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DİFÜZYON DENKLEMİNİN SINIRLAYICI TAYLOR YAKLAŞIMI YARDIMIYLA NÜMERİK ÇÖZÜMÜ

Ahmet BOZ

Dumlupınar Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Kütahya, ahmetboz@dumlupinar.edu.tr

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ÖZET

Bu çalışmada, lineer difüzyon denkleminin sınırlayıcı Taylor yaklaşımı yardımıyla nümerik çözümleri elde edilmiştir. Difüzyon denkleminin nümerik çözümü için $\exp(xA)$ üstel matris yaklaşımı kullanılmıştır. Bu yaklaşımın avantajı, bazı noktalarda denklemin tam değerine sahip olmasıdır. Difüzyon denklemi için uygulanan yöntem sonucunda elde edilen veriler, yöntemin tutarlı olduğunu göstermektedir.

Anahtar Kelimeler: *Sınırlayıcı Taylor yaklaşımı, Difüzyon denklemi, sonlu farklar*

NUMERICAL SOLUTION OF THE DIFFUSION EQUATION WITH RESTRICTIVE TAYLOR APPROXIMATION

ABSTRACT

In this paper, we solved linear diffusion equation using restrictive Taylor approximations. We use the restrictive Taylor approximation to approximate the exponential matrix $\exp(xA)$. The adventage is that has the exact value at certain point. We will use a new technique for solution of the Diffusion equation. The results show that the used numerical method produce the good results.

Keywords: *Restrictive Taylor approximation, Diffusion equation, Finite difference*

1. INTRODUCTION

The restrictive Taylor approximation are new techniques derived by the Hassan N.A. İsmail at all.[1-4].Gurarslan [8] construct the numerical modelling of linear and nonlinear diffusion equations by compact finite difference method. It is not only an approximation of a real function $f(x)$ but also for functions of matrices related to some difference operators. It has many useful applications for solving initial-boundary-value problems for parabolic and hyperbolic partial differential equations. It gives very fast and accurate results even if the solution is too large. Also it discussed the stability conditions for the mentioned methodsfor all that applications and it has given some estimations for the local truncation error upper bounds for all of the given algorithms.

In this work , we consider the following one dimensional diffusion equation ;

$$u_t = u_{xx} + f(u),\ 0 < x < L,\ t > 0 \tag{1}$$

$$u_t = (D(u)u_x)_x,\ \ 0 < x < L \tag{2}$$

subject to the initial condition

$$u(x,0) = f(x),\ 0 < x < L \tag{3}$$

and boundary conditions

$$u(0,t) = g_0(x),\ t > 0 \tag{4}$$

$$u(1,t) = g_1(x),\ t > 0 \tag{5}$$

The functions $f(u)$ are linear source functions. The function $D(u)$ is the diffusion term that plays a crucial role in a wide range of applications in diffusion proses [5-7]. The diffusion term $D(u)$ appears in several forms. Some of the well known diffusion proses are the fast and the slow diffusion proses where the diffusion term is of the form $D(u) = u^n$ where $n < 0$ and $n > 0$ respectively.

The Taylor series relates the value of a differentiable function at any point to its first and higher order derivatives at a reference point. In certain cases it may be difficult to find analytical solutions of complicated differential and partial differential equations describing the physical systems. In this cases numerical solutions can be obtained by replacing derivatives in the equation by approximations based on the Taylor series. The purpose of this paper is to present a very efficient finite difference method based on restrictive Taylor approximation for solving the diffusion equation.

2. METHOD

2.1. Restrictive Taylor Approximation

Consider a function $f(x)$ defined in a neighborhood of the point a ,and it has derivatives up to order (n+1) in this neighborhood. We use this derivatives to construct the function

$$RT_{n,f(x)}(x,a) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{\varepsilon f^{(n)}(a)}{n!}(x-a)^n \tag{6}$$

$RT_{n,f(x)}(x)$ is called restrictive Taylor approximation fort he function $f(x)$ at the point a. The parameter ε is to be determined such that $RT_{n,f(x)}(x_0) = f(x_0)$.It mean that the considered approximation is exact at two points a and x_0 .Let us put

$$f(x) = RT_{n,f(x)}(x) + \Re_{n+1}(x) \tag{7}$$

where $\Re_{n+1}(x)$ is remainder term of restrictive Taylor series. [2]

In the next theorem the remainder term $\Re_{n+1}(x)$ can be expressed in terms of ε n th and $(n+1)$ th derivatives of the function $f(x)$ at a point ξ lies between a and x .

THEOREM: Let the function $f(x): f(x) \in C^{n+1}; x \in I,\ I$ is the neighborhood of a point a .The error for approximation estimated by $\Re_{n+1}(x)$ is given by the formula (7) ,for which

$$\Re_{n+1}(x) = \frac{\varepsilon (x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi) - \frac{n(\varepsilon - 1)(x-a)^{n+1}}{(x-\xi)(n+1)!} f^{(n)}(\xi)$$

where $\xi \in [a, x]$ and ε is a restrictive parameter.

2.2. Restrictive Taylor Approximation of the Exponential Matrix

The exponential matrix $\exp(xA)$ can be formally defined by the convergent power series

$$\exp(xA) = \sum_{n=0}^{\infty} \frac{x^n}{n!} A^n, \quad A^0 = I \tag{8}$$

where A is an $(N-1)\times(N-1)$ matrix.

In the case of restrictive Taylor approximation of single function the term ε in Eq.(6) can be reduced to the square restrictive matrix T in the case of restrictive Taylor approximation for matrix function. Where

$$T = \begin{bmatrix} \varepsilon_1 & & & 0 \\ & \varepsilon_2 & & \\ & & \ddots & \\ 0 & & & \varepsilon_{N-1} \end{bmatrix}_{(N-1)\times(N-1)}$$

For $n=1$,

$$RT_{1,\exp(xA)}(x)=I+\varepsilon TA$$

2.3. Method of Solution

We consider the diffusion equation for $f(u)=-u$. [9] Thus we use the equation

$$u_t=u_{xx}-u \tag{9}$$

subject to the initial condition

$$u(x,0)=\sin x\ ,\ \ 0<x<1 \tag{10}$$

and boundary conditions

$$u(0,t)=0,\ t>0 \tag{11}$$

$$u(1,t)=0,\ t>0 \tag{12}$$

The exact solution of the equation (9) is given

$$u(x,t)=e^{-2t}\sin x \tag{13}$$

The open rectangular domain is covered by a rectangular grid with spacing h and k in the x, t direction respectively, the grid point (x,t) denoted by (ih,jk) and u(ih,jk)=$u_{i,j}$,where $i=0,1....,N$ and j is a non-negative integer.

The exact solution of a grid representation of Eq.(9) is given by[2]:

$$u_{i,j+1}=\exp(k[D_x^2-1])u_{i,j} \tag{14}$$

The approximation of the partial derivatives D_x^2 at the grid point (ih,jk) will take the usual form:

$$D_x^2u=\frac{1}{h^2}(u_{i-1,j}-2u_{i,j}+u_{i+1,j}) \tag{15}$$

and according to central finite difference formulation:

$$u=u_{i+1}-2u_i+u_{i-1} \tag{16}$$

The result of making this approximation is to replace Eq.(14) by the following equation

$$U^{j+1} = \exp(rA)U^j \ , \ r = \frac{k}{h^2} \tag{17}$$

where

$$U^j = (u_{1,j}, u_{2,j}, \ldots u_{N-1,j})^T \ , \ Nh = L \tag{18}$$

and

$$A = \begin{bmatrix} -2+2h^2 & 1-h^2 & \cdots & & & & 0 \\ 1-h^2 & -2+2h^2 & 1-h^2 & \cdots & & & \\ & 1-h^2 & -2+2h^2 & 1-h^2 & & & \\ & & \vdots & & & & \\ & & & & 1-h^2 & -2+2h^2 & 1-h^2 \\ 0 & & & & & 1-h^2 & -2+2h^2 \end{bmatrix}_{(N-1)\times(N-1)}$$

We use the $RT_{1,\exp(xA)}(x) = I + \varepsilon TA$ equation to approximate the exponential matrix in Eq.(17), then

$$U^{j+1} = (I + rTA)U^j = BU^j \tag{19}$$

$$u_{i,j+1} = [(1-h^2)r\varepsilon_i]u_{i-1,j} + [1+(-2+2h^2)r\varepsilon_i]u_{i,j} + [(1-h^2)r\varepsilon_i]u_{i+1,j} ,$$

$$i = 1,2,\ldots,N-1, \quad j \in Z^+ \tag{20}$$

3. FINDINGS

The accuracy of restrictive Taylor approximation method are compared in tables for various values of the time t. Tables give the exact value , approximate value for compact finite difference method, approximate value for restrictive Taylor approximation and absolute error for $\varepsilon = 0.0032408523$. Comparison of the RTA results with CFD6 method for $k = 0.0001$, $N = 6$, $r = 0.0036$ given in below tables.

Table 1.

t	x	Exact	RTA	AE [Present]	CFD6	AE[8]
0.01	1/6	0.162611	0.162611	1.1E-10		
	2/6	0.320715	0.320715	1.02E-10		
	3/6	0.469932	0.469932	1.01E-11	0.469932	1.02E-11
	4/6	0.606125	0.606125	2.02E-11		
	5/6	0.725520	0.725520	1.26E-10		
	1	0.824808	0.824808	1.14E-10		

Table 2.

t	x	Exact	RTA	AE [Present]	CFD6	AE [8]
0.1	1/6	0.135824	0.135824	1.34E-9		
	2/6	0.267884	0.267884	1.23E-9		
	3/6	0.392520	0.392520	1.96E-11	0.392520	2.04E-11
	4/6	0.506278	0.506278	1.36E-10		
	5/6	0.606005	0.606005	1.16E-9		
	1	0.688938	0.688938	1.71E-9		

Table 3.

t	x	Exact	RTA	AE[Present]	CFD6	AE [8]
1.00	1/6	0.022451	0.022451	2.71E-10		
	2/6	0.044280	0.044280	2.36E-10		
	3/6	0.064883	0.064883	4.16E-12	0.064883	4.91E-12
	4/6	0.083687	0.083687	1.84E-11		
	5/6	0.100172	0.100172	2.07E-10		
	1	0.113880	0.113880	2.66E-10		

4.CONCLUSION&DISCUSSION

In this work a numerical method was applied to the one dimensional diffusion equation We compare the computed results with other paper results in Table 1,Table 2 and Table 3.. In Table 1,for $t = 0.01, x = 0.5$ we compare our result with compact finite difference method. In Table 2,3 for $t = 0.1$ and $t = 1, x = 0.5$ we compare our result with compact finite difference method The proposed methods results are quite satisfactorily. We show that using the restrictive Taylor approximation for one dimensional diffusion equation describe our model well. This methods solution proses are very simple to implement and economical to use. We can applied this technique other linear partial differential equations and we hope that obtain satis factorily results.

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