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# HE'S VARIATIONAL ITERATION METHOD FOR SOLVING MODELLING THE POLLUTION OF A SYSTEM OF LAKES

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**ABSTRACT**

In this papers,He's variational iteration method(VIM) is implemented to for solving analytically systems of nonlinear ordinary differential equations such as modelling the pollution of a system of lakes. The proposed scheme is based on variational iteration method (VIM), Laplace transform and Padé approximants. The results to get the variational iteration method (VIM) are applied Padé approximants. Our proposed approach showed results to analytical solutions of nonlinear ordinary differential equation systems. The results are compared with the results obtained by MATLAB ode15s and the variational iteration method (VIM) are applied Padé approximants. Some plots are presented to show the reliability and simplicity of the methods.

**Key Words-** Padé approximants; Variational iteration method; Modelling the pollution of a system of lakes

## GÖLLER SİSTEMİNİN KİRLİLİK MODELİNİN ÇÖZÜMÜ İÇİN HE'NİN VARYASYONEL İTERASYON YÖNTEMİNİN UYGULANMASI

**ÖZET**

Bu çalışmada göller sisteminin kirlilik modeli gibi nonlineer adi diferensiyel denklem sisteminin analitik çözümü için He'nin varyasyonel iterasyon yöntemi uygulandı. Önerilen yaklaşım varyasyonel iterasyon yöntemi, Laplace dönüşümü ve Padé yaklaşımlarını baz almaktadır. Varyasyonel iterasyondan elde edilen sonuçlara Padé yaklaşımları uygulanmıştır. Önerdiğimiz yaklaşım ile nonlinear adi diferensiyel denklem sisteminin analitik çözümleri gösterildi. Matlab ode15s den elde edilen sonuçlar ile VIM'e Padé yaklaşımı uygulandıktan sonra elde edilen sonuçlar karşılaştırıldı. Yöntemlerin güvenilirliği ve basitliğini göstermek için bazı grafikler sunuldu.

**Anahtar Kelimeler**: Padé yaklaşımı; Varyasyonel iterasyon yöntemi; Göller sistemi için kirlilik modeli

## 1. INTRODUCTION

Modelling the pollution of a system of lakes is examined [1] at the study. The system of three lakes that are modeled in this study [2]. Each lake is considered to be a large compartment and the interconnecting channel as pipes between the compartments. The direction of flow in the channels or pipes is indicated by the arrows in [2]. A pollutant is introduced into the first lake where $p(t)$ denotes the rate at which the pollutant enters the lake per unit time. The function $p(t)$ may be constant or may vary with time. We are interested in knowing the levels of pollution in each lake at any time.

The components of the basic three-component model are the amount of the pollutant in lake 1 at any time $t \geq 0$, the amount of the pollutant in lake 2 at any time $t \geq 0$ and the amount of the pollutant in lake 3 at any time $t \geq 0$, are denoted respectively by $x(t)$, $y(t)$ and $z(t)$.These quantities satisfy

$$\begin{aligned} \frac{dx}{dt} &= \frac{F_{13}}{V_3} z(t) + p(t) - \frac{F_{31}}{V_1} x(t) - \frac{F_{21}}{V_1} x(t) \\ \frac{dy}{dt} &= \frac{F_{21}}{V_1} x(t) - \frac{F_{32}}{V_2} y(t) \\ \frac{dz}{dt} &= \frac{F_{31}}{V_1} x(t) + \frac{F_{32}}{V_2} y(t) - \frac{F_{13}}{V_3} z(t). \end{aligned} \tag{1}$$

with the initial conditions:

$x(0) = r_1, \quad y(0) = r_2, \quad z(0) = r_3$. Throughout this paper, we assume the following conditions:

Lake 1: $F_{13} = F_{21} + F_{31}$,

Lake 2: $F_{21} = F_{32}$,

Lake 3: $F_{31} + F_{32} = F_{13}$

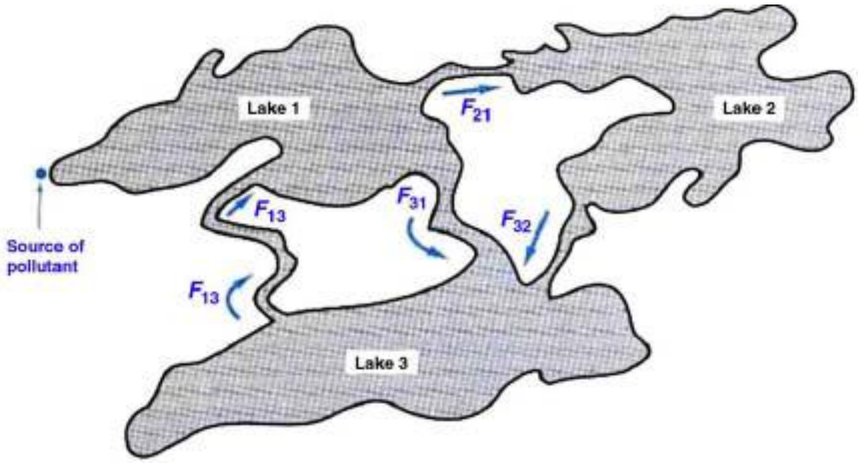


**Figure 1**. System of three lakes with interconnecting channels. A pollutant enters the first lake at the indicated source [1,4].

The variational iteration method (VIM) was first proposed by He [6-14] who was successfully applied to autonomous ordinary differential equation [13], to nonlinear partial differential equations with variable coefficients [14], Schrodinger–KdV, generalized KdV and shallow water equations [17], nonlinear evolution equations [18], nonlinear systems of partial differential equations [18], nonlinear heat transfer equations [19], Burger's and coupled Burger's equations [20], the epidemic model and the prey and predator problem [21], linear Helmholtz partial differential equation [22] and recently to nonlinear fractional differential equations with Caputo differential derivative [23], and other fields, [24-26].

In addition to the variational iteration method (VIM) proposed for obtaining exact and approximate analytic solutions for nonlinear problems, many different new methods have recently presented some techniques to eliminate the small parameter; for example, the homotopy analysis method [31], and the Adomian's decomposition method (ADM) [32-33], homotopy perturbation method [34–38].

The first connection between series solution methods such as an Adomian decomposition method and Padé approximants was established in [29-30]. The differential transform method is proposed for solving non-linear oscillatory systems[27-28].

In this paper, the variational iteration method (VIM) [6-14] and Padé approximants [15-16] used to solve of modelling the pollution of a system of lakes (1). The numerical solutions are compared with the available exact and by MATLAB ode23s.

## 2 PADÉ APPROXIMATON

A rational approximation to $f(x)$ on $[a, b]$ is the quotient of two polynomials $P_N(x)$ and $Q_M(x)$ of degrees N and M, respectively. We use the notation $R_{N,M}(x)$ to denote this quotient. The $R_{N,M}(x)$ Padé approximations to a function $f(x)$ are given by [15-16]

$$R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)} \quad \text{for } a \le x \le b. \tag{2}$$

The method of Padé requires that $f(x)$ and its derivative be continuous at $x = 0$. The polynomials used in (2) are

$$P_N(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_N x^N \tag{3}$$

$$Q_M(x) = 1 + q_1 x + q_2 x^2 + \ldots + q_M x^M \tag{4}$$

The polynomials in (3) and (4) are constructed so that $f(x)$ and $R_{N,M}(x)$ agree at $x = 0$ and their derivatives up to $N + M$ agree at $x = 0$. In the case $Q_0(x) = 1$, the approximation is just the Maclaurin expansion for $f(x)$. For a fixed value of $N + M$ the error is smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $P_N(x)$ has degree one higher then $Q_M(x)$.

Notice that the constant coefficient of $Q_M$ is $q_0 = 1$. This is permissible, because it notice be 0 and $R_{N,M}(x)$ is not changed when both $P_N(x)$ and $Q_M(x)$ are divided by the same constant. Hence the rational function $R_{N,M}(x)$ has $N + M + 1$ unknown coefficients. Assume that $f(x)$ is analytic and has the Maclaurin expansion

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k + \ldots, \tag{5}$$

And from the difference $f(x)Q_M(x) - P_N(x) = Z(x)$:

$$\left[\sum_{i=0}^{\infty} a_i x^i\right]\left[\sum_{i=0}^{M} q_i x^i\right] - \left[\sum_{i=0}^{N} p_i x^i\right] = \left[\sum_{i=N+M+1}^{\infty} c_i x^i\right], \tag{6}$$

The lower index $j = N + M + 1$ in the summation on the right side of (6) is chosen because the first $N + M$ derivatives of $f(x)$ and $R_{N,M}(x)$ are to agree at $x = 0$.

When the left side of (6) is multiplied out and the coefficients of the powers of $x^i$ are set equal to zero for $k = 0,1,2,\ldots, N + M$, the result is a system of $N + M + 1$ linear equations:

$$\begin{aligned} & a_0 - p_0 = 0 \\ & q_1 a_0 + a_1 - p_1 = 0 \\ & q_2 a_0 + q_1 a_1 + a_2 - p_2 = 0 \\ & q_3 a_0 + q_2 a_1 + q_1 a_2 + a_3 - p_3 = 0 \\ & q_M a_{N-M} + q_{M-1} a_{N-M+1} + a_N - p_N = 0 \end{aligned} \tag{7}$$

and

$$
\begin{aligned}
&q_M a_{N-M+1} + q_{M-1}a_{N-M+2} + \ldots + q_1 a_N \quad + a_{N+2} = 0\\
&q_M a_{N-M+2} + q_{M-1}a_{N-M+3} + \ldots + q_1 a_{N+1} \quad + a_{N+2} = 0\\
&. \qquad\qquad\qquad\qquad .\\
&. \qquad\qquad\qquad\qquad .\\
&. \qquad\qquad\qquad\qquad .\\
&q_M a_N + q_{M-1}a_{N+1} + \ldots + q_1 a_{N+M+1} \quad + a_{N+M} = 0
\end{aligned}
\tag{8}
$$

Notice that in each equation the sum of the subscripts on the factors of each product is the same, and this sum increases consecutively from 0 to $N+M$. The $M$ equations in (8) involve only the unknowns $q_1, q_2, q_3, \ldots, q_M$ and must be solved first. Then the equations in (7) are used successively to find $p_1, p_2, p_3, \ldots, p_N$ [15-16].

### 3. VARIATIONAL ITERATION METHOD

According to the variational iteration method [7], we consider the following differential equation:

$$Lu + Nu = g(x), \tag{9}$$

where $L$ is a linear operator, $N$ is a non-linear operator, and $g(x)$ is an inhomogeneous term. Then, we can construct a
correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left\{Lu_n(s) + N\tilde{u}_n(s) - g(s)\right\} ds, \tag{10}$$

where $\lambda$ is a general Lagrangian multiplier [9–11], which can be identified optimally via variational theory. The second term on the right is called the correction and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$.

### 3.1 Variational Iteration Method to modelling the pollution of a system of lakes

According to the variational iteration method, we derive a correct functional as follows:

$$
\begin{aligned}
x_{n+1}(t) &= x_n(t) + \int_0^t \lambda_1 \left\{ x_n'(\xi) - \frac{F_{13}}{V_3}\tilde{z}_n(\xi) - p(\xi) + \left(\frac{F_{31}}{V_1} + \frac{F_{21}}{V_1}\right)\tilde{x}_n(\xi) \right\} d\xi,\\
y_{n+1}(t) &= y_n(t) + \int_0^t \lambda_2 \left\{ y_n'(\xi) - \frac{F_{21}}{V_1}\tilde{x}_n(\xi) + \frac{F_{32}}{V_2}\tilde{y}_n(\xi) \right\} d\xi,\\
z_{n+1}(t) &= z_n(t) + \int_0^t \lambda_3 \left\{ z_n'(\xi) - \frac{F_{31}}{V_1}\tilde{x}_n(\xi) - \frac{F_{32}}{V_2}\tilde{y}_n(\xi) + \frac{F_{13}}{V_3}\tilde{z}_n(\xi) \right\} d\xi,
\end{aligned}
\tag{11}
$$

where $\lambda_1, \lambda_2$ and $\lambda_3$ are general Lagrange multipliers, $\tilde{x}_n(\xi), \tilde{y}_n(\xi)$ and $\tilde{z}_n(\xi)$ denote restricted variations, i.e. $\delta\tilde{x}_n(\xi) = \delta\tilde{y}_n(\xi) = \delta\,\tilde{z}_n(\xi) = 0$

Making the above correction functionals stationary, we can obtain following stationary conditions:

$$\begin{aligned}
&\lambda_1'(\xi) = 0,\\
&1 + \lambda_1(\xi)\big|_{\xi=t} = 0,\\
&\lambda_2'(\xi) = 0,\\
&1 + \lambda_2(\xi)\big|_{\xi=t} = 0,\\
&\lambda_3'(\xi) = 0,\\
&1 + \lambda_3(\xi)\big|_{\xi=t} = 0,
\end{aligned} \tag{12}$$

The Lagrange multipliers, therefore, can be identified as

$$\lambda_1 = \lambda_2 = \lambda_3 = -1. \tag{13}$$

Substituting Eq. (13) into the correction functional Eq. (11) results in the following iteration formula:

$$\begin{aligned}
x_{n+1}(t) &= x_n(t) - \int_0^t \left\{ x_n'(\xi) - \frac{F_{13}}{V_3} z_n(\xi) - p(\xi) + \left( \frac{F_{31}}{V_1} + \frac{F_{21}}{V_1} \right) x_n(\xi) \right\} d\xi,\\
y_{n+1}(t) &= y_n(t) - \int_0^t \left\{ y_n'(\xi) - \frac{F_{21}}{V_1} x_n(\xi) + \frac{F_{32}}{V_2} y_n(\xi) \right\} d\xi,\\
z_{n+1}(t) &= z_n(t) - \int_0^t \left\{ z_n'(\xi) - \frac{F_{31}}{V_1} x_n(\xi) - \frac{F_{32}}{V_2} y_n(\xi) + \frac{F_{13}}{V_3} z_n(\xi) \right\} d\xi,
\end{aligned} \tag{14}$$

We start with initial approximations $x_0(t) = r_1$, $y_0(t) = r_2$ and $z_0(t) = r_3$ . By the above iteration formula, we can obtain a few first terms being calculated:

$$\begin{aligned}
x_1(t) &= r_1 + \left[ \frac{F_{13} r_3}{V_3} + p - \left( \frac{F_{31} + F_{21}}{V_1} \right) r_1 \right] t,\\
y_1(t) &= r_2 + \left[ \frac{F_{21} r_1}{V_1} - \frac{F_{32} r_2}{V_2} \right] t,\\
z_1(t) &= r_1 + \left[ \frac{F_{31} r_1}{V_1} + \frac{F_{32} r_2}{V_2} - \frac{F_{31} r_3}{V_3} \right] t,
\end{aligned} \tag{15}$$

$$x_2(t) = r_1 + \left[\frac{F_{13}r_3}{V_3} + p - \left(\frac{F_{31}+F_{21}}{V_1}\right)r_1\right]t + \frac{1}{2!}\left[\begin{array}{l}\frac{F_{13}}{V_3}\left(\frac{F_{31}r_1}{V_1} + \frac{F_{32}r_2}{V_2} - \frac{F_{31}r_3}{V_3}\right) \\ -\left(\frac{F_{31}+F_{21}}{V_1}\right)\left(\frac{F_{13}r_3}{V_3} + p - \left(\frac{F_{31}+F_{21}}{V_1}\right)r_1\right)\end{array}\right]t^2$$

$$y_2(t) = r_2 + \left[\frac{F_{21}r_1}{V_1} - \frac{F_{32}r_2}{V_2}\right]t + \frac{1}{2!}\left[\begin{array}{l}\frac{F_{21}}{V_1}\left(\frac{F_{13}r_3}{V_3} + p - \left(\frac{F_{31}+F_{21}}{V_1}\right)r_1\right) \\ -\frac{F_{32}}{V_2}\left(\frac{F_{21}r_1}{V_1} - \frac{F_{32}r_2}{V_2}\right)\end{array}\right]t^2$$

$$z_2(t) = r_1 + \left[\frac{F_{31}r_1}{V_1} + \frac{F_{32}r_2}{V_2} - \frac{F_{31}r_3}{V_3}\right]t + \frac{1}{2!}\left[\begin{array}{l}\frac{F_{31}}{V_1}\left(\frac{F_{13}r_3}{V_3} + p - \left(\frac{F_{31}+F_{21}}{V_1}\right)r_1\right) \\ +\frac{F_{32}}{V_2}\left(\frac{F_{21}r_1}{V_1} - \frac{F_{32}r_2}{V_2}\right) \\ -\frac{F_{13}}{V_3}\left(\frac{F_{31}r_1}{V_1} + \frac{F_{32}r_2}{V_2} - \frac{F_{31}r_3}{V_3}\right)\end{array}\right]t^2 \quad (16)$$

.
.
.

Continuing in this manner, we can find the rest of components.
A five terms approximation to the solutions are considered

$$x(t) \approx x_4, \quad y(t) \approx y_4, \quad z(t) \approx z_4. \quad (17)$$

This was done with the standard parameter values given above and initial values $r_1 = 0$, $r_2 = 0$ and $r_3 = 0$ for the three-component model.
A few eight approximations for $x(t)$, $y(t)$ and $z(t)$ are calculated and presented below:

$$x_1(t) = 100t, \quad y_1(t) = 0, \quad z_1(t) = 0 \quad (18)$$

$$x_2(t) = 100t\text{-}.6552t^2, \quad y_2(t) = .3103t^2, \quad z_2(t) = .3448t^2 \quad (19)$$

.
.
.

$$
\begin{aligned}
x_7(t) &= 100t\text{-}.6552t^2+.399\text{e-}1t^3\text{-}.31\text{e-}2t^4+.19961\text{e-}3t^5\text{-}.60005\text{e-}6t^6 \\
&\quad \text{-}.14047\text{e-}9t^7, \\
y_7(t) &= .3103t^2\text{-}.35\text{e-}2t^3+.80652\text{e-}4t^4\text{-}.41347\text{e-}5t^5+.22109\text{e-}6t^6 \\
&\quad \text{-}.16894\text{e-}9t^7, \\
z_7(t) &= .3448t^2\text{-}.0363t^3+.003t^4\text{-}.19548\text{e-}3t^5+.37896\text{e-}6t^6 \\
&\quad +.30940\text{e-}9t^7,
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
x_8(t) &= 100t\text{-}.6552t^2+.399\text{e-}1t^3\text{-}.31\text{e-}2t^4+.19961\text{e-}3t^5\text{-}.60005\text{e-}6t^6 \\
&\quad \text{-}.14047\text{e-}9t^7+78137\text{e-}12t^8, \\
y_8(t) &= .3103t^2\text{-}.35\text{e-}2t^3+.80652\text{e-}4t^4\text{-}.41347\text{e-}5t^5+.22109\text{e-}6t^6 \\
&\quad \text{-}.16894\text{e-}9t^7\text{-}.37163\text{e-}12t^8, \\
z_8(t) &= .3448t^2\text{-}.0363t^3+.003t^4\text{-}.19548\text{e-}3t^5+.37896\text{e-}6t^6 \\
&\quad +.30940\text{e-}9t^7\text{-}.40974\text{e-}12t^8,
\end{aligned} \tag{21}
$$

In this section, we apply Laplace transformation to (21), which yields

$$
\begin{aligned}
L\left(x(s)\right) &= \frac{100}{s^2}-\frac{1.3104}{s^3}+\frac{.2394}{s^4}-\frac{.0744}{s^5}+\frac{.0239532}{s^6} \\
&\quad -\frac{.000432036}{s^7}-\frac{.7079688000\text{e-}6}{s^8}+\frac{.3150483840\text{e-}7}{s^9} \\
L\left(y(s)\right) &= \frac{.6206}{s^3}-\frac{.021}{s^4}+\frac{.001935648}{s^5}-\frac{.000496164}{s^6} \\
&\quad +\frac{.0001591848}{s^7}-\frac{.8514576000\text{e-}6}{s^8}-\frac{.1498412160\text{e-}7}{s^9} \\
L\left(z(s)\right) &= \frac{.6896}{s^3}-\frac{.2178}{s^4}+\frac{.072}{s^5}-\frac{.0234576}{s^6} \\
&\quad +\frac{.0002728512}{s^7}+\frac{.1559376000\text{e-}5}{s^8}-\frac{.1652071680\text{e-}7}{s^9}
\end{aligned} \tag{22}
$$

For simplicity, let $s=\frac{1}{t}$; then

$$
\begin{aligned}
L(x(t)) &= 100t^2\text{ -}1.3104t^3+.2394t^4\text{-}.0744t^5+.0239532t^6 \\
&\quad \text{-}.000432036t^7\text{-}.7079688000\text{e-}6t^8+.3150483840\text{e-}7t^9 \\
L(y(t)) &= .6206t^3\text{-}.021t^4+.001935648t^5\text{-}.000496164t^6 \\
&\quad +.0001591848t^7\text{-}.8514576000\text{e-}6t^8\text{-}.1498412160\text{e-}7t^9 \\
L(z(t)) &= .6896t^3\text{-}.2178t^4+.072t^5\text{-}.0234576t^6 \\
&\quad +.0002728512t^7+.1559376000\text{e-}5t^8\text{-}.1652071680\text{e-}7t^9
\end{aligned} \tag{23}
$$

Padé approximant $\left[4/4\right]$ of (23) and substituting $t=\dfrac{1}{s}$, we obtain $\left[4/4\right]$ in terms of s. By using the inverse Laplace transformation, we obtain

$$
\begin{aligned}
x(t) &= e^{.008713049932t}\left[-1.147803467\cos(.1347378635t)+264.3722030\sin(.1347378635t)\right] \\
&\quad -282.5854383e^{-.1289986783t}+283.7332416e^{-.0984583508t} \\
y(t) &= e^{.02356019737t}\left[-17.68106240\cos(.1143065208t)+13.61100134\sin(.1143065208t)\right] \\
&\quad +4.484042717e^{.1406499674t}+13.19701968e^{-.1341164440t} \\
z(t) &= e^{-.281996224t}\left[-1.77277691\cos(.1606153457t)-6.970923773\sin(.1606153457t)\right] \\
&\quad e^{.2707467141t}\left[1.77277691\cos(.1716767092t)+.8140149859\sin(.1716767092t)\right]
\end{aligned}
\tag{24}
$$

**Table 1:** Differences between the eighth degree VIM and the the Padé approximations solutions for the modelling the pollution of a system of lakes

| ***t(time)*** | ***x*** | ***y*** | ***z*** |
|---|---|---|---|
| **0** | 1.6700e-007 | 3.0000e-009 | 0 |
| **0.1** | 1.6555e-007 | 3.1004e-009 | 2.0028e-011 |
| **0.2** | 1.6413e-007 | 3.2036e-009 | 4.0210e-011 |
| **0.3** | 1.6275e-007 | 3.3097e-009 | 5.9511e-011 |
| **0.4** | 1.6140e-007 | 3.4186e-009 | 6.9646e-011 |
| **0.5** | 1.6009e-007 | 3.5301e-009 | 3.2838e-011 |
| **0.6** | 1.5882e-007 | 3.6441e-009 | 1.7374e-010 |
| **0.7** | 1.5762e-007 | 3.7596e-009 | 8.7076e-010 |
| **0.8** | 1.5650e-007 | 3.8750e-009 | 2.7770e-009 |
| **0.9** | 1.5555e-007 | 3.9869e-009 | 7.3333e-009 |
| **1** | 1.5489e-007 | 4.0891e-009 | 1.7192e-008 |

Table1 indicates that the differences among the VIM the Padé approximations. Considering this Table 1, we see that VIM the Padé approximations, the difference between the approach to the solutions obtained are very small.

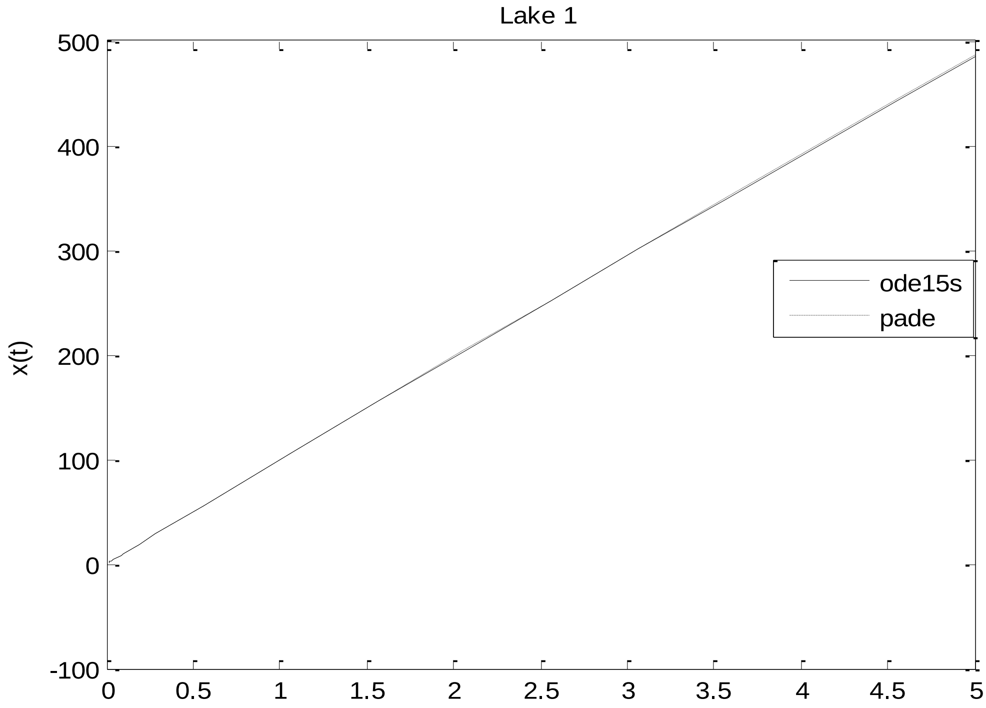


**Figure 2** The comparison of the results of $x(t)$ via the two methods for a system of lakes (1)

Figure 2 indicates that the differences among the ode15s with Padé approximations and the obtained results converge to the ode15s solution and the errors are reduced.

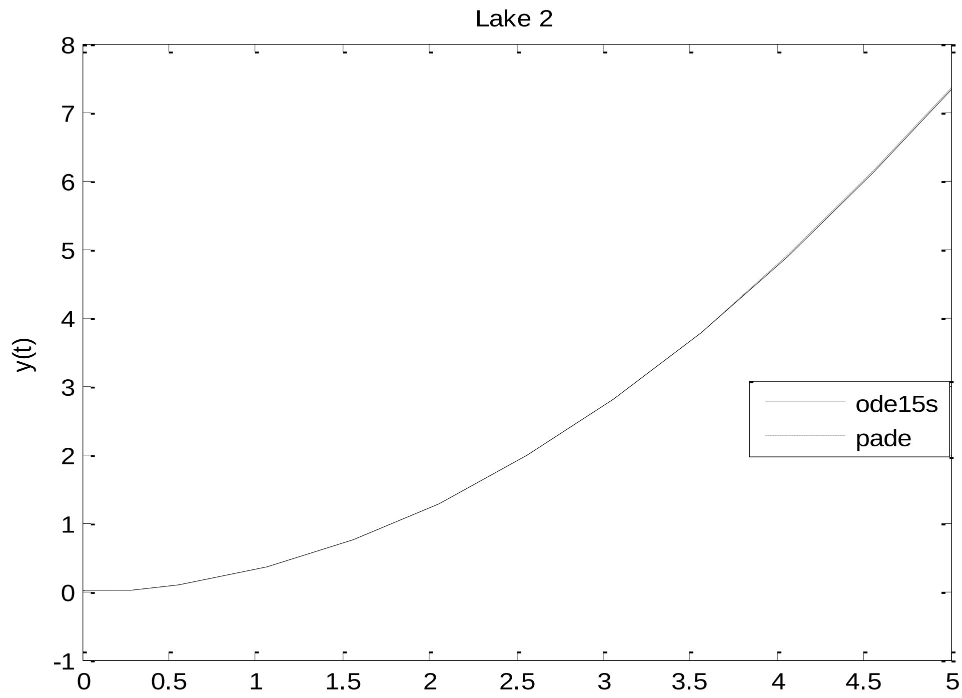


**Figure 3** The comparison of the results of $y(t)$ via the two methods for a system of lakes (1)

Figure 3 indicates that the differences among the ode15s with Padé approximations. Ode15s solution is obtained from the solution of Padé approximations is a good convergence and the error close to zero.

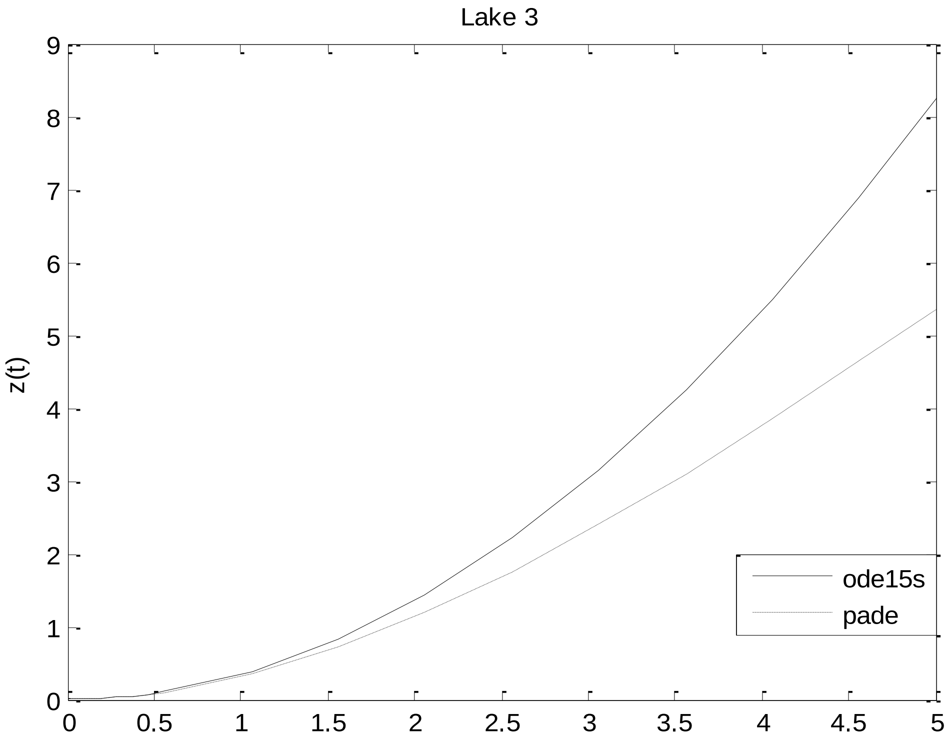


**Figure 4** The comparison of the results of $z(t)$ via the two methods for a system of lakes (1)

Figure 4 indicates that the differences among the ode15s with Padé approximations. The solution obtained from Padé approximations away from the solution is obtained ode15s.
As the plots state the amount of the pollutant in lake 1, lake 2 and lake 3 increase.

## 4. CONCLUSIONS

In this paper, He's variational iteration method was used for finding the solutions of nonlinear ordinary differential equation systems such as modelling the pollution of a system of lakes. The obtained solutions are shown graphically. we have presented an after treatment technique for the variational iteration method. Because the Pade´ approximant usually improves greatly the Maclaurin series in the convergence region and the convergence rate, the at leads to a better analytic approximate solution from variational iteration method truncated series We demonstrated the accuracy and efficiency of these methods by solving some ordinary differential equation systems. We use Laplace transformation and Padé approximant to obtain an analytic solution and to improve the accuracy of variational iteration method. The reliability of the method and reduction in the size of computational domain give this method a wider applicability. It is observed that the results to get the variational iteration method (VIM) applied Padé approximants is an effective and reliable tool for the solution of the nonlinear ordinary differential equation systems considered in the present paper. All the examples show that the results of the present method are in excellent consistency with those obtained by MATLAB ode15s and the variational iteration method.

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