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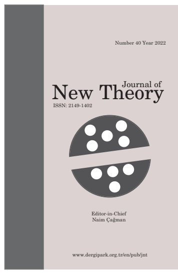

4-Dimensional 2-Crossed Modules

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Abstract — In this work, we defined a new category called 4-Dimensional 2-crossed modules. We identified the subobjects and ideals in this category. The notion of the subobject is a generalization of ideas like subsets from set theory, subspaces from topology, and subgroups from group theory. We then exemplified subobjects and ideals in the category of 4-Dimensional 2-crossed modules. A quotient object is the dual concept of a subobject. Concepts like quotient sets, spaces, groups, graphs, etc. are generalized with the notion of a quotient object. Using the ideal, we obtain the quotient of two subobjects and prove that the intersection of finite ideals is also an ideal in this category.

Keywords — *Crossed Module, subobject, ideal, category*

Mathematics Subject Classification (2020) — 18D99, 55P15

1. Introduction

In order to generalize the well-known conclusion that the category of crossed modules is equivalent to the category of simplicial groups with a Moore complex of length 2, Conduché introduced 2-crossed modules of groups in [1] demonstrating that the category of simplicial groups with a three-dimensional Moore complex and the category of 2-crossed modules are equivalent. Therefore, 2-crossed modules also serve as algebraic models for connected homotopy 3-types or pointed CW-complexes X such that $\pi(X) = 0$ if $i > 3$. The idea of 2-crossed modules is adapted for algebras by Grandjean and Vale [2].

The homotopy 3-types can also be represented by crossed squares [3] and quadratic modules [4]. The categories of braided regular crossed modules [5] and Gray 3-groupoids with a single object [6], are other categories that are equivalent to the category of 2-crossed modules. The category of 2-crossed modules is also shown in [7] to be equivalent to the categories of neat crossed squares and neat maps.

For the algebraic description of pointed relative CW–complexes with cells in dimensions 4, Baues and Bleile introduced the concept of 4–dimensional quadratic complexes [8] to investigate the presentation of a space X as mapping cone of a map $\partial(X)$ under a space D. The need for a proper understanding of the relevant algebraic and categorical structure of the 4-Dimensional 2-crossed modules are motivated by studies and examples [9–15] for higher categorical structures. In this work, we defined the notion of 4-Dimensional 2-crossed modules in order to look into any potential equivalence between homotopy 4-types, which was inspired by the work of Baues and Bleile. Examining how 4-Dimensional 2-crossed modules relate to an algebraic structure resembling 2-crossed modules is the main goal of this paper. In order to achieve this, we first introduce the category of 4-Dimensional 2-crossed modules before describing subobjects and ideals in full detail. In conclusion, we demonstrate that the quotient of the objects in this category is a 4-Dimensional 2-crossed module.

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The main ideas of this work can be given as:
- To construct a new category weaker than homotopy 4-types and stronger than homotopy 3-types,
- To fully describe the subobjects and ideals within this category,
- To construct the quotient object by using ideals in this category.

2. 4-Dimensional 2-Crossed Modules

Grandjeán and Vale [2] have given a definition of 2-crossed modules of algebras. The following is an equivalent formulation of that concept.

A 2-*crossed module* of k-algebras consists of a complex of P-algebras $L \overset{\partial_2}{\to} M \overset{\partial_1}{\to} P$ together with an action of P on all three algebras and a P-linear mapping

$$\{-,-\} : M \times M \to L$$

which is often called the Peiffer lifting such that the action of P on itself is by multiplication, ∂_2 and ∂_1 are P-equivariant.

PL1 : $\partial_2 \{m_0, m_1\} = m_0 m_1 - \partial_1 (m_1) \cdot m_0$
PL2 : $\{\partial_2 (l_0), \partial_2 (l_1)\} = l_0 l_1$
PL3 : $\{m_0, m_1 m_2\} = \{m_0 m_1, m_2\} + \partial_1 (m_2) \cdot \{m_0, m_1\}$
PL4 : $\{m, \partial_2 (l)\} + \{\partial_2 (l), m\} = \partial_1 (m) \cdot l$
PL5 : $\{m_0, m_1\} \cdot p = \{m_0 \cdot p, m_1\} = \{m_0, m_1 \cdot p\}$

for all $m, m_0, m_1, m_2 \in M, l, l_0, l_1 \in L$ and $p \in P$. Note that we have not specified that M acts on L. We could have done that as follows: if $m \in M$ and $l \in L$, define

$$m \cdot l = \{m, \partial_2 (l)\}$$

From this equation (L, M, ∂_2) becomes a crossed module. We can split **PL4** into two pieces:

PL4 :

$$\begin{array}{lll} (a) & \{m, \partial_2 (l)\} & = m \cdot l \\ (b) & \{\partial_2 (l), m\} & = m \cdot l - \partial_1 (m) \cdot l \end{array}$$

We denote such a 2-crossed module of algebras by $\{L, M, P, \partial_2, \partial_1\}$.

A morphism of 2-crossed modules is given by the following diagram

$$\begin{array}{ccccc} L & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & P \\ \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 \\ L' & \xrightarrow[\partial_2']{} & M' & \xrightarrow[\partial_1']{} & P' \end{array}$$

where $f_0 \partial_1 = \partial_1' f_1, f_1 \partial_2 = \partial_2' f_2$

$$f_1 (p \cdot m) = f_0 (p) \cdot f_1 (m) \quad , \quad f_2 (p \cdot l) = f_0 (p) \cdot f_2 (l)$$

for all $m \in M, l \in L, p \in P$ and

$$\{-,-\} (f_1 \times f_1) = f_2 \{-,-\}$$

We thus get the category of 2-crossed modules denoting it by $\mathsf{X_2Mod}$. In [4], Baues developed the concept of 4-dimensional quadratic modules after defining quadratic modules. Adapting this definition for 2-crossed modules we get a complex of algebras

$$\sigma : K \xrightarrow{\partial_4} L \xrightarrow{\partial_3} M \xrightarrow{\partial_2} P$$

such that

1. $(L, M, P, \partial_2, \partial_3)$ is a 2-crossed module with Peiffer lifting $\{-,-\} : M \times M \to L$;

2. K is a L–module such that $\partial_2(M)$ acts trivially and

3. ∂_4 is a homomorphism of Q_1–modules, such that $\partial_3\partial_4 = 0$.

A *morphism of* 4*–dimensional 2-crossed modules*, $f : \sigma \to \sigma'$, is a sequence of morphisms

$$\begin{array}{ccccccc} \sigma : K & \xrightarrow{\partial_4} & L & \xrightarrow{\partial_3} & M & \xrightarrow{\partial_2} & P \\ {\scriptstyle f_4}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow \\ \sigma' : K' & \xrightarrow[\partial_4]{} & L' & \xrightarrow[\partial_3']{} & M' & \xrightarrow[\partial_2']{} & P' \end{array}$$

such that (f_3, f_2, f_1) yields a morphism of 2-crossed modules, f_4 is an f_1–equivariant homomorphism of modules and $\partial_4 f_4 = f_3\partial_4$. We denote the category of 4-Dimensional 2-crossed modules by X_2Mod^{4D}.

Next, we will define the subobjects in X_2Mod^{4D}.

Definition 2.1. Let

$$\sigma : Q_4 \xrightarrow{\partial_4} Q_3 \xrightarrow{\partial_3} Q_2 \xrightarrow{\partial_2} Q_1$$

be an object in X_2Mod^{4D}. Then we say that

$$\sigma' : Q_4' \xrightarrow{\partial_4'} Q_3' \xrightarrow{\partial_3'} Q_2' \xrightarrow{\partial_2'} Q_1'$$

is a subobject of σ if

1. Q_4' is a subalgebra of Q_4, Q_3' is a subalgebra of Q_3 and Q_2' is a subring of Q_2;
2. $\partial_2' : Q_2 \to Q_1$ is a subpre-crossed module of $\partial_2 : Q_2 \to Q_1$;
3. The actions of Q_2' on Q_4' and Q_3' via Q_1' is induced from the actions of Q_2 on Q_4 and Q_3 via Q_1;
4. σ' is an object in X_2Mod^{4D} and
5. The diagram

$$\begin{array}{ccccccc} & Q_2 \times Q_2 & \xrightarrow{\{-,-\}} & & & & \\ & {\scriptstyle \mu_1\times\mu_1}\downarrow & & & & & \\ Q_4 & \xrightarrow{\partial_3} & Q_3 & \xrightarrow{\partial_2} & Q_2 & \xrightarrow{\partial_1} & Q_1 \\ & & {\scriptstyle \mu_2}\downarrow & & {\scriptstyle \mu_1}\downarrow & & {\scriptstyle \mu_0}\downarrow \\ & Q_2' \times Q_2' & \xrightarrow{\{-,-\}'} & & & & \\ Q_4' & \xrightarrow[\partial_3']{} & Q_3' & \xrightarrow[\partial_2']{} & Q_2' & \xrightarrow[\partial_1']{} & Q_1' \end{array}$$

is commutative where for i=1, 2, 3 μ_i are injections.

Example 2.2. Let

$$\sigma : Q_2 \otimes Q_2 \xrightarrow{\partial_4} Q_2 \otimes Q_2 \xrightarrow{\partial_3} Q_2 \xrightarrow{\partial_2} Q_1$$

be an object in X_2Mod^{4D} with $Id : Q_2 \otimes Q_2 \to Q_2 \otimes Q_2$ as Peiffer lifting. If K_2 is ideal of Q_2 and K_1 is a subring of Q_1 that is $\delta_2 : K_2 \to K_1$ is a subpre-crossed module of $\partial_2 : Q_2 \to Q_1$ then

$$\sigma' : K_2 \otimes K_2 \xrightarrow{Id} K_2 \otimes K_2 \xrightarrow{\delta_3} K_2 \xrightarrow{\delta_2} K_1$$

is a subobject of σ with $Id : K_2 \otimes K_2 \to K_2 \otimes K_2$ as Peiffer lifting.

Example 2.3. Let R be a k-algebra and

$$\begin{array}{ccccccc} & & R/R^2 \otimes R/R^2 & & & & \\ & & \Big\downarrow {\scriptstyle \{-,-\}=Id} & & & & \\ \sigma : R/R^2 \otimes R/R^2 & \xrightarrow[Id]{} & R/R^2 \otimes R/R^2 & \xrightarrow[\partial]{} & R & \xrightarrow[Id]{} & R \end{array}$$

be an object in X_2Mod^{4D}. If I is an ideal of R and J is a subring of I then,

$$\begin{array}{ccccccc} & & I/I^2 \otimes I/I^2 & & & & \\ & & \Big\downarrow {\scriptstyle \{-,-\}=Id} & & & & \\ \sigma' : I/I^2 \otimes I/I^2 & \xrightarrow[Id]{} & I/I^2 \otimes I/I^2 & \xrightarrow[\partial']{} & I & \xrightarrow[i]{} & J \end{array}$$

is a subobject of σ.

3. Ideals in X_2Mod^{4D}

In this section we will define ideals in X_2Mod^{4D} and intersections of two ideals is an ideal in this category.

Definition 3.1. Let

$$\sigma : K \xrightarrow{\partial_4} L \xrightarrow{\partial_3} M \xrightarrow{\partial_2} P$$

be an object in X_2Mod^{4D}. Then we say that

$$\sigma' : K' \xrightarrow{\partial_4'} L' \xrightarrow{\partial_3'} M' \xrightarrow{\partial_2'} P'$$

is an ideal of σ if

1. Let $L'L \subseteq L'$, $K'K \subset K'$ and M' be an ideal of M;
2. (a) $M'M \subset M$ and P' is an ideal of P;
   (b) for $p \in P'$ and $m \in M$, $p \cdot m \in M'$;
   (c) for $m' \in M'$ and $p \in P$, $p \cdot m' \in M'$;
3. For $m' \in M'$, $l \in L$, $k \in K$ $\partial_1(m') \cdot l \in L'$, $\partial_1(m') \cdot k \in K'$;
4. For $l' \in L'$, $m \in M$ and $k' \in K'$, $\partial_1(m) \cdot l' \in L'$, $\partial_1(m) \cdot k' \in K'$;
5. K' and L' are P-algebras. That is,
   (a) For $p' \in P'$, $l \in L$ and $k \in K$, $p' \cdot l \in L'$, $p' \cdot k \in K'$;
   (b) For $l' \in L'$, $p \in P$ and $k' \in K'$, $p \cdot l' \in L'$, $p \cdot k' \in K'$;

Example 3.2. Let I be an ideal of R if

$$\theta : K \longrightarrow L \longrightarrow I \xrightarrow{id} I$$

is an object in X_2Mod^{4D} then θ is an ideal of the 4-Dimensional 2-crossed module σ in Example 2.3.

Example 3.3. Let I and I' be ideals of R and $(I,R,\mu),(I',R,\mu')$ be two nil(2)-modules. Since $(I\cap I',I,\vartheta)$ is a nil(2)-module we have,

i)

$$\sigma' : K \xrightarrow{\partial_2} L \xrightarrow{\pi} I\cap I' \xrightarrow{\vartheta'} I'$$

is an ideal of

$$\sigma : K \xrightarrow{\delta_2} I\times I \xrightarrow{\pi} I \xrightarrow{\mu} R$$

ii)

$$\theta' : K \xrightarrow{\partial_2} L \xrightarrow{\pi} I\cap I' \xrightarrow{\vartheta} I$$

is an ideal of

$$\theta : K \xrightarrow{\delta_2'} I'\times I' \xrightarrow{\pi} I' \xrightarrow{\mu'} R$$

where $L=(I\cap I')\times(I\cap I')$ with Peiffer liftings as identities and π as projection.

Theorem 3.4. The intersection of finite ideals is an ideal in X_2Mod^{4D}.

Proof. Let

$$\sigma : K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$$

be an object in X_2Mod^{4D}. If

$$\sigma_1 : K_1 \xrightarrow{\partial_3'} L_1 \xrightarrow{\partial_2'} M_1 \xrightarrow{\partial_1'} P_1$$

and

$$\sigma_2 : K_2 \xrightarrow{\partial_3''} L_2 \xrightarrow{\partial_2''} M_2 \xrightarrow{\partial_1''} P_2$$

be two subobjects of σ. Then we have,

1. Since $L_1L\subset L_1$, $L_2L\subset L_2$ and $K_1K\subset K_1, K_2K\subset K_2$ we get

$$(L_1\cap L_2)L\subseteq L_1\cap L_2$$

and

$$(K_1\cap K_2)K\subset K_1\cap K_2$$

2. a. Since $M_1M\subseteq M_1$ and $M_2M\subseteq M_2$ we get

$$(M_1\cap M_2)M\subseteq M_1\cap M_2$$

where $N_1\trianglelefteq P$ and $P_2\trianglelefteq P$ imply $P_1\cap N_2\trianglelefteq P$.

b. Since σ_1 is an ideal of σ for $x\in P_1$ we have $x\cdot m\in M_1$ and σ_2 is an ideal of σ for $x\in P_2$ we have $x\cdot m\in M_2$. Then we get $x\cdot m\in M_1\cap M_2$.

c. Since σ_1 and σ_2 are ideals of σ, for $y\in M_1, y\in M_2$, and $p\in P$ we have $p\cdot y\in M_1$ and $p\cdot y\in M_2$ which implies $p\cdot y\in M_1\cap M_2$.

3. Since σ_1 is an ideal of σ for $y\in M_1$ and $l\in L$ we have $\partial(y)\cdot l\in L_1$ and since ∂_2 is an ideal of σ for $\partial(y)\in M_2$ and $l\in L$ we have $\partial(y)\cdot l\in L_2$. Then we get $\partial(y)\cdot l\in L_1\cap L_2$. Similarly for $y\in M_1\cap M_2$ and $k\in K$ we get $\partial(y)\cdot k\in K_1\cap K_2$.

4. Since for $z\in L_1$ and $m\in M$ $\partial_1(m)\cdot z\in L_1$ and for $z\in L_2, m\in M$ $\partial_1(m)\cdot z\in L_2$ we have $\partial_1(m)\cdot z\in L_1\cap L_2$. Similarly for $t\in K_1\cap K_2$ we get $\partial_1(m)\cdot t\in K_1\cap K_2$.

5. a. Let $x\in P_1\cap P_2$ for $x\in P_1$ and $l\in L$ we have $x\cdot l\in L_1$ and for $x\in P_2$ and $l\in L$ we have $x\cdot l\in L_2$. Then we get $x\cdot l\in L_1\cap L_2$. Simlarly for $x\in P_1\cap P_2$ and $k\in K$ we have $x\cdot k\in K_1\cap K_2$.

b. Let $z\in L_1\cap L_2$ for $p\in P$ and $z\in L_1$ we have $p\cdot z\in L_1$ and for $z\in L_2$ and $p\in P$ we have $p\cdot z\in L_2$. Then we have $p\cdot z\in L_1\cap L_2$. Similarly for $t\in K_1\cap K_2$ we have $p\cdot k\in K_1\cap K_2$.

As a result the object

$$K_1\cap K_2 \xrightarrow{(\partial_3',\partial_3'')} L_2\cap L_2 \xrightarrow{(\partial_2',\partial_2'')} M_1\cap M_2 \xrightarrow{(\partial_1',\partial_1'')} P_1\cap P_2$$

in X_2Mod^{4D} is an ideal of

$$K \xrightarrow{\partial_3} L \xrightarrow{\partial_3} M \xrightarrow{\partial_3} P$$

□

4. Quotient object in X_2Mod^{4D}

In this section using the ideal σ' of an object σ in X_2Mod^{4D}, we prove that the quotient σ/σ' is an object in X_2Mod^{4D}.

Let

$$\sigma' : K' \xrightarrow{\partial_3'} L' \xrightarrow{\partial_2'} M' \xrightarrow{\partial_1'} P'$$

be an ideal of

$$\sigma : K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$$

in X_2Mod^{4D}. Since $M \trianglelefteq M$ and $P' \trianglelefteq P$, the action of P/P' on M/M' can be given as

$$(x + P') \cdot (y + M') = x \cdot y + M'$$

and for $p' \in P'$

$$\begin{aligned} p' \cdot (y + M') &= p' \cdot y + M' \\ &= 0 + M' \qquad (\because p' \cdot y \in M) \end{aligned}$$

P' acts on M/M' trivially. Next, we will show that

$$\begin{aligned} \delta : M/M' &\to P/P' \\ y + M' &\mapsto \partial_1(y) + P' \end{aligned}$$

is a well defined nil(2)-module morphism.

∂_1' is the restriction of ∂_1 to M' implies $\partial_1' \subset P'$. For $m' \in M'$ we have $\partial_1'(m') \cdot (l + L') = \partial_1'(m') \cdot l + L' = L'(\sigma'$ is an ideal of σ then $\partial_1'(m') \cdot l \in L'$). That is the action of M' on L/L' via P' must be trivial. Therefore M/M' acts on L/L' via P/P'. This action can be defined as:

$$(\partial_1(m) + M) \cdot (l + L') = \partial_1(m) \cdot l + L'$$

for $l + L' \in L/L'$ and $m + M' \in M/M'$.

Theorem 4.1. Let

$$\sigma' : K' \xrightarrow{\partial_3'} L' \xrightarrow{\partial_2'} M' \xrightarrow{\partial_1'} P'$$

be an ideal of

$$\sigma : K \xrightarrow{\partial_3} L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$$

in X_2Mod^{4D}. Then

$$\sigma/\sigma' : K/K' \xrightarrow{\delta_3} L/L' \xrightarrow{\delta_2} M/M' \xrightarrow{\delta_1} P/P'$$

is an object in X_2Mod^{4D}.

Proof. 1. PL1. For $x_1 + M', x_2 + M' \in M/M'$:

$$\begin{aligned} \delta\{x_1 + M', x_2 + M'\}_\delta &= \partial_2\{x_1, x_2\}_\sigma + L' \\ &= x_1x_2 - x_1 \cdot \partial_1(x_2) + L' \end{aligned}$$

PL2. For $y_1 + M', y_2 + M' \in L/L'$:

$$\begin{aligned} \{\delta_2(y_1 + L'), \delta(y_2 + L')\}_\delta &= \{\partial_2(y_1) + M', \partial_2(y_2) + M'\}_\sigma \\ &= \{\partial_2(y_1), \partial_2(y_2)\}_\sigma + M' \\ &= y_1y_2 + M' \\ &= (y_1 + M')(y_2 + M') \end{aligned}$$

PL3. For $x_o + M', x_1 + M', x_2 + M' \in M/M'$:

$$\begin{aligned}
\{x_0 + M', x_1x_2 + M'\}_\delta &= \{x_0, x_1x_2\}_\sigma + M' \\
&= \{x_0x_1, x_2\}_\sigma + \partial_1(x_2)\{x_0, x_1\}_\sigma + M' \\
&= \{x_0x_1, x_2\}_\sigma + M' + \partial_1(x_2)\{x_0, x_1\}_\sigma + M' \\
&= \{x_0x_1 + M', x_2 + M'\}_\delta + \partial_1(x_2)\{x_0 + M', x_1 + M'\}_\delta
\end{aligned}$$

PL4. a. For $y + L' \in L/L'$ and $x + M' \in M/M'$:

$$\begin{aligned}
\{\delta_2(y + L'), x + M'\}_\delta &= \{\partial_2(y) + M', x + M'\}_\delta \\
&= \{\partial_2(y), x\}_\sigma + M' \\
&= (x \cdot y - \partial_1(x) \cdot y) + M'
\end{aligned}$$

b. For $y + L' \in L/L'$ and $x + M' \in M/M'$:

$$\begin{aligned}
\{x + M', \delta_2(y + L')\}_\delta &= \{x + M', \partial_2(y) + M'\}_\delta \\
&= \{x, \partial_2(y)\}_\sigma + M' \\
&= (x \cdot y) + M'
\end{aligned}$$

PL5. For $x_o + M', x_1 + M' \in M/M'$ and $t + P' \in P/P'$:

$$\begin{aligned}
\{x_0 + M', x_1 + M'\}_\delta \cdot (t + P') &= (\{x_0, x_1\}_\sigma \cdot (t + P')_\delta) + M' \\
&= (\{x_0 \cdot t, x_1\}_\sigma) + M' \\
&= \{(x_0 \cdot t) + M', x_1 + M'\}_\delta
\end{aligned}$$

and

$$\begin{aligned}
\{x_0 + M', x_1 + M'\}_\delta \cdot (t + P') &= (\{x_0, x_1\}_\sigma \cdot (t + P')_\delta) + M' \\
&= (\{x_0, x_1 \cdot t\}_\sigma) + M' \\
&= \{x_0 + M', (x_1 \cdot t) + M'\}_\delta
\end{aligned}$$

2. Since σ' is an ideal of σ/σ' for $m+M' \in M/M'$ we have $\delta_1(m+M')\cdot(k+K') = \delta_1(m+M')\cdot(k+K') = K'$. That is M/M' acts on K/K' via P' trivially. Therefore K/K' is a P/P'-module.
3. For $k + K' \in K/K'$:

$$\begin{aligned}
\delta_2\delta_3(k + K') &= \delta_2(\partial_3(k) + L') \\
&= \partial_2(\partial_3(k) + M') \\
&= 0 + M' \\
&= 0_{M/M'}
\end{aligned}$$

□

5. Conclusion

In this work, we introduced a new category weaker than homotopy 4-types and stronger than homotopy 3-types. As an intriguing result 4–dimensional 2-crossed modules serve as a bridge to investigate categorical equivalences between homotopy 3-types and homotopy 4-types. The categorical equivalences of the category X_2Mod^{4D} and other homotopy 3–4 types from the various models can be explored as further research. The research presented in this study has addressed fundamentals of the category X_2Mod^{4D}, and these provide guidance for future work in the following:

- Constructing categorical properties such as limit, product, pullback, pushout, etc.,
- Embedding theorem can be adapted for the category X_2Mod^{4D},
- Freeness conditions and simplicial properties can be examined.

In addition, categorical equivalences and properties are also reference points for further work of X_2Mod^{4D}.

Author Contributions

The author read and approved the last version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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