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SECTIONS IN GAP

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ABSTRACT

In this paper we describe a share package XMOD (Alp, Wensley, 1997) of functions for computing with finite, permutation crossed modules, their morphisms and derivations; cat^1-groups, their morphisms and their sections, written using the GAP (Schönert, 1993) group theory programming language. We also give the implementation method of sections to the GAP.

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ÖZET

Bu makalede GAP programının ortak paketi XMod (Alp, Wensley, 1997) tanımlamanın yanı sıra Section ların GAP programına uygulanması incelenmiştir.

1. Introduction

A starting point for this paper was to consider the possibility of implementing functions for doing calculations with crossed modules, derivations, actor crossed

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modules, cat1-groups, sections, induced crossed modules and induced cat1-groups in GAP (Schönert,1993).

We should first explain the importance of crossed modules. The general points are:

- *crossed modules may be thought of as 2-dimensional groups;*
- *a number of phenomena in group theory are better seen from a crossed module point of view;*
- *crossed modules occur geometrically as* $\pi_2(X,A) \rightarrow \pi_1 A$ *when* A *is a subspace of* X *or as* $\pi_1 F \rightarrow \pi_1 E$ *where* $F \rightarrow E \rightarrow B$ *is a fibration;*
- *crossed modules are usefully related to forms of double groupoids.*

Particular constructions, such as induced crossed modules, are important for the applications of the 2-dimensional Van-Kampen Theorem of Brown and Higgins (Brown, Higgins,1978) , and so for the computation of homotopy 2-types.

For all these reasons, the facilitation of the computations with crossed modules should be advantageous. It should help to solve specific problems, and it should make it easier to construct examples and see relations with better known theories.

The powerful computer algebra system GAP provides a high level programming language with several advantages for the coding of new mathematical structures. The GAP system has been developed over the last 15 years at RWTH in Aachen. Some of its most exciting features are:

• *it has a highly developed, easy to understand programming language incorporated;*

• *it is especially powerful for group theory;*

• *it is portable to a wide variety of operating systems on many hardware platforms.*

• *it is public domain and it has a lively forum, with open discussion. These make it increasingly used by the mathematical community.*

On the other hand, GAP has some disadvantages, too:

• *the built in programming language is an interpreted language, which makes GAP programs relatively slow compared to compiled languages such as C or Pascal. GAP source can not be compiled. This will change in version 4 to be released during 1997;*

• *the demands on system resources are quite high for the serious calculations.*

However, the advantages outweigh the disadvantages, and so GAP was chosen.

Our aim in this paper is to describe a share package XMOD (Alp, Wensley, 1997) for the GAP group theory language which enables computations with the equivalent notions of finite, permutation *crossed modules* and *cat1-groups*.

The term crossed module was introduced by J. H. C. Whitehead in (Whitehead,1946) . Most references of crossed modules state the axioms of a crossed module using left actions, but we shall use right actions since this is the convention used by most computational group packages.

In (Loday,(1982) Loday reformulated the notion of a crossed module as a 1-cat group and showed that the category **XMod** is equivalent to the category **Cat1** of cat1-groups. Loday also generalised the notion of cat1-group to that of catn-group, for all $n \geq 1$ (although he used the term n-cat-group).Crossed modules and their higher analogues were considered by Ellis in (Ellis,1984).

The category **XMod** is also equivalent to the category **GpGpd** of group-groupoids and to the category of 1-truncated simplicial groups with trivial Moore complex.

In section 2 we recall the basic properties of crossed modules and their derivations and of cat^1-groups and their sections. We also gave the implementation method and implementation algorithms in section 3.

2. Crossed Modules and Cat^1-Groups

In this section we recall the descriptions of three equivalent categories: **XMod**, the category of crossed modules and their morphisms; **Cat1**, the category of cat^1-groups and their morphisms; and

GpGpd, the subcategory of groups in the category **Gpd** of groupoids. We also describe functors between these categories which exhibit the equivalences.

A crossed module $X = (\partial : S \to R)$ consists of a group homomorphism ∂, called the *boundary* of X , together with an action $\alpha : R \to \mathrm{Aut}\,(S)$ satisfying, for all $s, s' \in S$ and $r \in R$,

XMod 1: $\partial\left(s^{r}\right) = r^{-1}(\partial\, s)r$,

XMod 2: $s^{\partial s'} = s'^{-1} s s'$.

The kernel of ∂ is abelian.

The standard examples of crossed modules are:

1. *Any homomorphism* $\partial : R \to R$ *of abelian groups with* R *acting trivially on* S *may be regarded as a crossed module.*

2. *A conjugation crossed module is an inclusion of a normal subgroup* $S \trianglelefteq R$, *where* R *acts on* S *by conjugation.*
3. *A central extension crossed module has as boundary a surjection* $\partial : R \rightarrow R$ *with central kernel, where* $r \in R$ *acts on* S *by conjugation with* $\partial^{-1} r$.
4. *An automorphism crossed module has as range a subgroup* R *of the automorphism group Aut(*S*) of* S *which contains the inner automorphism group of* S. *The boundary maps* $s \in S$ *to the inner automorphism of* S *by* s.
5. *An* R*-Module crossed module has an* R*-module as source and* ∂ *is the zero map.*
6. *The direct product* $X_1 \times X_2$ *of two crossed modules has source* $S_1 \times S_2$, $R_1 \times R_2$ *and boundary* $\partial_1 \times \partial_2$, *with* R_1, R_2 *acting trivially on* S_1, S_2 *respectively.*

An important motivating topological example of crossed module due to Whitehead (Whitehead,1949) is the boundary $\partial : \pi_2(X, A, x) \rightarrow \pi_1(A, x)$ from the second relative homotopy group of a based pair (X, A, x) of topological spaces, with the usual action of the fundamental group $\pi_1(A, x)$.

A morphism between two crossed modules X_1 and X_2 is a pair (σ, ρ), where $\sigma : S_1 \rightarrow S_2$ and $\rho : R_1 \rightarrow R_2$ are homomorphisms satisfying

$$\partial_2 \sigma = \rho \partial_1, \sigma(s^r) = (\sigma s)^{\rho r}.$$

When $X_2 \times X_1$ and σ, ρ are automorphisms then (σ, ρ) is an automorphism of X_1. The group of automorphisms is denoted by Aut(X_1).

The Whitehead monoid Der(X) of X was defined in (Whitehead,1949) to be the monoid of all *derivations*

from R to S, that is the set of all maps $R \rightarrow S$, with composition o, satisfying

$$\textbf{Der 1}: \chi(qr) = (\chi q)^r (\chi r)$$
$$\textbf{Der 2}: (\chi_1 \circ \chi_2)(r) = (\chi_1 r)(\chi_2 r)(\chi_1 \partial \chi_2 r).$$

Invertible elements in the monoid are called *regular*. The Whitehead group $W(X)$ is the group of Der(X). The *actor* of X is a crossed module $(\Delta : W(X) \rightarrow \mathrm{Aut}(X))$

Which was shown by Lue and Norrie, in (Lue,1979) and (Norrie,1987), to be the automorphism object of X in the category **XMod.**

The standard examples of Whitehead groups (Gilbert, 1990) are :

1. *If S is a R - module, then the trivial homomorphism $S \to R$ is a crossed module and Der(X) = $W(X)$ is the usual abelian group of derivations.*
2. *Together with the conjugation action of a group R on itself, the identity map $X=(id=R \to R)$ is a crossed module. An automorphism α of R determines its displacement derivation $\delta_\alpha \in W(X)$ given by $\delta_\alpha(r)=\alpha(r)r^{-1}$, and the correspondence $\alpha \to \delta_\alpha$ is an isomorphism $\delta : Aut R \to W(X)$.*
3. *Generalising (ii) we have the inclusion $S \to R$ of a normal S of a group R, with R acting by conjugation. Then $W(X)$ is isomorphic to the subgroup of $Aut R$ consisting of all those α whose displacement derivations take values in S,*
$$W(X) \cong \{\alpha \in Aut R \mid \forall \ r \in R, \alpha(r)r^{-1} \in S \}$$
In particular, if S is a characteristic subgroup of R, then $W(X)$ is the kernel of the canonical map from $Aut R$ to $Aut(R/S)$.
4. *A crossed module $X=(\partial : S \to R)$ with surjective boundary map amounts to a central extension of $\ker\partial$ by R; so let E be a group and K a central subgroup of E. Let $Aut_K E$ be the subgroup of $AutE$ consisting of those automorphism of E that act trivially on K. The natural map $N = (\nu : E \to E/K)$ is a crossed module and $W(N)$ is isomorphic to $Aut_K E$.*

In (Loday , 1982) Loday reformulated the notion of a crossed module as a cat^1-group, namely a group G with a pair of homomorphisms $t,h : G \to G$ having a common image R and satisfying certain axioms. We find it convenient to define a cat^1-group $C=(e;t,h : G \to R)$ as having source group G, range group R, and three homomorphisms: two surjections $t,h:G \to R$ and an embedding $e : R \to G$ satisfying:

Cat 1 : $te=he=\mathrm{id}_R$,

Cat 2 : $[\ker t, \ker h]=\{1_G\}$.

The maps t and h are usually referred to as the *source* and *target*, but we choose to call them the *tail* and *head* of C, because *source* is the GAP term for the domain of a function.

A morphism $C_1 \to C_2$ of cat^1-groups is a pair (γ, ρ) where $\gamma : G_1 \to G_2$ and $\rho : R_1 \to R_2$ are homomorphisms satisfying

$$h_2\gamma = \rho\ h_1,\quad t_2\gamma = \rho\ t_1,\quad e_2\gamma = \rho\ e_1$$

The construction for cat^1-groups equivalent to the derivation of a crossed module is the *section.* The monoid of sections of C is the set of group homomorphisms $\xi : R \to G$, with composition $\circ$, satisfying:

Sect 1 : $t\xi = \mathrm{id}_R$,
Sect 2 : $(\xi_1 \circ \xi_2)(r) = (\xi_2 r)(eh\xi_2 r)^{-1}(\xi_1 h\xi_2 r)$

The embedding e is the identity for this composition, and $h(\xi_1 \circ \xi_2) = (h\xi_1)(h\xi_2)$. A section is *regular* when $h\xi$ is an automorphism and, of course, the group of regular sections is isomorphic to the Whitehead group.

The crossed module X associated to C has $S = \ker t$ and $\partial = h\backslash s$. The cat^1-group associated to X has $G = R \propto S$, using the action from X, and

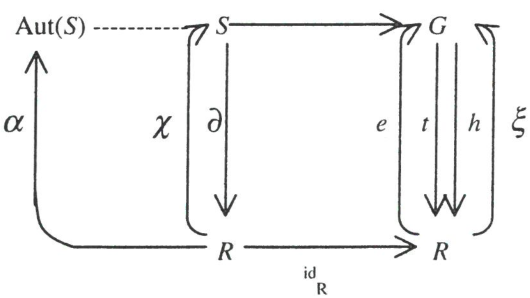
$$t(r,s) = r, \ ; \ h(r,s) = r\,(\partial\, s), \ ; \ er = (r,1).$$

We denote by ε the inclusion of S in G.

The equation $\xi r = (er)(\chi r)$ defines a section ξ of C, given a derivation χ of X, and conversely. Each χ or ξ determines endomorphisms of R, S, G,X and C, namely

ρ :	$R \to R$,	$r \mapsto r(\partial\,\chi r) = h\xi\, r$,
σ :	$S \to S$,	$s \mapsto s(\chi \partial\, s)$,
γ :	$G \to G$,	$g \mapsto (eh\xi\, tg)(\xi\, tg^{-1})g(ehg^{1})(\xi\, hg)$
(σ, ρ) :	$X \to X$,	
(γ, ρ) :	$C \to C$	

The accompanying diagram shows the relationship between the various groups and homomorphisms.



3. GAP Implementation

In order to represent crossed modules, their derivations and actors, cat^1-groups and their sections within GAP we utilise the record structure in GAP3. The standard method in GAP is to represent a new algebraic structure as a record with a set of fields. All records with the same structure are allocated an operations record, namely a record whose fields are functions which operate on the structure. Thus the operations record **XModOps** contains functions such as DirectProduct, IsSimplyConnected and Actor. Similarly the operations record **XModMorphismOps** contains functions such as Kernel and CompositeMorphism. We describe algorithms for constructing crossed modules and their derivations in (Alp, 1998), (Alp, 1997) and (Alp 1999) We also describe algorithms for constructing sections of cat^1-groups in section 4.

4. Algorithms for Sections

Sections are group homomorphism which satisfy the section conditions **Sect1** and **Sect2**. In the implementation a section is stored as a Group Homomorphism By Images. However, sections are provided with a modified set of operations, **Cat1 Section By Images Ops**, which includes a special

Print function to display the section. There are two more functions, **SectionDerivation** and **DerivationSection**, which convert derivations to sections and vice-versa.

4.1. Record structures for Sections

A section $\xi : R \rightarrow G$ is stored as a record with fields:

xi.source	the range group R of cal C,
xi.range	the source group G of cal C,
xi.generators	a fixed generating set for R,
xi.genimages	the chosen images of the generators,
xi.cat1	the cat^1-group C,
xi.operations	special set of operations **Cat1SectionBy Images Ops**,
xi.isSection	a boolean flag normally true.

There are two functions to calculate sections, **RegularSections** and **AllSections**. Both create or modify a record **Sec = C.sections** with fields:

Sec.areSections,	a boolean flag, normally **true**,
Sec.isReg,	true when only the regular sections are known,
Sec.isAll,	**true** when all the sections have been found,
Sec.generators,	a emphcopy of **R.generators**,
Sec.genimageList,	a list of **.genimages** lists for the sections,
Sec.regular,	the number of regular sections (if known),
Sec.cat1,	the cat^1-group cal C,
Sec.operations,	a special set of operations **Cat1SectionsOps**.

4.2. Algorithm for Cat1SectionByImages

The function **Cat1SectionByImages** is called as:

gap> Cat1SectionByImages(C, im);

The input parameters are a cat^1-group and a list of images. As output, the function returns a record as described in section 4.1.

Step 1 Check that the given arguments are of the correct form.
Step 2 Construct a map $\xi : R \rightarrow G$ using **GroupHomomorphismByImages(R, G, genR, im);**.
Step 3 Set up the record fields as described in section 4.1.
Step 4 **Call IsSection(xi)**; to verify axioms SECT1 and SECT2.

5. RegularSections and AllSections

It is easier to test that a prospective ξ is a homomorphism than that a map χ satisfies axiom **Der 1.** However, since $\xi r = (er)(\chi r)$, a section ξ is determined by a choice of χr_i for each r_i in a generating set $\{r_1, r_2, ...\}$ for R.

Since $r^{-1}(\rho r) = \partial \chi r$ it follows that $\chi r \in \partial^{-1}(r^{-1}(\rho r))$. In order to find the regular sections, we use the standard GAP> function **AutomorphismGroup(R)** to obtain Aut (R). Then, for each $\rho \in$ Aut (R), we make a list of preimages

$$[\partial^{-1}(r_1^{-1}(\rho r_1)), \partial^{-1}(r_2^{-1}(\rho r_2)), ...].$$

A backtrack procedure is then used to select χr_1 , χr_2 , ... from these preimage lists, with each selection being tested to see whether it provides a partial homomorphism $R \rightarrow G$.

A similar strategy is used to find emphall the sections, replacing **Aut(R)** by the endomorphism monoid **End(R)**. Since no standard GAP> function yet exists for computing **End(R)**, we have added a function **EndomorphismClasses(R)**. An endomorphism of R is determined by

- *a normal subgroup N of R, a permutation representation of the quotient $\theta : R/N \rightarrow Q$, giving a projection $\theta \circ \nu : R \rightarrow Q$, where $\nu : R \rightarrow R/N$ is the natural homomorphism;*
- *an automorphism α of Q;*
- *a subgroup H' in a conjugacy class [H] of subgroups of R isomorphic to Q having representative H, an isomorphism $\phi : Q \cong H$, and a conjugating elemert $r \in R$ such that $H^r = H'$.*

Endomorphisms are placed in the same class if they have the same choice of N and $[H]$, so the number of endomorphisms is

$$\left| \text{End(R)} \right| = \sum_{classes} \left| \text{Aut (Q)} \right| \left| \text{[H]} \right|.$$

The function returns a record **E = R.endomorphismClasses.classes** with fields

E.quotient,	the group $Q \cong R/N$,
E.autoGroup,	the automorphism group of Q,
E.isomorphism,	the isomorphism phicircthetacircnu,
E.representative,	the subgroup H,
E.conj,	the list of conjugating elements r'.

Functions **RegularSections** and **AllSections** are called as:

gap> RegularSections(C [, method]);
gap> AllSections(C, [, method]);

where method is one of **"endo"** or **"xmod"**. The default method is **"endo"** uses the method described in the previous section. When **"xmod"** is specified the following procedure is used.

Step 1 Call **X := XModCat1(C)**;.
Step 2 Call **D := RegularDerivation(X)**;.
Step 3 For each image **im** in **D.genimageList** call **SectionDerivation(C,im);** to construct the corresponding section.

6. Cat1SectionByImages

Cat1SectionByImages(*C, im*)

This function takes a list of images in G = C.source for the generators of R = C.range and constructs a homomorphism $\xi : R \longrightarrow G$ which is then tested to see whether the axioms of a section are satisfied.

```
gap> SC;
cat1-group [c3^2|Xc2 == >s3]
gap> imxi := [(1,2,3), (1,2)(4,6)];;
gap> xi := Cat1SectionByImages(SC, imxi);
Cat1SectionByImages(s3, c3^2|Xc2, [(4,5,6), (2,3)(5,6)],
        [(1,2,3), (1,2)(4,6)])
```

7. IsSection

IsSection(*C, im*)
IsSection(*xi*)

This function may be called in two ways, and tests that the section given by the images of its generators is well-defined.

```
gap> im0 := [(1,2,3), 2,3)(4,5)];;
gap> IsSection(SC, im0);
false
```

8. RegularSections

RegularSections(*C ["endo" or "xmod"]*)

By default, this function computes the set of idempotent automorphisms from $R \longrightarrow R$ and takes these as possible choices for $h\xi$. A backtrack procedure then calculates possible images for such a section. The result is stored in a sections record C.sections with fields similar to those of a serivations record. The alternative strategy, for which "xmod" option should be specified is to calculate the regular derivations of the associated crossed module first, and convert the resulting derivations to sections.

```
gap> Unbind(XSC.derivations);
gap> regSC := RegularSections(SC);
RegularSections record for cat1-group [c3^2|Xc2 == >s3],
: 6 regular sections, others not found.
```

9. AllSections

AllSections(*C [,"endo"or "xmod"]*)

By default, this function computes the set of idempotent endomorphisms from $R \longrightarrow R$ and takes these as possible choices for the composite homomorphism $h\xi$. A backtrack procedure then calculates possible images for such a section. This function calculates all the sections of C and overwrites any existing subfields of C.sections.

```
gap> allSC := AllSections(SC);
AllSections record for cat1-group [c3^2|Xc2 ==> s3],
: 6 regular sections, 3 irregular ones found.
gap> RecFields(allSC);
["areSections", "isReg", "isAll", "regular", "genimageList",
"generators", "cat1", "operations"]
gap> PrintList(allSC.genimageList);
[(4, 5, 6), (2, 3)(5, 6)]
[(4, 5, 6), (1, 3)(4, 5)]
[(4, 5, 6), (1, 2)(4, 6)]
[(1, 3, 2)(4, 6, 5), (2, 3)(5, 6)]
[(1, 3, 2)(4, 6, 5), (1, 3)(4, 5)]
[(1, 3, 2)(4, 6, 5), (1, 2)(4, 6)]
[(1, 2, 3), (2, 3)(5, 6)]
[(1, 2, 3), (1, 2)(4, 6)]
[(1, 2, 3), (1, 3)(4, 5)]
gap> allXSC := AllDerivations(XSC, "cat1");
AllDerivations record for crossed module [c3- >s3],
: 6 regular derivations, 3 irregular ones found.
```

10. AreSections

AreSections(*S*)

This function checks that the record S has the correct fields for a sections record (regular or all).

```
gap> AreSections(allSC);
true
```

11. SectionDerivation

SectionDerivation(*D, i*)

This function converts a derivation of X to a section of the associated cat1-group C. This function is inverse to **DerivationSection**. In the following examples we note that allXSC has been obtained using allSC, so the derivations and sections correspond in the same order.

```
gap> chi8 := XModDerivationByImages(XSC, allXSC.genimageList[8]);
XModDerivationByImages(s3, c3, [(4,5,6), (2,3)(5,6)],
        [(1,2,3)(4,6,5), (1,2,3)(4,6,5)])
gap> xi8 := SectionDerivation(chi8);
GroupHomomorphismByImages(s3, c3^2|Xc2,
        [(4,5,6), (2,3)(5,6)], [(1,2,3), (1,2)(4,6)])
```

12. DerivationSection

DerivationSection(*C, xi*)

This function converts a section of C to a derivation of the associated crossed module X. This function is inverse to SectionDerivation.

```
gap> xi4 := Cat1SectionByImages(SC, allSC.genimageList[4]);
Cat1SectionByImages(s3, c3^2|Xc2, [(4,5,6), (2,3)(5,6)],
        [(1,3,2)(4,6,5), (2,3)(5,6)])
gap> chi4 := DerivationSection(xi4);
XModDerivationByImages(s3, c3, [(4,5,6), (2,3)(5,6)],
        [(1,3,2)(4,5,6), ()])
```

13. CompositeSection

CompositeSection(*xi, xj*)

This function applies the Whitehead composition to two sections and returns the composite.

```
gap> xi48 := CompositeSection(xi4, xi8);
Cat1SectionByImages(s3, c3^2|Xc2, [(4,5,6), (2,3)(5,6)],
        [(1,2,3), (1,3)(4,5)])
gap> SectionDerivation(chi48) = xi48;
true
```

14. SourceEndomorphismSection

SourceEndomorphismSection(*xi*)

Each section ξ determines an endomorphism γ of G such that

$$\gamma g = (eh\xi\, tg)(\xi\, tg^{-1})\, g\, (ehg^{-1})\, (\xi\, hg).$$

```
gap> gamma4 := SourceEndomorphismDerivation(xi4);
GroupHomomorphismByImages(c3^2Xc2, c3^2Xc2,
        [(1,2,3), (4,5,6), (2,3)(5,6)], [(1,3,2), (4,6,5), (2,3)(5,6)])
```

15. RangeEndomorphismSection

RangeEndomorphismDerivation(*xi*)

Each derivation ξ determines an endomorphism ρ of R such that $\rho r = h\xi r$.

```
gap> rho4 := RangeEndomorphismDerivation(XSC, 4);
GroupHomomorphismByImages(s3, s3, [(4,5,6), (2,3)(5,6)],
        [(4,6,5), (2,3)(5,6)])
```

16. Cat1EndomorphismSection

Cat1EndomorphismSection(*xi*)

The endomorphisms gamma4, rho4 together determine a pair which may be used to construct an endomorphism of *C*. When the derivation is regular, the resulting morphism is an automorphism, and this construction determines a homomorphism from the Whitehead group to the automorphism group of C.

```
gap> psi4 := Cat1EndomorphismSection(xi4);
Morphism of cat1-groups < [c3^2|Xc2 == >s3]-- >[c3^2|Xc2 == >s3] >
```

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