{ds}u_\varepsilon(s))ds \\
&= \int_0^x \frac{d}{ds}u_\varepsilon(s)ds \\
&= u_\varepsilon(x) - u_\varepsilon(0)
\end{aligned}$$

then

$$I^\alpha D^\alpha u \in \mathcal{G}([0,\infty))$$

□

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