RATING ACADEMY

JSP

Journal of Scientific Perspectives

Volume **3,** Issue **2,** Year **2019**, pp. **141-158**
E - ISSN: **2587-3008**
URL: http://ratingacademy.com.tr/ojs/index.php/jsp
DOİ: https://doi.org/10.26900/jsp.3.015
Research Article

HADAMARD, SIMPSON AND OSTROWSKI TYPE INEQUALITIES FOR E-CONVEXITY

Musa ÇAKMAK*

* *Reyhanlı Vocational School of Social Sciences, Mustafa Kemal University, Hatay 31000, Turkey,*
E-mail: enkucukcakmak@gmail.com
ORCID ID: https://orcid.org/0000-0002-8794-4797

Received: 13 March 2019; Accepted: 29 April 2019

ABSTRACT

In this study, we proposed a new definition to give a different perspec-tive to convex functions. We have introduced the expansion of Hadamard, midpoint Hadamard, trapezoid Hadamard, Simpson and Ostrowski inequalities for the newly defined classes of convex functions.

Keywords: *(a; m; e)-convexity, Hadamard's inequality, Simpson inequality, Ostrowski inequality.*

2010 AMS Subject Classification: *26A33, 26D07, 26D15, 26D60*

1. INTRODUCTION

1.1. **Theorems.** If φ is integrable on $[u_1, u_2]$, then the average value of φ on $[u_1, u_2]$ is

$$\frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \varphi(z)dz. \tag{1}$$

Let $\varphi : D \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $u_1, u_2 \in D$ with $u_1 < u_2$. Then the following double inequality:

$$\varphi\left(\frac{u_1 + u_2}{2}\right) \le \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \varphi(z)dz \le \frac{\varphi(u_1) + \varphi(u_2)}{2} \tag{2}$$

is known as Hermite-Hadamard inequality for convex mappings. For particular choice of the function φ in (2) yields some classical inequalities of means. Both inequalities in (2) hold in reversed direction if φ is concave. The refinement of the second inequality in (2) is due to Bullen as follows:

$$\frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \varphi(z)dz \le \frac{1}{2}\left[\varphi\left(\frac{u_1 + u_2}{2}\right) + \frac{\varphi(u_1) + \varphi(u_2)}{2}\right] \le \frac{\varphi(u_1) + \varphi(u_2)}{2} \tag{3}$$

where φ is as above. This (3) integral inequality is well known in the literature as **Bullen Inequality** [21, Pečarić, Proschan and Tong, 1991]. For some recent results in connection with Hermite-Hadamard inequality and its applications we refer to [[1], [4]-[9], [11]-[25], [27]] where further references are given.

The following inequality is well known in the literature as **Simpson's inequality** [10, Dragomir, Agarwal, and Cerone, 2000];

$$\begin{aligned}(4)\qquad & \int_{u_1}^{u_2}\varphi(z)\,dz-\frac{u_2-u_1}{3}\left[\frac{\varphi(u_1)+\varphi(u_2)}{2}+2\varphi\left(\frac{u_1+u_2}{2}\right)\right]\\ \leq\ & \frac{1}{1280}\left\|\varphi^{(4)}\right\|_{\infty}(u_2-u_1)^5,\end{aligned}$$

where the mapping $\varphi : [u_1, u_2] \to \mathbb{R}$ is assumed to be four times continuously differentiable on the interval and $\varphi^{(4)}$ to be bounded on (u_1, u_2) , that is,

$$\left\|\varphi^{(4)}\right\|_{\infty}=\sup_{r\in(u_1,u_2)}\left|\varphi^{(4)}(r)\right|<\infty.$$

Let $\varphi : D \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable mapping on D° (the interior of D) such that $\varphi' \in L[u_1.u_2]$, where $u_1, u_2 \in D$ with $u_1 < u_2$. If $|\varphi'| \leq M$, then the following double inequality:

$$(5)\qquad \left|\varphi(z)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(u)du\right|\leq\frac{M}{u_2-u_1}\frac{(z-u_1)^2+(u_2-z)^2}{2}$$

holds. This result is known in the literature as the **Ostrowski inequality** [26].

1.2. **Definitions.** Let $h : J \to \mathbb{R}$ be a nonnegative function, $h \not\equiv 0$. We say that $\varphi : D \to \mathbb{R}$ is an h-convex function, or that φ belongs to the class $SX(h,D)$, if φ is nonnegative and for all $z,\mu \in D$ and $r \in (0,1)$ we have

$$(6)\qquad \varphi(rz+(1-r)\mu)\leq h(r)\varphi(z)+h(1-r)\varphi(\mu).$$

If inequality (6) is reversed, then φ is said to be h-concave, i.e. $\varphi \in SV(h,D)$. Obviously, if $h(r) = r$, then all nonnegative convex functions belong to $SX(h,D)$ and all nonnegative concave functions belong to $SV(h,D)$; if $h(r) = 1$, then $SX(h,D) \supseteq P(D)$; and if $h(r) = r^s$, where $s \in (0,1)$, then $SX(h,D) \supseteq K_s^2$, where K_s^2 is $s-$convex in the second sense [26, 27].

The function $\varphi : [0,v] \to \mathbb{R}$ is said to be $(\alpha,m)-$convex, where $(\alpha,m) \in [0,1]^2$, if for every $z,\mu \in [0,v]$ and $r \in [0,1]$, we have

$$\varphi(rz+m(1-r)\mu)\leq r^{\alpha}\varphi(z)+m(1-r^{\alpha})\varphi(\mu).$$

Denote by $K_m^{\alpha}(v)$ the set of the (α, m) −convex functions on $[0, v]$ for which $\varphi(0) \leq 0$. We say that φ is (α, m) −concave if $-\varphi$ is (α, m) −convex. Denote by $K_m^{\alpha}(v)$ the class of all (α, m) −convex functions on $[0, v]$ for which $\varphi(0) \leq 0$ [24, 25].

1.3. Lemmas.

Lemma 1.1. [14] *Let $\varphi : D \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on D°, $u_1, u_2 \in D^{\circ}$ with $u_1 < u_2$. If $\varphi' \in L[u_1, u_2]$, then the following equality holds:*

$$\frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \varphi(z)\,dz - \varphi\left(\frac{u_1+u_2}{2}\right)$$
$$= (u_2 - u_1)\int_0^{\frac{1}{2}} r\varphi'(ru_1 + (1-r)u_2)\,dr + \int_{\frac{1}{2}}^{1} (r-1)\varphi'(ru_1 + (1-r)u_2)\,dr.$$

Lemma 1.2. [1]*Let $\varphi : D \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on D° where $u_1, u_2 \in D$ with $u_1 < u_2$. If $\varphi' \in L[u_1, u_2]$, then the following equality holds:*

$$\frac{\varphi(u_1) + \varphi(u_2)}{2} - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \varphi(z)\,dz$$
$$= \frac{u_2 - u_1}{4}\left[\int_0^1 (-r)\varphi'\left(\frac{1+r}{2}u_1 + \frac{1-r}{2}u_2\right)dr + \int_0^1 r\varphi'\left(\frac{1+r}{2}u_2 + \frac{1-r}{2}u_1\right)dr\right].$$

Lemma 1.3. [2]*Let $\varphi : D \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping D° where $u_1, u_2 \in D$ with $u_1 < u_2$. If $\varphi' \in L[u_1, u_2]$, then the following equality holds:*

$$\frac{1}{6}\left[\varphi(u_1) + 4\varphi\left(\frac{u_1+u_2}{2}\right) + \varphi(u_2)\right] - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \varphi(z)\,dz$$
$$= (u_2 - u_1)\int_0^1 p(r)\varphi'(ru_2 + (1-r)u_1)\,dr,$$

where

$$p(r) = \begin{cases} r - \frac{1}{6}, & r \in \left[0, \frac{1}{2}\right) \\ r - \frac{5}{6}, & r \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Lemma 1.4. [3]*Let $\varphi : D \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on D°, where $u_1, u_2 \in D$ with $u_1 < u_2$. If $\varphi' \in L[u_1, u_2]$, then the following equality holds:*

$$\varphi(z) - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2} \varphi(u)\,du = (u_2 - u_1)\int_0^1 p(r)\varphi'(ru_1 + (1-r)u_2)\,dr$$

for each $r \in [0, 1]$, where

$$p(r) = \begin{cases} r, & r \in \left[0, \frac{u_2 - z}{u_2 - u_1}\right] \\ r - 1, & r \in \left(\frac{u_2 - z}{u_2 - u_1}, 1\right] \end{cases},$$

for all $z \in [u_1, u_2]$.

The main aim of this paper is to establish refinements inequalities of Hadamard's type, midpoint-Hadamard type, trapezoid-Hadamard type, Simpson's type and Ostrowski's type for $e-$convex functions.

2. MAIN RESULTS

Firstly, Let us define the following special definition on convex functions. Secondly we establish a few different intelligent integral inequalities.

Definition 2.1. *Let* $h : J \subseteq \mathbb{R} \to \mathbb{R}$ *be a nonnegative function. A function* $\varphi : D \subseteq \mathbb{R} \to (0, \infty)$ *is said to be* $(\alpha, m, e^h) -$ *convex function if*

$$\varphi(rz + m(1-r)\mu) \leq e^{h(r^\alpha)}\varphi(z) + me^{h(1-r^\alpha)}\varphi(\mu)$$

for all $z, \mu \in D$ *and* $r \in [0,1]$ *with some fixed* $(\alpha, m) \in (0,1]^2$. *Let us suggest several special cases of new definition;*

(1) If $h(r) = r^s$ for $s \in [0,1]$, then Definition 2.1 reduces to definition of $(\alpha, m, e^s)-$convexity.
(2) If $h(r) = r^{\frac{1}{s}}$ for $s \in [0,1]$, then Definition 2.1 reduces to definition of $(\alpha, m, e^{1/s})-$convexity.
(3) If $h(r) = r^{\frac{1}{\alpha}}$ and $m = 1$, then Definition 2.1 reduces to definition of $e-$convexity.
(4) If $h(r) = r$, then Definition 2.1 reduces to definition of $(\alpha, m, e)-$convexity in the first sense.
(5) If $h(r) = 1$, then Definition 2.1 reduces to definition of $(m, e)-$convexity.
(6) If $h(r) = r^s$ for $s \in [0,1]$,and $\alpha = 1$, then Definition 2.1 reduces to definition of $(\alpha, m, e)-$convexity in the second sense.
(7) If $h(r) = r(1-r)$ and $\alpha = 1$, then Definition 2.1 reduces to definition of $(m, tgs, e)-$convexity.
(8) If $h(r) = \frac{\sqrt{(1-r)}}{2\sqrt{r}}$ and $\alpha = 1$, then Definition 2.1 reduces to definition of $(m, e_{MT})-$convexity.

Now, we prove the following theorems.

Theorem 2.2. *Let* φ *be* $e-$*convex function,* $u_1, u_2 \in D \subset \mathbb{R}$ *with* $u_1 < u_2$, $\varphi \in L[u_1, u_2]$. *Then the following inequalities hold:*

$$\text{(7)} \qquad \frac{1}{2\sqrt{e}}\varphi\left(\frac{u_1+u_2}{2}\right) \leq \frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz \leq (e-1)(\varphi(u_1)+\varphi(u_2)).$$

Proof. Since φ is $e-$convex, we have

$$\begin{aligned}\varphi\left(\frac{u_1+u_2}{2}\right) &= \varphi\left(\frac{ru_1+(1-r)u_2}{2}+\frac{ru_2+(1-r)u_1}{2}\right)\\ &\leq e^{\frac{1}{2}}\varphi(ru_1+(1-r)u_2)+e^{\frac{1}{2}}\varphi(ru_2+(1-r)u_1)\end{aligned}$$

for all $r\in[0,1]$. By integrating, we get

$$\begin{aligned}\varphi\left(\frac{u_1+u_2}{2}\right) &\leq e^{\frac{1}{2}}\int_0^1\varphi(ru_1+(1-r)u_2)\,dr+e^{\frac{1}{2}}\int_0^1\varphi(ru_2+(1-r)u_1)\,dr\\ &= e^{\frac{1}{2}}\left(\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z_1)\,dz_1+\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z_2)\,dz_2\right)\\ &= \frac{2e^{\frac{1}{2}}}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz.\end{aligned}$$

For the proof of right hand side of (7), we can write

$$\begin{aligned}\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz &\leq \int_0^1 e^r\varphi(u_1)\,dr+\int_0^1 e^{1-r}\varphi(u_2)\,dr\\ &= \varphi(u_1)\,e^r\big|_0^1-\varphi(u_2)\,e^{1-r}\big|_0^1\\ &= (e-1)(\varphi(u_1)+\varphi(u_2)).\end{aligned}$$

Thus, the theorem is proven. □

Theorem 2.3. *(Midpoint Hadamard) Let $\varphi: D\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on D°, $u_1,u_2\in D$ with $u_1<u_2$ and let $1/r_1+1/r_2=1$ with $r_1>1$. If $|\varphi'|^{r_2}$ is $e-$convex on $[u_1,u_2]$, then the following inequalities hold:*

$$\begin{aligned}&\left|\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz-\varphi\left(\frac{u_1+u_2}{2}\right)\right|\\ \leq\ &\frac{u_2-u_1}{(r_1+1)\,2^{r_1+1}}\left[\left(\left(\sqrt{e}-1\right)|\varphi'(u_1)|^{r_2}+\left(e-\sqrt{e}\right)|\varphi'(u_2)|^{r_2}\right)^{\frac{1}{r_2}}\right.\\ &\left.+\left(\left(e-\sqrt{e}\right)|\varphi'(u_1)|^{r_2}+\left(\sqrt{e}-1\right)|\varphi'(u_2)|^{r_2}\right)^{\frac{1}{r_2}}\right]\\ \leq\ &\frac{(u_2-u_1)(e-1)}{(r_1+1)\,2^{r_1+1}}\left(|\varphi'(u_1)|+|\varphi'(u_2)|\right).\end{aligned}$$

Proof. From Lemma 1.1 and $e-$convexity of $|\varphi'|^{r_2}$ function on $[u_1,u_2]$ and by using Hölder's integral inequality, we obtain

$$
\begin{aligned}
&\left|\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz-\varphi\left(\frac{u_1+u_2}{2}\right)\right| = (u_2-u_1)\times \\
&\int_0^{\frac{1}{2}} r\left|\varphi'(ru_1+(1-r)u_2)\right|dr+\int_{\frac{1}{2}}^1(1-r)\left|\varphi'(ru_1+(1-r)u_2)\right|dr \\
\leq\ & (u_2-u_1)\left[\left(\int_0^{\frac{1}{2}} r^{r_1}dr\right)^{\frac{1}{r_1}}\left(\int_0^{\frac{1}{2}}\left|\varphi'(ru_1+(1-r)u_2)\right|^{r_2}dr\right)^{\frac{1}{r_2}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1(1-r)^{r_1}dr\right)^{\frac{1}{r_1}}\left(\int_{\frac{1}{2}}^1\left|\varphi'(ru_1+(1-r)u_2)\right|^{r_2}dr\right)^{\frac{1}{r_2}}\right] \\
\leq\ & \frac{u_2-u_1}{(r_1+1)\,2^{r_1+1}}\left[\left(\int_0^{\frac{1}{2}}\left(e^r\left|\varphi'(u_1)\right|^{r_2}+e^{1-r}\left|\varphi'(u_2)\right|^{r_2}\right)dr\right)^{\frac{1}{r_2}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1\left(e^r\left|\varphi'(u_1)\right|^{r_2}+e^{1-r}\left|\varphi'(u_2)\right|^{r_2}\right)dr\right)^{\frac{1}{r_2}}\right] \\
=\ & \frac{u_2-u_1}{(r_1+1)\,2^{r_1+1}}\left[\left(\left(e^{\frac{1}{2}}-1\right)\left|\varphi'(u_1)\right|^{r_2}+\left(e-e^{\frac{1}{2}}\right)\left|\varphi'(u_2)\right|^{r_2}\right)^{\frac{1}{r_2}}\right. \\
&\left.+\left(\left(e-e^{\frac{1}{2}}\right)\left|\varphi'(u_1)\right|^{r_2}+\left(e^{\frac{1}{2}}-1\right)\left|\varphi'(u_2)\right|^{r_2}\right)^{\frac{1}{r_2}}\right].
\end{aligned}
$$

For the proof of second inequality, using the fact that

$$
\sum\nolimits_{i=1}^{m}(p_i+q_i)^v\leq\sum\nolimits_{i=1}^{m}p_i^v+\sum\nolimits_{i=1}^{m}q_i^v \tag{8}
$$

for $(0\leq v<1)$, $u_{11},u_{12},...,u_{1i}\geq 0$, $u_{21},u_{22},...,u_{2i}\geq 0$, we get

$$
\begin{aligned}
&\left[\left(\left(e^{\frac{1}{2}}-1\right)\left|\varphi'(u_1)\right|^{r_2}+\left(e-e^{\frac{1}{2}}\right)\left|\varphi'(u_2)\right|^{r_2}\right)^{\frac{1}{r_2}}\right. \\
&\left.+\left(\left(e-e^{\frac{1}{2}}\right)\left|\varphi'(u_1)\right|^{r_2}+\left(e^{\frac{1}{2}}-1\right)\left|\varphi'(u_2)\right|^{r_2}\right)^{\frac{1}{r_2}}\right] \\
\leq\ & (e-1)\left|\varphi'(u_1)\right|+(e-1)\left|\varphi'(u_2)\right|
\end{aligned}
$$

Thus, the proof is done. □

Theorem 2.4. *(Trapezoid-Hadamard) Let $\varphi: D\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on D°, $u_1,u_2\in D$ with $u_1<u_2$ and let $1/r_1+1/r_2=1$ with $r_1>1$. If*

$|\varphi'|^{r_2}$ *is* $e-$*convex on* $[u_1, u_2]$, *then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{\varphi(u_1)+\varphi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz\right| \\
\leq\ & \frac{u_2-u_1}{4(r_1+1)^{1/r_1}}\left[\left((2e-2\sqrt{e})\,|\varphi'(u_1)|^{r_2}+(2\sqrt{e}-2)\,|\varphi'(u_2)|^{r_2}\right)^{\frac{1}{r_2}}\right. \\
&\left.+\left((2e-2\sqrt{e})\,|\varphi'(u_2)|^{r_2}+(2\sqrt{e}-2)\,|\varphi'(u_1)|^{r_2}\right)^{\frac{1}{r_2}}\right] \\
\leq\ & \frac{(u_2-u_1)(e-1)}{2(r_1+1)^{1/r_1}}\left(|\varphi'(u_1)|+|\varphi'(u_2)|\right).
\end{aligned}$$

Proof. From Lemma 1.2 and $e-$convexity of $|\varphi'|^{r_2}$ function and by using Hölder's integral inequality, we obtain

$$\begin{aligned}
&\left|\frac{\varphi(u_1)+\varphi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz\right| \\
\leq\ & \frac{u_2-u_1}{4}\times \\
&\left[\left(\int_0^1 r^{r_1}dr\right)^{\frac{1}{r_1}}\left(\int_0^1\left|\varphi'\left(\frac{1+r}{2}u_1+\frac{1-r}{2}u_2\right)\right|^{r_2}dr\right)^{\frac{1}{r_2}}\right. \\
&\left.+\left(\int_0^1 r^{r_1}dr\right)^{\frac{1}{r_1}}\left(\int_0^1\left|\varphi'\left(\frac{1+r}{2}u_2+\frac{1-r}{2}u_1\right)\right|^{r_2}dr\right)^{\frac{1}{r_2}}\right] \\
\leq\ & \frac{u_2-u_1}{4(r_1+1)^{1/r_1}}\times \\
&\left[\left(\int_0^1\left(e^{\frac{1+r}{2}}|\varphi'(u_1)|^{r_2}+e^{\frac{1-r}{2}}|\varphi'(u_2)|^{r_2}\right)dr\right)^{\frac{1}{r_2}}\right. \\
&\left.+\left(\int_0^1\left(e^{\frac{1+r}{2}}|\varphi'(u_2)|^{r_2}+e^{\frac{1-r}{2}}|\varphi'(u_1)|^{r_2}\right)dr\right)^{\frac{1}{r_2}}\right] \\
=\ & \frac{u_2-u_1}{4(r_1+1)^{1/r_1}}\left[\left(\left(2e-2e^{\frac{1}{2}}\right)|\varphi'(u_1)|^{r_2}+\left(2e^{\frac{1}{2}}-2\right)|\varphi'(u_2)|^{r_2}\right)^{\frac{1}{r_2}}\right. \\
&\left.+\left(\left(2e-2e^{\frac{1}{2}}\right)|\varphi'(u_2)|^{r_2}+\left(2e^{\frac{1}{2}}-2\right)|\varphi'(u_1)|^{r_2}\right)^{\frac{1}{r_2}}\right].
\end{aligned}$$

By using Inequality (8), we get

$$\begin{aligned} & \left(\left(2e-2e^{\frac{1}{2}}\right)\left|\varphi'\left(u_1\right)\right|^{r_2}+\left(2e^{\frac{1}{2}}-2\right)\left|\varphi'\left(u_2\right)\right|^{r_2}\right)^{\frac{1}{r_2}} \\ & +\left(\left(2e-2e^{\frac{1}{2}}\right)\left|\varphi'\left(u_2\right)\right|^{r_2}+\left(2e^{\frac{1}{2}}-2\right)\left|\varphi'\left(u_1\right)\right|^{r_2}\right)^{\frac{1}{r_2}} \\ \leq \; & (2e-2)\left|\varphi'\left(u_1\right)\right|+(2e-2)\left|\varphi'\left(u_2\right)\right|, \end{aligned}$$

which completes the proof.

Theorem 2.5. *(Trapezoid-Hadamard) Let $\varphi : D \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on D°, $u_1, u_2 \in D$ with $u_1 < u_2$. If $|\varphi'|^{r_2}$ is $e-$convex on $[u_1, u_2]$, $r_2 \geq 1$, then the following inequalities hold:*

$$\begin{aligned} & \left|\frac{\varphi\left(u_1\right)+\varphi\left(u_2\right)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz\right| \\ \leq \; & \frac{u_2-u_1}{2^{(3r_2-1)/r_2}}\left[\left(\left(4\sqrt{e}-2e\right)\left|\varphi'\left(u_1\right)\right|^{r_2}+\left(4\sqrt{e}-6\right)\left|\varphi'\left(u_2\right)\right|^{r_2}\right)^{\frac{1}{r_2}}\right. \\ & \left.+\left(\left(4\sqrt{e}-2e\right)\left|\varphi'\left(u_2\right)\right|^{r_2}+\left(4\sqrt{e}-6\right)\left|\varphi'\left(u_1\right)\right|^{r_2}\right)^{\frac{1}{r_2}}\right] \\ \leq \; & \frac{u_2-u_1}{2^{(2r_2-1)/r_2}}\left(4\sqrt{e}-e-3\right)\left(\left|\varphi'\left(u_1\right)\right|+\left|\varphi'\left(u_2\right)\right|\right). \end{aligned}$$

Proof. From Lemma 1.2 and $e-$convexity of $|\varphi'|^{r_2}$ function and by using power mean inequality, we establish

$$
\begin{aligned}
& \left|\frac{\varphi(u_1)+\varphi(u_2)}{2}-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz\right| \\
\leq\ & \frac{u_2-u_1}{4}\times \\
& \left[\left(\int_0^1 r dr\right)^{1-\frac{1}{r_2}}\left(\int_0^1 r\left|\varphi'\left(\frac{1+r}{2}u_1+\frac{1-r}{2}u_2\right)\right|^{r_2}dr\right)^{\frac{1}{r_2}}\right. \\
& \left.+\left(\int_0^1 r dr\right)^{1-\frac{1}{r_2}}\left(\int_0^1 r\left|\varphi'\left(\frac{1+r}{2}u_2+\frac{1-r}{2}u_1\right)\right|^{r_2}dr\right)^{\frac{1}{r_2}}\right] \\
\leq\ & \frac{u_2-u_1}{2^{3-1/r_2}}\times \\
& \left[\left(\int_0^1\left(re^{\frac{1+r}{2}}\left|\varphi'(u_1)\right|^{r_2}+re^{\frac{1-r}{2}}\left|\varphi'(u_2)\right|^{r_2}\right)dr\right)^{\frac{1}{r_2}}\right. \\
& \left.+\left(\int_0^1\left(re^{\frac{1+r}{2}}\left|\varphi'(u_2)\right|^{r_2}+re^{\frac{1-r}{2}}\left|\varphi'(u_1)\right|^{r_2}\right)dr\right)^{\frac{1}{r_2}}\right] \\
=\ & \frac{u_2-u_1}{2^{3-1/r_2}}\left[\left(\left(4e^{\frac{1}{2}}-2e\right)\left|\varphi'(u_1)\right|^{r_2}+\left(4e^{\frac{1}{2}}-6\right)\left|\varphi'(u_2)\right|^{r_2}\right)^{\frac{1}{r_2}}\right. \\
& \left.+\left(\left(4e^{\frac{1}{2}}-2e\right)\left|\varphi'(u_2)\right|^{r_2}+\left(4e^{\frac{1}{2}}-6\right)\left|\varphi'(u_1)\right|^{r_2}\right)^{\frac{1}{r_2}}\right].
\end{aligned}
$$

By using Inequality (8), we get

$$
\begin{aligned}
& \left(\left(4e^{\frac{1}{2}}-2e\right)\left|\varphi'(u_1)\right|^{r_2}+\left(4e^{\frac{1}{2}}-6\right)\left|\varphi'(u_2)\right|^{r_2}\right)^{\frac{1}{r_2}} \\
& +\left(\left(4e^{\frac{1}{2}}-2e\right)\left|\varphi'(u_2)\right|^{r_2}+\left(4e^{\frac{1}{2}}-6\right)\left|\varphi'(u_1)\right|^{r_2}\right)^{\frac{1}{r_2}} \\
\leq\ & \left(8e^{\frac{1}{2}}-2e-6\right)\left(\left|\varphi'(u_1)\right|+\left|\varphi'(u_2)\right|\right)
\end{aligned}
$$

which completes the proof. □

Theorem 2.6. *($Simpson_{r_1,r_2}$) Let $\varphi : D\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on D°, $u_1,u_2\in D$ with $u_1<u_2$ and let $1/r_1+1/r_2=1$ with $r_1>1$. If $|\varphi'|^{r_2}$ is*

$e-$convex on $[u_1, u_2]$, then the following inequality holds;

$$\left|\frac{1}{6}\left[\varphi(u_1)+4\varphi\left(\frac{u_1+u_2}{2}\right)+\varphi(u_2)\right]-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz\right|$$
$$\leq (u_2-u_1)\left(\frac{1+2^{r_1+1}}{2^{r_1+1}(r_1+1)}\right)^{\frac{1}{r_1}}\left[\left(\left(\sqrt{e}-1\right)|\varphi'(u_2)|^{r_2}+\left(e-\sqrt{e}\right)|\varphi'(u_1)|^{r_2}\right)^{\frac{1}{r_2}}\right.$$
$$\left.+\left(\left(e-\sqrt{e}\right)|\varphi'(u_2)|^{r_2}+\left(\sqrt{e}-1\right)|\varphi'(u_1)|^{r_2}\right)^{\frac{1}{r_2}}\right].$$

Proof. From Lemma 1.3 and by using Hölder's integral inequality, we can write

$$\left|\frac{1}{6}\left[\varphi(u_1)+4\varphi\left(\frac{u_1+u_2}{2}\right)+\varphi(u_2)\right]-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz\right|$$
$$\leq (u_2-u_1)\left[\int_0^{\frac{1}{2}}\left|r-\frac{1}{6}\right||\varphi'(ru_2+(1-r)u_1)|\,dr\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\left|r-\frac{5}{6}\right||\varphi'(ru_2+(1-r)u_1)|\,dr\right]$$
$$\leq (u_2-u_1)\left[\left(\int_0^{\frac{1}{2}}\left|r-\frac{1}{6}\right|^{r_1}dr\right)^{\frac{1}{r_1}}\left(\int_0^{\frac{1}{2}}|\varphi'(ru_2+(1-r)u_1)|^{r_2}dr\right)^{\frac{1}{r_2}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}\left|r-\frac{5}{6}\right|^{r_1}dr\right)^{\frac{1}{r_1}}\left(\int_{\frac{1}{2}}^{1}|\varphi'(ru_2+(1-r)u_1)|^{r_2}dr\right)^{\frac{1}{r_2}}\right]$$
$$= (u_2-u_1)\left(\frac{1+2^{r_1+1}}{2^{r_1+1}(r_1+1)}\right)^{\frac{1}{r_1}}\left[\left(\int_0^{\frac{1}{2}}|\varphi'(ru_2+(1-r)u_1)|^{r_2}dr\right)^{\frac{1}{r_2}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^{1}|\varphi'(ru_2+(1-r)u_1)|^{r_2}dr\right)^{\frac{1}{r_2}}\right].$$

Since $|\varphi'|^{r_2}$ is $e-$convex, we get

$$\int_0^{\frac{1}{2}}|\varphi'(ru_2+(1-r)u_1)|^{r_2}dr = \int_0^{\frac{1}{2}}\left(e^r|\varphi'(u_2)|^{r_2}+e^{1-r}|\varphi'(u_1)|^{r_2}\right)dr$$
$$= \left(\sqrt{e}-1\right)|\varphi'(u_2)|^{r_2}+\left(e-\sqrt{e}\right)|\varphi'(u_1)|^{r_2},$$

$$\int_{\frac{1}{2}}^{1}|\varphi'(ru_2+(1-r)u_1)|^{r_2}dr = \int_{\frac{1}{2}}^{1}\left(e^r|\varphi'(u_2)|^{r_2}+e^{1-r}|\varphi'(u_1)|^{r_2}\right)dr$$
$$= \left(e-\sqrt{e}\right)|\varphi'(u_2)|^{r_2}+\left(\sqrt{e}-1\right)|\varphi'(u_1)|^{r_2}.$$

Thus, we establish

$$\left|\frac{1}{6}\left[\varphi(u_1)+4\varphi\left(\frac{u_1+u_2}{2}\right)+\varphi(u_2)\right]-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz\right|$$
$$\leq (u_2-u_1)\left(\frac{1+2^{r_1+1}}{2^{r_1+1}(r_1+1)}\right)^{\frac{1}{r_1}}\left[\left((\sqrt{e}-1)\left|\varphi'(u_2)\right|^{r_2}+(e-\sqrt{e})\left|\varphi'(u_1)\right|^{r_2}\right)^{\frac{1}{r_2}}\right.$$
$$\left.+\left((e-\sqrt{e})\left|\varphi'(u_2)\right|^{r_2}+(\sqrt{e}-1)\left|\varphi'(u_1)\right|^{r_2}\right)^{\frac{1}{r_2}}\right].$$

The proof is done. □

The corresponding version of the midpoint inequality is given in the following result:

Corollary 2.7. *If we take $\varphi(u_1)=4\varphi\left(\frac{u_1+u_2}{2}\right)=\varphi(u_2)$ in Theorem 2.6, then we get*

$$\left|\varphi\left(\frac{u_1+u_2}{2}\right)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz\right|$$
$$\leq (u_2-u_1)\left(\frac{1+2^{r_1+1}}{2^{r_1+1}(r_1+1)}\right)^{\frac{1}{r_1}}\left[\left((\sqrt{e}-1)\left|\varphi'(u_2)\right|^{r_2}+(e-\sqrt{e})\left|\varphi'(u_1)\right|^{r_2}\right)^{\frac{1}{r_2}}\right.$$
$$\left.+\left((e-\sqrt{e})\left|\varphi'(u_2)\right|^{r_2}+(\sqrt{e}-1)\left|\varphi'(u_1)\right|^{r_2}\right)^{\frac{1}{r_2}}\right].$$

Theorem 2.8. *(Simpson$_{r_2}$) Let $\varphi: D\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on D°, $u_1,u_2\in D$ with $u_1<u_2$ and let $r_2\geq 1$. If $|\varphi'|^{r_2}$ is $e-$convex on $[u_1,u_2]$, then the following inequality holds;*

$$\left|\frac{1}{6}\left[\varphi(u_1)+4\varphi\left(\frac{u_1+u_2}{2}\right)+\varphi(u_2)\right]-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz\right|$$
$$\leq (u_2-u_1)\left(\frac{5}{72}\right)^{\frac{r_2-1}{r_2}}$$
$$\times\left[\left(\left(2\sqrt[6]{e}-\frac{2}{3}\sqrt{e}-\frac{7}{6}\right)\left|\varphi'(u_2)\right|^{r_2}+\left(2\sqrt[6]{e^5}-\frac{4}{3}\sqrt{e}-\frac{5}{6}e\right)\left|\varphi'(u_1)\right|^{r_2}\right)^{\frac{1}{r_2}}\right.$$
$$\left.+\left(\left(2\sqrt[6]{e^5}-\frac{4}{3}\sqrt{e}-\frac{5}{6}e\right)\left|\varphi'(u_2)\right|^{r_2}+\left(2\sqrt[6]{e}-\frac{2}{3}\sqrt{e}-\frac{7}{6}\right)\left|\varphi'(u_1)\right|^{r_2}\right)^{\frac{1}{r_2}}\right].$$

Proof. From Lemma 1.3 and by using power mean inequality, we can write

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi(u_1)+4\varphi\left(\frac{u_1+u_2}{2}\right)+\varphi(u_2)\right]-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(z)\,dz\right| \\
\leq\; & (u_2-u_1)\left[\int_0^{\frac{1}{2}}\left|r-\frac{1}{6}\right|\left|\varphi'(ru_2+(1-r)u_1)\right|dr\right. \\
& \left.+\int_{\frac{1}{2}}^{1}\left|r-\frac{5}{6}\right|\left|\varphi'(ru_2+(1-r)u_1)\right|dr\right] \\
\leq\; & (u_2-u_1)\left[\left(\int_0^{\frac{1}{2}}\left|r-\frac{1}{6}\right|dr\right)^{1-\frac{1}{r_2}}\left(\int_0^{\frac{1}{2}}\left|r-\frac{1}{6}\right|\left|\varphi'(ru_2+(1-r)u_1)\right|^{r_2}dr\right)^{\frac{1}{r_2}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^{1}\left|r-\frac{5}{6}\right|dr\right)^{1-\frac{1}{r_2}}\left(\int_{\frac{1}{2}}^{1}\left|r-\frac{5}{6}\right|\left|\varphi'(ru_2+(1-r)u_1)\right|^{r_2}dr\right)^{\frac{1}{r_2}}\right] \\
=\; & (u_2-u_1)\left(\frac{5}{72}\right)^{\frac{r_2-1}{r_2}}\left[\left(\int_0^{\frac{1}{2}}\left|r-\frac{1}{6}\right|\left|\varphi'(ru_2+(1-r)u_1)\right|^{r_2}dr\right)^{\frac{1}{r_2}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^{1}\left|r-\frac{5}{6}\right|\left|\varphi'(ru_2+(1-r)u_1)\right|^{r_2}dr\right)^{\frac{1}{r_2}}\right].
\end{aligned}$$

Since $|\varphi'|^{r_2}$ is $e-$convex, we get

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|r-\frac{1}{6}\right|\left|\varphi'(ru_2+(1-r)u_1)\right|^{r_2}dr \\
=\; & \left|\varphi'(u_2)\right|^{r_2}\int_0^{\frac{1}{2}}\left|r-\frac{1}{6}\right|e^{r}dr+\left|\varphi'(u_1)\right|^{r_2}\int_0^{\frac{1}{2}}\left|r-\frac{1}{6}\right|e^{1-r}dr \\
=\; & \left(2\sqrt[6]{e}-\frac{2}{3}\sqrt{e}-\frac{7}{6}\right)\left|\varphi'(u_2)\right|^{r_2}+\left(2\sqrt[6]{e^5}-\frac{4}{3}\sqrt{e}-\frac{5}{6}e\right)\left|\varphi'(u_1)\right|^{r_2},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left|r-\frac{5}{6}\right|\left|\varphi'(ru_2+(1-r)u_1)\right|^{r_2}dr \\
=\; & \left|\varphi'(u_2)\right|^{r_2}\int_{\frac{1}{2}}^{1}\left|r-\frac{5}{6}\right|e^{r}dr+\left|\varphi'(u_1)\right|^{r_2}\int_{\frac{1}{2}}^{1}\left|r-\frac{5}{6}\right|e^{1-r}dr \\
=\; & \left(2\sqrt[6]{e^5}-\frac{4}{3}\sqrt{e}-\frac{5}{6}e\right)\left|\varphi'(u_2)\right|^{r_2}+\left(2\sqrt[6]{e}-\frac{2}{3}\sqrt{e}-\frac{7}{6}\right)\left|\varphi'(u_1)\right|^{r_2}.
\end{aligned}$$

Thus, by combining all the above terms, we obtain the required result. □

Theorem 2.9. *(Ostrowski$_{r_1,r_2}$) Let $\varphi : D \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on D°, $u_1, u_2 \in D$ with $u_1 < u_2$ and let $1/r_1 + 1/r_2 = 1$ with $r_1 > 1$. If $|\varphi'|^{r_2}$ is $e-$convex on $[u_1, u_2]$, then the following inequality holds;*

$$\begin{aligned}
&\left|\varphi(z) - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2}\varphi(u)\,du\right| \\
\leq\ & \frac{(u_2 - z)^{\frac{r_1+1}{r_1}}}{(u_2 - u_1)^{\frac{1}{r_1}}(r_1+1)^{\frac{1}{r_1}}}\left(\left(e^{\frac{u_2-z}{u_2-u_1}} - 1\right)|\varphi'(u_1)|^{r_2} + \left(e - e^{\frac{z-u_1}{u_2-u_1}}\right)|\varphi'(u_2)|^{r_2}\right)^{1/r_2} \\
&+\frac{(z - u_1)^{\frac{r_1+1}{r_1}}}{(u_2 - u_1)^{\frac{1}{r_1}}(r_1+1)^{\frac{1}{r_1}}}\left(\left(e - e^{\frac{u_2-z}{u_2-u_1}}\right)|\varphi'(u_1)|^{r_2} + \left(e^{\frac{z-u_1}{u_2-u_1}} - 1\right)|\varphi'(u_2)|^{r_2}\,dr\right)^{1/r_2}.
\end{aligned}$$

Proof. From Lemma 1.4 and by using the Hölder's integral inequality and since $|\varphi'|^{r_2}$ is $e-$convex, we can write

$$\begin{aligned}
&\left|\varphi(z) - \frac{1}{u_2 - u_1}\int_{u_1}^{u_2}\varphi(u)\,du\right| \\
\leq\ & (u_2 - u_1)\left[\left(\int_0^{\frac{u_2-z}{u_2-u_1}} r^{r_1}dr\right)^{1/r_1}\left(\int_0^{\frac{u_2-z}{u_2-u_1}}|\varphi'(ru_1 + (1-r)u_2)|^{r_2}dr\right)^{1/r_2}\right. \\
&\left.+\left(\int_{\frac{u_2-z}{u_2-u_1}}^1 (1-r)^{r_1}dr\right)^{1/r_1}\left(\int_{\frac{u_2-z}{u_2-u_1}}^1|\varphi'(ru_1 + (1-r)u_2)|^{r_2}dr\right)^{1/r_2}\right] \\
\leq\ & \frac{(u_2 - z)^{\frac{r_1+1}{r_1}}}{(u_2 - u_1)^{\frac{1}{r_1}}(r_1+1)^{\frac{1}{r_1}}}\left(\int_0^{\frac{u_2-z}{u_2-u_1}}\left(e^r|\varphi'(u_1)|^{r_2} + e^{1-r}|\varphi'(u_2)|^{r_2}\right)dr\right)^{1/r_2} \\
&+\frac{(z - u_1)^{\frac{r_1+1}{r_1}}}{(u_2 - u_1)^{\frac{1}{r_1}}(r_1+1)^{\frac{1}{r_1}}}\left(\int_{\frac{u_2-z}{u_2-u_1}}^1\left(e^r|\varphi'(u_1)|^{r_2} + e^{1-r}|\varphi'(u_2)|^{r_2}\right)dr\right)^{1/r_2} \\
=\ & \frac{(u_2 - z)^{\frac{r_1+1}{r_1}}}{(u_2 - u_1)^{\frac{1}{r_1}}(r_1+1)^{\frac{1}{r_1}}}\left(\left(e^{\frac{u_2-z}{u_2-u_1}} - 1\right)|\varphi'(u_1)|^{r_2} + \left(e - e^{\frac{z-u_1}{u_2-u_1}}\right)|\varphi'(u_2)|^{r_2}\right)^{1/r_2} \\
&+\frac{(z - u_1)^{\frac{r_1+1}{r_1}}}{(u_2 - u_1)^{\frac{1}{r_1}}(r_1+1)^{\frac{1}{r_1}}}\left(\left(e - e^{\frac{u_2-z}{u_2-u_1}}\right)|\varphi'(u_1)|^{r_2} + \left(e^{\frac{z-u_1}{u_2-u_1}} - 1\right)|\varphi'(u_2)|^{r_2}\,dr\right)^{1/r_2}.
\end{aligned}$$

The proof is completed. □

Corollary 2.10. *If we take $z=\frac{u_1+u_2}{2}$ in Theorem 2.9, then we get*

$$\begin{aligned}&\left|\varphi\left(\frac{u_1+u_2}{2}\right)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(u)\,du\right|\\ \leq\ &\frac{u_2-u_1}{2^{\frac{r_1+1}{r_1}}(r_1+1)^{\frac{1}{r_1}}}\left[\left(\left(\sqrt{e}-1\right)\left|\varphi'(u_1)\right|^{r_2}+\left(e-\sqrt{e}\right)\left|\varphi'(u_2)\right|^{r_2}\right)^{1/r_2}\right.\\ &\left.+\left(\left(e-\sqrt{e}\right)\left|\varphi'(u_1)\right|^{r_2}+\left(\sqrt{e}-1\right)\left|\varphi'(u_2)\right|^{r_2}dr\right)^{1/r_2}\right].\end{aligned}$$

Theorem 2.11. *(Ostrowski$_{r_2}$) Let $\varphi: D\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on D°, $u_1,u_2\in D$ with $u_1<u_2$ and let $r_2\geq 1$. If $|\varphi'|^{r_2}$ is $e-$convex on $[u_1,u_2]$, then the following inequality holds;*

$$\begin{aligned}&\left|\varphi(z)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(u)\,du\right|\\ =\ &\frac{(u_2-z)^{2\left(1-\frac{1}{r_2}\right)}}{2(u_2-u_1)^{1-\frac{2}{r_2}}}\\ &\times\left(\left(1+\frac{z-u_1}{u_2-u_1}e^{\frac{u_2-z}{u_2-u_1}}\right)\left|\varphi'(u_1)\right|^{r_2}+\left(\frac{(u_1-2u_2+z)}{u_2-u_1}e^{\frac{z-u_1}{u_2-u_1}}+e\right)\left|\varphi'(u_2)\right|^{r_2}\right)^{1/r_2}\\ &+\frac{(z-u_1)^{2\left(1-\frac{1}{r_2}\right)}}{2(u_2-u_1)^{1-\frac{2}{r_2}}}\\ &\times\left(\left(e^{\frac{u_2-z}{u_2-u_1}}\frac{(2u_1-u_2-z)}{u_2-u_1}+e\right)\left|\varphi'(u_1)\right|^{r_2}+\left(1-\frac{e^{\frac{z-u_1}{u_2-u_1}}(u_2-z)}{u_2-u_1}\right)\left|\varphi'(u_2)\right|^{r_2}dr\right)^{1/r_2}.\end{aligned}$$

Proof. From Lemma 1.4 and using the power mean integral inequality and since $|\varphi'|^{r_2}$ is $e-$convex , we can write

$$\begin{aligned}
&\left|\varphi(z)-\frac{1}{u_2-u_1}\int_{u_1}^{u_2}\varphi(u)\,du\right| \\
\leq\;& (u_2-u_1)\left[\left(\int_0^{\frac{u_2-z}{u_2-u_1}} r\,dr\right)^{1-1/r_2}\left(\int_0^{\frac{u_2-z}{u_2-u_1}} r\left|\varphi'(ru_1+(1-r)u_2)\right|^{r_2}dr\right)^{1/r_2}\right. \\
&\left.+\left(\int_{\frac{u_2-z}{u_2-u_1}}^1 (1-r)\,dr\right)^{1-1/r_2}\left(\int_{\frac{u_2-z}{u_2-u_1}}^1 (1-r)\left|\varphi'(ru_1+(1-r)u_2)\right|^{r_2}dr\right)^{1/r_2}\right] \\
\leq\;& \frac{(u_2-z)^{2\left(1-\frac{1}{r_2}\right)}}{2(u_2-u_1)^{1-\frac{2}{r_2}}}\left(\int_0^{\frac{u_2-z}{u_2-u_1}}\left(re^r\left|\varphi'(u_1)\right|^{r_2}+re^{1-r}\left|\varphi'(u_2)\right|^{r_2}\right)dr\right)^{1/r_2} \\
&+\frac{(z-u_1)^{2\left(1-\frac{1}{r_2}\right)}}{2(u_2-u_1)^{1-\frac{2}{r_2}}}\left(\int_{\frac{u_2-z}{u_2-u_1}}^1\left((1-r)e^r\left|\varphi'(u_1)\right|^{r_2}+(1-r)e^{1-r}\left|\varphi'(u_2)\right|^{r_2}\right)dr\right)^{1/r_2} \\
=\;& \frac{(u_2-z)^{2\left(1-\frac{1}{r_2}\right)}}{2(u_2-u_1)^{1-\frac{2}{r_2}}} \\
&\times\left(\left(1+\frac{z-u_1}{u_2-u_1}e^{\frac{u_2-z}{u_2-u_1}}\right)\left|\varphi'(u_1)\right|^{r_2}+\left(\frac{(u_1-2u_2+z)}{u_2-u_1}e^{\frac{z-u_1}{u_2-u_1}}+e\right)\left|\varphi'(u_2)\right|^{r_2}\right)^{1/r_2} \\
&+\frac{(z-u_1)^{2\left(1-\frac{1}{r_2}\right)}}{2(u_2-u_1)^{1-\frac{2}{r_2}}} \\
&\times\left(\left(e^{\frac{u_2-z}{u_2-u_1}}\frac{(2u_1-u_2-z)}{u_2-u_1}+e\right)\left|\varphi'(u_1)\right|^{r_2}+\left(1-\frac{e^{\frac{z-u_1}{u_2-u_1}}(u_2-z)}{u_2-u_1}\right)\left|\varphi'(u_2)\right|^{r_2}dr\right)^{1/r_2}
\end{aligned}$$

The proof is completed. □

Corollary 2.12. *If we take* $z=\frac{u_1+u_2}{2}$ *in Theorem 2.11, then we get*

$$\begin{aligned}
&\left|\varphi\left(\frac{u_1+u_2}{2}\right)-\frac{1}{u_2-u_1}\int_{u_1}^{u2}\varphi(u)\,du\right| \\
\leq\;& \frac{u_2-u_1}{2^{3-\frac{2}{r_2}}}\left[\left(\left(1+\frac{1}{2}\sqrt{e}\right)\left|\varphi'(u_1)\right|^{r_2}+\left(\frac{3}{2}\sqrt{e}+e\right)\left|\varphi'(u_2)\right|^{r_2}\right)^{1/r_2}\right. \\
&\left.+\left(\left(e-\frac{3}{2}\sqrt{e}\right)\left|\varphi'(u_1)\right|^{r_2}+\left(1-\frac{1}{2}\sqrt{e}\right)\left|\varphi'(u_2)\right|^{r_2}dr\right)^{1/r_2}\right] \\
\leq\;& \frac{u_2-u_1}{2^{3-\frac{2}{r_2}}}\left(\left(1+e-\sqrt{e}\right)\left|\varphi'(u_1)\right|+\left(1+e+\sqrt{e}\right)\left|\varphi'(u_2)\right|\right).
\end{aligned}$$

REFERENCES

[1] ALOMARİ M., DARUS M., KIRMACIU.S., Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Comp. and Math. with Appl., V.59, 2010, pp. 225-232.

[2] ALOMARİ M., DARUS M., DRAGOMİR S.S., New inequalities of Simpson's type for s-convex functions with applications, RGMIA Research Report Collection Volume 12 (4), 2010.

[3] ALOMARİ M., DARUS M., Some Ostrowski type inequalities for quasi-convex functions with applications to special means, RGMIA Research Report Collection Volume 13, article 2, 2010.

[4] BESSENYEİ M., The Hermite-Hadamard inequality on simplices, Amer. Math. Monthly 115 (2008), no. 4, 339-345. MR 2009b:52023

[5] BESSENYEİ M., Hermite-Hadamard-type inequalities for generalized convex functions, J. Inequal. Pure Appl. Math. 9 (2008), no. 3, Article 63, pp. 51 (electronic).

[6] BESSENYEİ M., The Hermite-Hadamard inequality in Beckenbach's setting, J. Math. Anal. Appl. 364 (2010), no. 2, 366-383. MR MR2576189

[7] BESSENYEİ M. and PÁLES Zs., Higher-order generalizations of Hadamard's inequality, Publ. Math. Debrecen 61 (2002), no. 3-4, 623-643. MR 2003k:26021

[8] BESSENYEİ M. and PÁLES Zs., Characterizations of convexity via Hadamard's inequality, Math. Inequal. Appl. 9 (2006), no. 1, 53-62. MR 2007a:26010

[9] BOMBARDELLİ M. and VAROŠANEC S., Properties of h -convex functions related to the Hermite-Hadamard-Fejér inequalities, Comput. Math. Appl. 58 (2009) 1869-1877.

[10] DRAGOMİR S. S., AGARWAL R. P., and CERONE P., On Simpson's inequality and applications, J. Inequal. Appl. 5 (2000), no. 6, 533{579; Available online at http://dx.doi.org/10.1155/S102558340000031X.

[11] DRAGOMİR S. S., PEČARİĆ J. and PERSSON L.E., Some inequalities of Hadamard type, Soochow J.Math., 21, 335-241, 1995.

[12] GÖZPİNAR A., SET E., DRAGOMİR S. S., some generalized Hermite-Hadamard type inequalities involving fractional integral operator for functions whose second derivatives in absolute value are s-convex, Acta Math. Univ. Comenianae, in press

[13] HUDZİK H. and MALİGRANDA L., Some remarks on s-convex functions, Aequationes Math., 48, 100-111, 1994.

[14] KIRMACI U.S., Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula, Appl.Math.Comput. 147 (1), 137-146 (2004).

[15] MAKÓ J. and PÁLES Zs., Approximate Hermite-Hadamard type inequalities for approximately convex functions, Math. Inequal. Appl., 16 (2), 507-526, (2013).

[16] MAKÓ J. and PÁLES Zs., Implications between approximate convexity properties and approximate Hermite-Hadamard inequalities, Cent. Eur. J. Math. 10 (2012), no. 3, 1017-1041.

[17] MİTRİNOVİĆ D. S., PEČARİĆ J. , and FİNK A.M., Classical and new inequalities in analysis, KluwerAcademic, Dordrecht, 1993.

[18] ÖZDEMİR M.E., GÜRBÜZ M. and AKDEMİR A.O., Inequalities for h-Convex Functions via Further Properties, RGMIA Research Report Collection Volume 14, article 22, 2011.

[19] SARIKAYA M.Z., SAĞLAM A. and YILDIRIM H., some Hadamard-type inequalities for h-convex functions, J. Math. Inequal. 2 (3) (2008) 335-341.

[20] NOOR M.A., NOOR K.I., IFTİKHAR S. On Integral Inequalities of Hermite-Hadamard Type for Harmonic (h,s)-Convex Functions, International Journal of Analysis and Applications 11(1) (2016), 61-69.

[21] SARIKAYA M.Z., SET E. and ÖZDEMİR M.E., On some new inequalities of Hadamard-type involving h-convex functions, Acta Math. Univ. Comenian LXXIX (2) (2010) 265-272.

[22] PACHPATTE B. G., Mathematical Inequalities, North-Holland Mathematical Library, Elsevier Science B.V. Amsterdam, 2005.

[23] PEČARİĆ J.E., PROSCHAN F., TONG Y.L., Convex Functions, Partial Orderings and Statistical Applications, Academic Press, 1991.

[24] TOADER G.H., ON a generalisation of the convexity, Mathematica, 30 (53) (1988), 83-87.

[25] TOADER G.H., Some generalisations of the convexity, Proc. Colloq. Approx. Optim, Cluj-Napoca (Romania), 1984, 329-338.

[26] TUNÇ M. , Ostrowski-type inequalities via h-convex functions with applications to special means, Jour. Ineq. and Appl. 2013 (1), 326, 2013.

[27] VAROŠANEC S., On h-convexity, J. Math. Anal. Appl., Volume 326, Issue 1 (2007), 303-311.