Journal of Mathematical Sciences and Modelling, 5 (1) (2022) 35-39
Research Article

Journal of Mathematical Sciences and Modelling

Journal Homepage: www.dergipark.gov.tr/jmsm
ISSN 2636-8692
DOI: http://dx.doi.org/10.33187/jmsm.993823

Approximating Fixed Points of Generalized $\alpha-$Nonexpansive Mappings by the New Iteration Process

Seyit Temir[1*] and Öznur Korkut[1]

[1]*Department of Mathematics, Art and Science Faculty, Adıyaman University, 02040, Adıyaman, Turkey*
[*]*Corresponding author*

Article Info

Keywords: *Convergence, Fixed point, Generalized $\alpha-$nonexpansive mappings, Iteration processes, Uniformly convex Banach spaces.*
2010 AMS: *47H09, 47H10.*
Received: *10 September 2021*
Accepted: *27 December 2021*
Available online: *30 April 2022*

Abstract

In this paper we introduce a new iteration process for approximation of fixed points. We numerically compare convergence behavior of our iteration process with other iteration process like M-iteration process. We also prove weak and strong convergence theorems for generalized $\alpha-$nonexpansive mappings by using new iteration process. Furthermore we give an example for generalized $\alpha-$nonexpansive mapping but does not satisfy (C) *condition*.

1. Introduction and Preliminaries

Let be X be a real Banach space and K be a nonempty subset of X, and $T: K \to K$ be a mapping. A point $x \in K$ is called a fixed point of $T: K \to K$ if $x = Tx$. We denote $F(T)$ the set of all fixed points of T. A mapping $T: K \to K$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. T is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$. In the last 60 years, many iteration processes have been developed regarding the fixed point approach. Recently, with the development of iteration processes, a faster approach to the fixed point has gained importance. Some of well-known iteration processes are Mann iterative scheme [1], Ishikawa [2], Noor [3], S-iteration process [4], Abbas and Nazir [5], Picard-S [6], Thakur et al. [7] and so on.
In 2018, Ullah and Arshad [8] introduced the following iteration process called M-iteration process : for arbitrary $x_1 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = (1-a_n)x_n + a_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n, \forall n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\{a_n\} \in [0,1]$.
Motivated by above, in this paper, we introduce new iteration scheme:for arbitrary $x_1 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = T((1-b_n)x_n + b_n T x_n), \\ y_n = T z_n, \\ x_{n+1} = T((1-a_n)T x_n + a_n T y_n), \forall n \in \mathbb{N}, \end{cases} , \tag{1.2}$$

where $\{a_n\}$ and $\{b_n\} \in [0,1]$.
In order to show numerically that our new iteration process (1.2) have a good speed of convergence comparatively to (1.1), we consider the following example.

Example 1.1. *Let us define a function $T: [0,10) \to [0,10)$ by $T(x) = \sqrt{2x+3}$. Then clearly T is a contraction map. Let $a_n = 0.70, b_n = 0.30$ for all n. Set the stop parameter to $\|x_n - 3\| \leq 10^{-6}$, 3 is the fixed point of T. The iterative values for initial value $x_1 = 4$ are given in Table 1. The efficiency of new iteration process is clear. We can see that our new iteration process (1.2) have a good speed of convergence comparatively to (1.1) iteration process.*

Email addresses and ORCID numbers: seyittemir@adiyaman.edu.tr, 0000-0001-9056-2354 (S. Temir), oevrenxxx79@hotmail.com (Ö. Korkut)

Table 1: Sequences generated by M-iteration and New iteration processes for mapping T of Example 1.1.

	M-iteration	New iteration
x_1	4	4
x_2	3.083577194937360	3.037893699789630
x_3	3.007388352660220	3.001521367442330
x_4	3.000656421483590	3.000061224295530
x_5	3.000058346040820	3.000002464079130
x_6	3.000005186294710	3.000000099171560
x_7	3.000000461003820	3.000000003991350
x_8	3.000000040978120	3.000000000160640
x_9	3.000000003642500	3.000000000006470
x_{10}	3.000000000323780	3.000000000000260
x_{11}	3.000000000028780	3.000000000000010
x_{12}	3.000000000002560	3.000000000000000
x_{13}	3.000000000000230	3.000000000000000
x_{14}	3.000000000000020	3.000000000000000
x_{15}	3.000000000000000	3.000000000000000

In the recent years, several generalizations of nonexpansive mappings and related fixed point have have been studied by many authors (see [7], [8], [9], [10], [12], [14], [15], [16], [17], [20]). In 2008, Suzuki [17] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called (C) *condition*. Let K be a nonempty convex subset of a Banach space X, a mapping $T : K \to K$ is satisfy *condition* (C) if for all $x,y \in K$, $\frac{1}{2}\|x-Tx\| \le \|x-y\|$ implies $\|Tx-Ty\| \le \|x-y\|$. Suzuki [17] showed that the mapping satisfying *condition* (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. The mapping satisfy *condition* (C) is called Suzuki generalized nonexpansive mapping. In 2011, Aoyama and Kohsaka [9] introduced the class of $\alpha-$nonexpansive mappings in the setting of Banach spaces and obtained some fixed point results for such mappings. A mapping $T : K \to K$ is called a $\alpha-$nonexpansive mapping if there exists an $\alpha \in [0,1)$ such that for each $x,y \in K$,

$$\|Tx-Ty\|^2 \le \alpha\|Tx-y\|^2 + \alpha\|x-Ty\|^2 + (1-2\alpha)\|x-y\|^2.$$

In [14], authors introduced the following class of nonexpansive type mappings and obtained some fixed point results for this class of mappings. A mapping $T : K \to K$ is called a generalized α-nonexpansive mapping if there exists an $\alpha \in [0,1)$ and for each $x,y \in K$, $\frac{1}{2}\|x-Tx\| \le \|x-y\|$ implies

$$\|Tx-Ty\| \le \alpha\|Tx-y\| + \alpha\|Ty-x\| + (1-2\alpha)\|x-y\|.$$

In 2019, Şahin [15] studied the M-iteration process in hyperbolic spaces and proved some strong and Δ-convergence theorems of this iteration process for generalized nonexpansive mappings. In 2021, Ullah et al. [20] introduced some convergence results for generalized α-nonexpansive mappings using M-iteration process in the framework of Banach spaces.
Inspired and motivated by these facts, we consider generalized α-nonexpansive mappings which properly contains, the α-nonexpansive mappings. Also we give an example for generalized $\alpha-$nonexpansive mapping but does not satisfy (C) *condition*. Further we prove some convergence theorems of new iterative process (1.2) to fixed point for generalized α-nonexpansive mappings in a Banach space.
The following definitions will be needed in proving our main results.
A Banach space X is said to be uniformly convex [11] if for each $\varepsilon \in (0,2]$ there exists $\delta > 0$ such that $\|\frac{(x+y)}{2}\| \le 1-\delta$ for all $x,y \in X$ with $\|x\| = \|y\| = 1$ and $\|x-y\| > \varepsilon$.
Recall that a Banach space X is said to satisfy *Opial's condition* [13] if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ weakly as $n \to \infty$ and for all $y \in X$ with $y \neq x$ imply that $\liminf_{n\to\infty}\|x_n-x\| < \liminf_{n\to\infty}\|x_n-y\|$.
In what follows, we give some definitions and lemmas to be used in main results:
Let $\{x_n\}$ be a bounded sequence in a Banach space X. For $x \in X$, we set

$$r(x,\{x_n\}) = \limsup_{n\to\infty} \|x_n-x\|.$$

The asymptotic radius of $\{x_n\}$ relative to K is defined by

$$r(K,\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in K\}.$$

The asymptotic center of $\{x_n\}$ relative to K is the set

$$A(K,\{x_n\}) = \{x \in K : r(x,\{x_n\}) = r(K,\{x_n\})\}.$$

It is known that, in a uniformly convex Banach space, $A(K,\{x_n\})$ consists of exactly one-point.

Lemma 1.2. *[18] Suppose that X is a uniformly convex Banach space and $0 < k \le t_n \le m < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequence of X such that $\limsup_{n\to\infty}\|x_n\| \le \xi$, $\limsup_{n\to\infty}\|y_n\| \le \xi$ and $\lim_{n\to\infty}\|t_nx_n+(1-t_n)y_n\| = \xi$ hold for $\xi \ge 0$. Then $\lim_{n\to\infty}\|x_n-y_n\| = 0$.*

Let $\{u_n\}$ in K be a given sequence. $T : K \to K$ with the nonempty fixed point set $F(T)$ in K is said to satisfy *Condition (I)* [19] with respect to the $\{u_n\}$ if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|u_n-Tu_n\| \ge f(d(u_n,F(T))$ for all $n \ge 1$.
Now we give the following well-known facts about generalized α-nonexpansive mapping, which can be found in [14].

Lemma 1.3. *(1) If T is Suzuki generalized nonexpansive mapping then T is a generalized α -nonexpansive mapping.*

(2) If T is a generalized α -nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.

(3) If T is a generalized α -nonexpansive mapping, then $F(T)$ is closed. Moreover if X is strictly convex and K is convex, then $F(T)$ is also convex.

(4) If T is a generalized α -nonexpansive mapping, then for each $x,y \in K$,

$$\|x-Ty\| \leq (\frac{3+\alpha}{1-\alpha})\|Tx-x\| + \|x-y\|.$$

(5) If X has Opial property, T is a generalized α -nonexpansive mapping, $\{x_n\}$ converges weakly to a point $x*$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $x* \in F(T)$.That is, $I-T$ is demiclosed at zero, where I is the identity mapping on X.

Now we give an example where T is a generalized α-nonexpansive mapping but does not satisfy condition (C).

Example 1.4. *Let $K = [0,2]$ be a subset of $\mathbb{R}$ endowed with the usual norm. Define a mapping $T : K \to K$ by*

$$Tx = \begin{cases} 0, & x \neq 2, \\ 1, & x = 2. \end{cases}$$

For $x \in (1, 1.33]$ and $y = 2$, then we have $\frac{1}{2}|x-Tx| \leq |x-y|$ and $|Tx-Ty| = 1 > 2-x = |x-y|$. Thus T does not satisfy Suzuki's condition (C). However, T is a generalized $\alpha-$nonexpansive mapping with $\alpha \geq \frac{1}{3}$.

2. Weak and Strong Convergence Theorems of New Iteration Process for Generalized $\alpha-$Nonexpansive Mapping

In this section, we prove weak and strong convergence theorems of new iterative scheme defined by (1.2) for generalized $\alpha-$nonexpansive mapping in a uniformly convex Banach space.

Lemma 2.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X , T be a generalized $\alpha-$nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrary chosen $x_1 \in K$, let $\{x_n\}$ be a sequence generated by (1.2) with $\{a_n\}$ and $\{b_n\}$ real sequences in $[0,1]$, then $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F(T)$.*

Proof. For any $p \in F(T)$, using (1.2), we have,

$$\begin{aligned} \|z_n - p\| &= \|T((1-b_n)x_n + b_nTx_n) - p\| \\ &\leq \|(1-b_n)(x_n-p) + b_n(Tx_n - p)\| \\ &\leq (1-b_n)\|x_n - p\| + b_n\|x_n - p\| = \|x_n - p\|. \end{aligned} \tag{2.1}$$

Using (1.2) and (2.1), we get

$$\|y_n - p\| = \|Tz_n - p\| \leq \|z_n - p\| \leq \|x_n - p\| \tag{2.2}$$

By using (1.2) and (2.2) , we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|T((1-a_n)Tx_n + a_nTy_n) - p\| \\ &\leq \|(1-a_n)(Tx_n - p) + a_n(Ty_n - p)\| \\ &\leq (1-a_n)\|x_n - p\| + a_n\|y_n - p\| \\ &\leq (1-a_n)\|x_n - p\| + a_n\|x_n - p\| = \|x_n - p\| \end{aligned} \tag{2.3}$$

This implies that $\{\|x_n - p\|\}$ is bounded and non-increasing for all $p \in F(T)$. It follows that $\lim_{n\to\infty}\|x_n - p\|$ exists. □

Theorem 2.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X , T be a generalized $\alpha-$nonexpansive mapping. For arbitrary chosen $x_1 \in K$, let $\{x_n\}$ be a sequence in K defined by (1.2) with the real sequences $\{a_n\}$ in $(0,1]$ and $\{b_n\}$ in $[k,m]$ for some $k,m \in (0,1)$, then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and by Lemma 2.1, $\lim_{n\to\infty}\|x_n - p\|$ exists. Put $\lim_{n\to\infty}\|x_n - p\| = \xi$. From (2.1) and (2.2) we have

$$\limsup_{n\to\infty}\|z_n - p\| \leq \limsup_{n\to\infty}\|x_n - p\| \leq \xi,$$

and

$$\limsup_{n\to\infty}\|y_n - p\| \leq \limsup_{n\to\infty}\|x_n - p\| \leq \xi,$$

and also we have

$$\limsup_{n\to\infty}\|Tx_n - p\| \leq \limsup_{n\to\infty}\|x_n - p\| \leq \xi.$$

On the other hand,

$$\begin{aligned}\|x_{n+1}-p\| &= \|T((1-a_n)Tx_n+a_nTy_n)-p\| \leq \|(1-a_n)(x_n-p)+a_n(Ty_n-p)\| \\ &\leq (1-a_n)\|x_n-p\|+a_n\|Ty_n-p\| \leq (1-a_n)\|x_n-p\|+a_n\|y_n-p\|.\end{aligned}$$

$$\|x_{n+1}-p\|-\|x_n-p\| \leq \frac{\|x_{n+1}-p\|-\|x_n-p\|}{a_n} \leq \|y_n-p\|-\|x_n-p\|.$$

So we can get $\|x_{n+1}-p\| \leq \|y_n-p\|$. Therefore $\xi \leq \liminf_{n\to\infty}\|y_n-p\|$. Thus we have $\lim_{n\to\infty}\|y_n-p\|=\xi$. Also,

$$\begin{aligned}\xi=\lim_{n\to\infty}\|y_n-p\| &= \lim_{n\to\infty}\|Tz_n-p\| \\ &\leq \lim_{n\to\infty}\|T(T((1-b_n)x_n+b_nTx_n))-p\| \\ &\leq \lim_{n\to\infty}\|T((1-b_n)x_n-p+b_nTx_n)-p\| \\ &\leq \lim_{n\to\infty}\|(1-b_n)(x_n-p)+b_n(Tx_n-p)\| \\ &\leq \lim_{n\to\infty}(1-b_n)\|x_n-p\|+\lim_{n\to\infty}b_n\|Tx_n-p\| \leq \xi.\end{aligned}$$

Hence we have $\lim_{n\to\infty}\|(1-b_n)(x_n-p)+b_n(Tx_n-p)\|=\xi$. Thus by Lemma 1.2, we have $\lim_{n\to\infty}\|x_n-Tx_n\|=0$.
Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. Let $p\in A(K,\{x_n\})$. By Lemma 1.3 (4), we have

$$\begin{aligned}r(Tp,\{x_n\}) &= \limsup_{n\to\infty}\|x_n-Tp\| \\ &\leq \limsup_{n\to\infty}((\frac{3+\alpha}{1-\alpha})\|Tx_n-x_n\|+\|x_n-p\|+\|p-Tp\| \\ &= \limsup_{n\to\infty}\|x_n-p\|=r(p,\{x_n\})\end{aligned}$$

This implies that $Tp=p\in A(K,\{x_n\})$. Since X is a uniformly Banach space, $A(K,\{x_n\})$ is consists of a unique element.Thus, we have $Tp=p$. This completes the proof. □

In the next result, we prove strong convergence theorems as follows.

Theorem 2.3. *Let X be a real uniformly convex Banach space and K be a nonempty compact convex subset of X and T be a generalized $\alpha-$nonexpansive mapping on K and $F(T)\neq\emptyset$. Assume that $p\in F(T)$ is a fixed point of T and let $\{x_n\}$ be as in Theorem 2.2. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. $F(T)\neq\emptyset$, so by Theorem 2.2, we have $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\longrightarrow p$ as $k\to\infty$ for $p\in K$. Then for $(\frac{3+\alpha}{1-\alpha})\geq 1$ we have

$$\|x_{n_k}-Tp\| \leq (\frac{3+\alpha}{1-\alpha})\|Tx_{n_k}-x_{n_k}\|+\|x_{n_k}-p\| \text{ for all } k\geq 0.$$

Letting $k\to\infty$, we get $Tp=p$, $p\in F(T)$. $\lim_{n\to\infty}\|x_n-p\|$ exists for every $p\in F(T)$, so $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 2.4. *Let the conditions of Theorem 2.2 be satisfied. Also if T satisfies condition (I), then $\{x_n\}$ defined by (1.2) converges strongly to a fixed point of T.*

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n-p\|$ exists and so $\lim_{n\to\infty}d(x_n,p)$ exists for all $p\in F(T)$. Also by Theorem 2.2, $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. It follows from condition (I) that $\lim_{n\to\infty}f(d(x_n,F(T))\leq\lim_{n\to\infty}\|x_n-Tx_n\|$. That is, $\lim_{n\to\infty}f(d(x_n,F(T))=0$. Since $f:[0,\infty)\to[0,\infty)$ is a nondecreasing function satisfying $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$, we have $\lim_{n\to\infty}d(x_n,F(T))=0$. So, all the assumptions of Theorem 2.5 in [20] are satisfied. The rest of the proof is similar to the proof of Theorem 2.5 in [20] and therefore it is omitted. Thus, we can easily see that $\{x_n\}$ strongly converges to an element of $F(T)$. □

Finally, we prove the weak convergence of the iterative scheme (1.2) for generalized $\alpha-$nonexpansive mapping in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.5. *Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. Let T be a generalized $\alpha-$nonexpansive mapping on K with $F(T)\neq\emptyset$. Assume that $p\in F(T)$ is a fixed point of T and let $\{x_n\}$ be as in Theorem 2.2. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since $F(T)\neq\emptyset$, it follows from Theorem 2.2 that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|Tx_n-x_n\|=0$. Let v_1,v_2 be weak limits of subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By $\lim_{n\to\infty}\|x_n-Tx_n\|$ and $I-T$ is demiclosed with respect to zero , therefore we obtain $Tv_1=v_1$. Again in the same manner, we can $Tv_2=v_2$. Next we prove the uniqueness. By Lemma 2.1, $\lim_{n\to\infty}\|x_n-v_1\|$ and $\lim_{n\to\infty}\|x_n-v_2\|$ exist. For suppose that $v_1\neq v_2$, then by the Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-v_1\| &= \lim_{j\to\infty}\|x_{n_j}-v_1\|<\lim_{j\to\infty}\|x_{n_j}-v_2\|=\lim_{n\to\infty}\|x_n-v_2\| \\ &= \lim_{k\to\infty}\|x_{n_k}-v_2\|<\lim_{k\to\infty}\|x_{n_k}-v_1\|=\lim_{n\to\infty}\|x_n-v_1\|\end{aligned}$$

which is a contradiction. So, $v_1=v_2$. Therefore $\{x_n\}$ converges weakly to a fixed point of T. This completes the proof. □

3. Conclusions

We introduce a new iteration process to approximate fixed points of a new type of nonexpansive mappings. We noticed from Table 1 that our new iteration process is faster than M-iteration process for contraction mapping. We also illustrated an example of a mapping that is generalized $\alpha-$nonexpansive mapping but not Suzuki's generalized nonexpansive mapping.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] W. R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc. **4** (1953), 506-510.
[2] I. Ishikawa, *Fixed point by a new iteration method*, Proc. Am. Math. Soc. **44** (1974), 147-150.
[3] M. A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl., **251** (2000), 217-229.
[4] R. P. Agarwal, D. O'Regan, D. R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal., **8**(1) (2007), 61-79.
[5] M. Abbas, T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Mat. Vesnik, **66**(2) (2014), 223-234.
[6] F. Gürsoy, V. Karakaya, *A Picard-S hybrid type iteration method for solving a differential equation with retarded argument*, (2014), Preprint arXiv 1403.2546, 1-16.
[7] B. S. Thakur, D. Thakur, M. Postolache, *A new iteration scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput., **275** (2016), 147-155.
[8] K. Ullah, M. Arshad, *Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process*, Filomat, **32**(1) (2018), 187-196.
[9] K. Aoyama, F. Kohsaka, *Fixed point theorem for α-nonexpansive mappings in Banach spaces*, Nonlinear Analy., **74**(13) (2011), 4378-4391.
[10] M. Başarır, A. Şahin, *On the strong and Δ-convergence of S-iteration process for generalized nonexpansive mappings on $CAT(0)$ space*, Thai J. Math., **12**(3) (2014), 549-559.
[11] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc., **40** (1936), 396-414.
[12] N. Hussain, K. Ullah, M. Arshad, *Fixed point approximation of Suzuki generalized non-expansive mappings via new faster iteration process*, J. Nonlinear Convex Anal., **19** (2018), 1383-1393.
[13] Z. Opial, *Weak convergence of successive approximations for nonexpansive mappings*, Bull. Ame. Math.Soc., **73** (1967), 591-597.
[14] R. Pant, R. Shukla, *Approximating fixed points of generalized α-nonexpansive mappings in Banach spaces*, Numer. Funct. Anal. Optim., **38**(2) (2017), 248-266.
[15] A. Şahin, *Some new results of M-iteration process in hyperbolic spaces*, Carpathian J. Math., **35**(2) (2019), 221-232.
[16] A. Şahin, M. Başarır, *Some convergence results of the K*-iteration process in $CAT(0)$ spaces* , In: J. L. Cho, Y.L. Jleli, M. Mursaleen, B.Samet, C.Vetro, (Eds.), Advances in Metric Fixed Point Theory and Applications, Springer, Singapore, (2021).
[17] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **340**(2) (2008), 1088-1095.
[18] J. Schu, *Weak and strong convergence of fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc., **43** (1991), 153-159.
[19] H. F. Senter, W. G. Dotson Jr., *Approximating fixed points of nonexpansive mappings*, Proc. Am. Math. Soc. **44** (1974), 375-380.
[20] K. Ullah, F. Ayaz, J. Ahmad, *Some convergence results of M iterative process in Banach spaces*, Asian-European Journal of Mathematics (2021), 2150017, 12 pages, doi:10.1142/S1793557121500170.