Stochastic Extended Korteweg-De Vries Equation

Anna Karczewska[1*] **and Maciej Szczeciński**[1]

[1]*Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Poland*
Corresponding author

Abstract

In the paper, we consider stochastic Korteweg-de Vries - type equation. We give sufficient conditions for the existence and uniqueness of the local mild solution to the equation with additive noise. We discuss the possibility of the globalization of mild solution, as well.

1. Introduction

Nonlinear wave equations attracted enormous attention in many fields, e.g. physics (hydrodynamics, plasma physics, optics), technology (electric circuits, light impulses propagation) and biology (neuroscience models, protein, and DNA motion). Usually, such equations are obtained as a kind of approximation and/or simplification of the set of several more fundamental equations governing the system with their boundary and initial conditions. Approximations are usually based on the perturbative approach in which some small parameters, related to particular properties of the considered system, appear. Then the relevant quantities are expanded in power series of these small parameters. The limitation to terms of the first or second order allows deriving approximate nonlinear wave equations describing the evolution of a given system.

In several fields the lowest (first) order equation takes form of the Korteveg-de Vries equation (commonly denoted as KdV) [1]

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1.1}$$

It was derived firstly for surface gravity waves on shallow water but later found in many other systems, see, e.g. [2, 3, 4, 5].

Although the KdV equation displays dominant features of weekly dispersive nonlinear waves, it is a valid approximation only for constant water depth and waves with small amplitudes. For waves with a larger amplitudes perturbative approach to Euler equations should be applied up to second order in small parameters. Then linear terms with fifth order derivatives and new nonlinear terms appear in final nonlinear wave equation. This equation was derived by Marchant and Smyth and called the **extended KdV** in [6]. For short we call this equation **KdV2** stressing second order perturbation expansion. Contrary to KdV this equation is non-integrable. Despite this fact, we found three kinds of analytic solutions to KdV2, namely single soliton solutions, periodic cnoidal solutions, and periodic superposition solutions, see. e.g. [7, 8, 9].

Nonlinear dispersive waves attracted the considerable attention of mathematicians. Among many examples of mathematical description of those problems, we point out books of Linares and Ponce [10] and Tao [11].

Surface water waves are subjected to some unpredictable influences of the environment, like winds, bottom fluctuations, etc. These unknown factors can be accounted for by introducing a forcing term of stochastic nature into wave equation.

In the current paper, we study stochastic version of KdV2. We supply sufficient conditions for the existence and uniqueness of a local mild solution to the Korteweg-de Vries type equation of the form (2.1) below. We follow and generalize the approach of de Bouard and Debussche [12] and Kenig, Ponce and Vega [13, 14] to such equation.

Email addresses and ORCID numbers: A.Karczewska@wmie.uz.zgora.pl, 0000-0001-7160-7908 (A. Karczewska), M.Szczecinski@wmie.uz.zgora.pl (M. Szczeciński)

We obtained the existence and uniqueness results on a random interval. The generalization of these results to any time interval with the approach due to de Bouard and Debussche [12] is not possible since they use some properties of classical KdV equation and its invariants. In our case, for extended KdV equation, there exists only one (the lowest) exact invariant, the other ones are only adiabatic (approximate) [15]. In Section 3 we discuss the possibility for some globalization of obtained mild solution to stochastic extended KdV equation studied. We use the near-identity transformation (NIT for short) Kodama [16], Dullin et al. [17] to transform original non-integrable extended KdV equation into the asymptotically equivalent equation which has Hamiltonian form and therefore is integrable. The term *asymptotic equivalence* means that solutions of both equations coincide when physically relevant coefficients of the equations tend to zero (for details, see Section 3).

2. Existence and uniqueness

In this section, we prove the existence and uniqueness of mild solution on a random interval to the stochastic extended KdV-type equation of the form

$$\mathrm{d}u + \left(\frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x} + u\frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} \right) \mathrm{d}t = \Phi \, \mathrm{d}W, \quad x \in \mathbb{R}, \quad t \geq 0. \tag{2.1}$$

Motivation for studying the equation (2.1) is given in Section 3. In (2.1), W is a cylindrical Wiener process defined on the stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with values on $L^2(\mathbb{R})$ adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$. The operator Φ belongs to L_2^0, where $L_2^0 := L_2^0(L^2(\mathbb{R}); H^\sigma(\mathbb{R}))$ is the space of Hilbert-Schmidt operators acting from $L^2(\mathbb{R})$ into $H^\sigma(\mathbb{R})$ and $H^\sigma(\mathbb{R})$ is the Sobolev space (see, e.g., Adams [18]), $\sigma > 0$. The equation (2.1) is supplemented with an initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \quad t \geq 0. \tag{2.2}$$

Definition 2.1. *A stochastic process* $u(t)$, $t \geq 0$, *defined on the basis* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ *is said to be a mild solution to (2.1)-(2.2), if*

$$u(t) = V(t)u_0 + \int_0^t V(t-s) \left(u\frac{\partial u}{\partial x} + u\frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} \right) \mathrm{d}s + \int_0^t V(t-s) \Phi \, \mathrm{d}W(s). \tag{2.3}$$

In (2.3), $V(t), t \geq 0$, is a unitary group generated by the linear part of the KdV equation (1.1).
To simplify notation we will use the following abbreviation for stochastic convolution

$$W_V(t) := \int_0^t V(t-s) \Phi \, dW(s), \quad t \geq 0. \tag{2.4}$$

Definition 2.2. *For a given set A by $_nA$ we shall denote the biggest subset of A defined as* $_nA := \left\{ u \in A : \frac{\partial^{kn} u}{\partial x^{kn}} \in A, k \in \mathbb{N} \right\}$.

In the paper we shall use the following notation

$$X_\sigma(T) := \left\{ u \in L^\infty(0,T; H^\sigma(\mathbb{R})) \cap L^2(\mathbb{R}; L^\infty([0,T])), D^\sigma \partial_x u \in L^\infty(\mathbb{R}, L^2([0,T])), \partial_x u \in L^4([0,T]; L^\infty(\mathbb{R})) \right\}.$$

Lemma 2.3. *If* $(u_0)_{2x} \in H^\sigma(\mathbb{R})$, *then* $V(t)(u_0)_{2x} \in X_\sigma(T)$, $\sigma > \frac{3}{4}$.

Proof. Proof comes from Proposition 3.5 in de Bouard and Debussche [12]. □

Now, we can formulate first result.

Theorem 2.4. *Assume that* $\Phi \in L_2^0(L^2(\mathbb{R}); H^\sigma(\mathbb{R}))$ *with* $\alpha > \frac{3}{4}$ *and*

$$\frac{\partial^2}{\partial x^2} \left[\int_0^t V(t-s) \Phi \, \mathrm{d}W(s) \right] \in L^2 \left(\Omega; L_x^2 (L_t^\infty) \right). \tag{2.5}$$

Then $\frac{\partial^2}{\partial x^2} W_V \in \widehat{X}_\sigma(T)$, $\mathbb{P}$-*almost surely, where* $\widehat{X}_\sigma(T) := \left\{ u : L^2(\mathbb{R}; L^\infty([0,T])), D^\sigma \partial_x u \in L^\infty(\mathbb{R}, L^2([0,T])), \partial_x u \in L^4([0,T]; L^\infty(\mathbb{R})) \right\}$, *for any* $T > 0$ *and all* σ, *such that* $\frac{3}{4} < \sigma < 1$.

Proof. For reader's convenience the proof of Theorem 2.4 is postponed to the section 4. □

Corollary 2.5. *Assume that* $\Phi \in L_2^0(L^2(\mathbb{R}); H^\sigma(\mathbb{R}))$ *and for* $\sigma > \frac{3}{4}$ *holds*

$$\frac{\partial^2}{\partial x^2} W_V \in L^2 \left(\Omega; L_t^\infty (H_x^\sigma) \right).$$

Then $\frac{\partial^2}{\partial x^2} W_V \in X_\sigma(T)$, $\mathbb{P}$-*almost surely for any* $T > 0$, *and* σ *such that* $\frac{3}{4} < \sigma < 1$.

Now, we are able to formulate the existence and uniqueness result.

Theorem 2.6. *Assume that* $u_0 \in {}_2L^2 \left(\Omega; H^1(\mathbb{R}) \right) \cap {}_2L^4 \left(\Omega; L^2(\mathbb{R}) \right)$ *and it is* $\mathscr{F}_0$-*measurable and* $\Phi \in L_2^0 \left(L^2(\mathbb{R}); H^1(\mathbb{R}) \right)$. *If (2.5) holds then there exists a unique mild solution to the equation (2.1) with initial condition (2.2), such that* $u \in {}_2X_\sigma(T)$ *almost surely for some* $T > 0$ *and for any* $\sigma \in \left(\frac{3}{4}, 1 \right)$.

Proof. As we have already written, in the proof we follow the method used in de Bouard and Debussche [12].
We introduce the mapping $\mathscr{T}$ defined as follows

$$\mathscr{T}(u) := V(t)u_0 + \int_0^t V(t-\tau)(u\frac{\partial u}{\partial x} + u\frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2})\,d\tau + W_V(t), \quad t \geq 0. \tag{2.6}$$

Then

$$\begin{aligned}
\mathscr{T}(u) &= V(t)u_0 + \int_0^t V(t-\tau)(u\partial_x u + u\partial_x u_{2x} + u_{2x}\partial_x u)\,d\tau + W_V(t)) = \\
&= V(t)u_0 + \int_0^t V(t-\tau)(u\partial_x u)\,d\tau + \int_0^t V(t-\tau)(u\partial_x u_{2x})\,d\tau + \int_0^t V(t-\tau)(u_{2x}\partial_x u)\,d\tau + W_V(t), \quad t \geq 0.
\end{aligned} \tag{2.7}$$

We want to obtain the following condition

$$[\,u \in \{u : u \in X_\sigma(T), u_{2x} \in X_\sigma(T)\}\,] \implies [\,\mathscr{T}(u) \in \{u : u \in X_\sigma(T), u_{2x} \in X_\sigma(T)\}\,]. \tag{2.8}$$

From Theorem 3.2 and Proposition 3.5 in de Bouard and Debussche [12] and because $u, u_{2x} \in X_\sigma(T)$, the mapping $\mathscr{T}$ maps $X_\sigma(T)$ into itself if $u_0 \in H^\sigma(\mathbb{R})$. We will check when $\frac{\partial^2}{\partial x^2}\mathscr{T}(u) \in X_\sigma(T)$. We have

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\mathscr{T}(u) &= \frac{\partial^2}{\partial x^2}V(t)u_0 + \frac{\partial^2}{\partial x^2}\int_0^t V(t-\tau)(uu_x)\,d\tau + \frac{\partial^2}{\partial x^2}\int_0^t V(t-\tau)(uu_{3x})\,d\tau + \frac{\partial^2}{\partial x^2}\int_0^t V(t-\tau)(u_x u_{2x})\,d\tau + \frac{\partial^2}{\partial x^2}\int_0^t V(t-\tau)\Phi dW(\tau) \\
&= V(t)(u_0)_{2x} + \int_0^t V(t-\tau)(uu_{3x} + 3u_x u_{2x})\,d\tau + \int_0^t V(t-\tau)(u_{2x}u_{3x} + 2u_x u_{4x} + uu_{5x})\,d\tau \\
&\quad + \int_0^t V(t-\tau)(3u_{2x}u_{3x} + u_x u_{4x})\,d\tau + \frac{\partial^2}{\partial x^2}\int_0^t V(t-\tau)\Phi dW(\tau).
\end{aligned}$$

Let $u \in {}_2X_\sigma(T)$. Then $v = u_{2x} \in {}_2X_\sigma(T)$ and $v_{2x} \in {}_2X_\sigma(T)$. We have

$$\int_0^t V(t-\tau)(uu_{3x} + 3u_x u_{2x})\,d\tau = \int_0^t V(t-\tau)u\partial_x u_{2x}\,d\tau + 3\int_0^t V(t-\tau)u_{2x}\partial_x u\,d\tau = \int_0^t V(t-\tau)u\partial_x v\,d\tau + 3\int_0^t V(t-\tau)v\partial_x u\,d\tau; \tag{2.9}$$

$$\begin{aligned}
\int_0^t V(t-\tau)(u_{2x}u_{3x} + 2u_x u_{4x} + uu_{5x})\,d\tau &= \int_0^t V(t-\tau)u_{2x}\partial_x u_{2x}\,d\tau + 2\int_0^t V(t-\tau)u_{4x}\partial_x u\,d\tau + \int_0^t V(t-\tau)u\partial_x u_{4x}\,d\tau \\
&= \int_0^t V(t-\tau)v\partial_x v\,d\tau + 2\int_0^t V(t-\tau)v_{2x}\partial_x u\,d\tau + \int_0^t V(t-\tau)u\partial_x v_{2x}\,d\tau;
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\int_0^t V(t-\tau)(3u_{2x}u_{3x} + u_x u_{4x})\,d\tau &= 3\int_0^t V(t-\tau)u_{2x}\partial_x u_{2x}\,d\tau + \int_0^t V(t-\tau)u_{4x}\partial_x u\,d\tau \tag{2.11} \\
&= 3\int_0^t V(t-\tau)v\partial_x v\,d\tau + \int_0^t V(t-\tau)v_{2x}\partial_x u\,d\tau. \tag{2.12}
\end{aligned}$$

From Theorem 3.2, Proposition 3.5 de Bouard and Debussche [12], Lemma 2.3 and Theorem 2.4 above, and equations (2.9)-(2.11) we obtain that the mapping $\mathscr{T}$ maps the set ${}_2X_\sigma(T)$ into itself if $u_0 \in {}_2H^\sigma(\mathbb{R})$ and $\Phi \in {}_2\mathscr{L}_2\left(L^2(\mathbb{R}, H^\sigma(\mathbb{R}))\right)$. We want to find a ball $\mathscr{B}$ in ${}_2X_\sigma(T)$ centered at point 0 and radius $2R$ such that the mapping $\mathscr{T}|_{\mathscr{B}}$ is contraction. More precisely, we want to have the following conditions

$$\text{(i)} \;\; |u|_{X_\sigma(T)} < 2R \implies |\mathscr{T}(u)|_{X_\sigma(T)} < 2R; \quad \text{(ii)} \;\; |\mathscr{T}(u) - \mathscr{T}(v)|_{X_\sigma(T)} < |u - v|_{X_\sigma(T)}, \quad |u|_{X_\sigma(T)}, |v|_{X_\sigma(T)} < 2R. \tag{2.13}$$

First, let us note that for any $u \in X_\sigma(T)$ there exists $M_u > 0$ such that $|u_{2x}|_{X_\sigma(T)} = M_u |u|_{X_\sigma(T)}$. Denote $M := \sup\{M_u : u \in X_\sigma(T)\}$. Then

$$|u_{2x}|_{X_\sigma(T)} \leq M|u|_{X_\sigma(T)}. \tag{2.14}$$

From (2.14) and Proposition 3.5. de Bouard and Debussche [12] we obtain the following estimate

$$\begin{aligned}
|\mathscr{T}(u)|_{X_\sigma(T)} &\leq C_1(\sigma,T)|u_0|_{H^\sigma(\mathbb{R})} + C_2(\sigma,T)T^{\frac{1}{2}}|u|^2_{X_\sigma(T)} + C_3(\sigma,T)T^{\frac{1}{2}}|u|_{X_\sigma(T)}|u_{2x}|_{X_\sigma(T)} + C_4(\sigma,T)T^{\frac{1}{2}}|u_{2x}|_{X_\sigma(T)}|u|_{X_\sigma(T)} + |W_V|_{X_\sigma(T)} \\
&\leq C_1(\sigma,T)|u_0|_{H^\sigma(\mathbb{R})} + C_2(\sigma,T)T^{\frac{1}{2}}|u|^2_{X_\sigma(T)} + C_3(\sigma,T)T^{\frac{1}{2}}M|u|^2_{X_\sigma(T)} + C_4(\sigma,T)T^{\frac{1}{2}}M|u|^2_{X_\sigma(T)} + |W_V|_{X_\sigma(T)}.
\end{aligned}$$

Here and below we write for shortening W_V instead of $W_V(t)$, $t \geq 0$.
Since $C_i(\sigma,T)$, $i = 1,2,3,4$, are nondecreasing with respect to T, we can use $C(\sigma,T) := \max_T\{C_1(\sigma,T), C_2(\sigma,T), C_3(\sigma,T), C_4(\sigma,T)\}$, which is nondecreasing with respect to T to our estimate. We obtain

$$\begin{aligned}
|\mathscr{T}(u)|_{X_\sigma(T)} &\leq C(\sigma,T)|u_0|_{H^\sigma(\mathbb{R})} + C(\sigma,T)T^{\frac{1}{2}}|u|^2_{X_\sigma(T)} + C(\sigma,T)T^{\frac{1}{2}}M|u|^2_{X_\sigma(T)} + C(\sigma,T)T^{\frac{1}{2}}M|u|^2_{X_\sigma(T)} + |W_V|_{X_\sigma(T)} \\
&= C(\sigma,T)|u_0|_{H^\sigma(\mathbb{R})} + C(\sigma,T)T^{\frac{1}{2}}|u|^2_{X_\sigma(T)}(1 + 2M) + |W_V|_{X_\sigma(T)}.
\end{aligned}$$

Now, we shall find R fulfilling condition (2.13)(i). Assume that $|u|_{X_\sigma(T)} < 2R$. Then we have

$$|\mathscr{T}(u)|_{X_\sigma(T)} \leq C(\sigma,T)|u_0|_{H^\sigma(\mathbb{R})} + C(\sigma,T)T^{\frac{1}{2}}4R^2(1 + 2M) + |W_V|_{X_\sigma(T)}.$$

We want to receive $C(\sigma,T)|u_0|_{H^\sigma(\mathbb{R})} + C(\sigma,T)T^{\frac{1}{2}}4R^2(1+2M) + |W_V|_{X_\sigma(T)} \leq 2R$. This is equivalent to $C(\sigma,T)|u_0|_{H^\sigma(\mathbb{R})} + |W_V|_{X_\sigma(T)} \leq 2R - C(\sigma,T)T^{\frac{1}{2}}4R^2(1+2M)$. Let us note that it is enough to have such R that

$$C(\sigma,T)|u_0|_{H^\sigma(\mathbb{R})} + |W_V|_{X_\sigma(T)} \leq R \leq 2R - C(\sigma,T)T^{\frac{1}{2}}4R^2(1+2M).$$

From the second inequality we obtain $R \leq 2R - C(\sigma,T)T^{\frac{1}{2}}4R^2(1+2M)$, then $0 \leq R - C(\sigma,T)T^{\frac{1}{2}}4R^2(1+2M)$ and $0 \leq R[1 - 4RC(\sigma,T)T^{\frac{1}{2}}(1+2M)]$, so $0 \leq 1 - 4RC(\sigma,T)T^{\frac{1}{2}}(1+2M)$, and finally $1 \geq 4RC(\sigma,T)T^{\frac{1}{2}}(1+2M)$. Hence, in order to obtain (2.13) (i), the following inequalities must hold

$$C(\sigma,T)|u_0|_{H^\sigma(\mathbb{R})} + |W_V|_{X_\sigma(T)} \leq R \qquad \text{and} \qquad 4RC(\sigma,T)T^{\frac{1}{2}}(1+2M) \leq 1. \tag{2.15}$$

Let us note that the second condition in (2.15) will hold too, if

$$\kappa 4RC(\sigma,T)T^{\frac{1}{2}}(1+2M) \leq 1 \quad \text{for any fixed constant } \kappa > 1. \tag{2.16}$$

Now, we will check when the condition (2.13)(ii) holds. First, we shall estimate the norm $|\mathscr{T}(u) - \mathscr{T}(v)|_{X_\sigma(T)}$. We can write

$$|\mathscr{T}(u) - \mathscr{T}(v)|_{X_\sigma(T)} = \left| \int_0^t V(t-\tau)(u\partial_x u - v\partial_x v)\,d\tau + \int_0^t V(t-\tau)(u\partial_x u_{2x} - v\partial_x v_{2x})\,d\tau + \int_0^t V(t-\tau)(u_{2x}\partial_x u - v_{2x}\partial_x v)\,d\tau \right|_{X_\sigma(T)}$$

$$= \frac{1}{2}\left| \int_0^t V(t-\tau)\left[u\partial_x(u-v) + (u-v)\partial_x u + v\partial_x(u-v) + (u-v)\partial_x v\right]d\tau \right.$$

$$+ \int_0^t V(t-\tau)\left[u\partial_x(u-v)_{2x} + (u-v)\partial_x u_{2x} + v\partial_x(u-v)_{2x} + (u-v)\partial_x v_{2x}\right]d\tau$$

$$+ \left. \int_0^t V(t-\tau)\left[u_{2x}\partial_x(u-v) + (u-v)_{2x}\partial_x u + v_{2x}\partial_x(u-v) + (u-v)_{2x}\partial_x v\right]d\tau \right|_{X_\sigma(T)}$$

$$\leq \frac{1}{2}\left| \int_0^t V(t-\tau)\left[u\partial_x(u-v) + (u-v)\partial_x u + v\partial_x(u-v) + (u-v)\partial_x v\right]d\tau \right|_{X_\sigma(T)}$$

$$+ \frac{1}{2}\left| \int_0^t V(t-\tau)\left[u\partial_x(u-v)_{2x} + (u-v)\partial_x u_{2x} + v\partial_x(u-v)_{2x} + (u-v)\partial_x v_{2x}\right]d\tau \right|_{X_\sigma(T)}$$

$$+ \frac{1}{2}\left| \int_0^t V(t-\tau)\left[u_{2x}\partial_x(u-v) + (u-v)_{2x}\partial_x u + v_{2x}\partial_x(u-v) + (u-v)_{2x}\partial_x v\right]d\tau \right|_{X_\sigma(T)}.$$

Then

$$|\mathscr{T}(u) - \mathscr{T}(v)|_{X_\sigma(T)} \leq \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|u|_{X_\sigma(T)}|u-v|_{X_\sigma(T)} + \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}|u|_{X_\sigma(T)}$$

$$+ \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|v|_{X_\sigma(T)}|u-v|_{X_\sigma(T)} + \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}|v|_{X_\sigma(T)}$$

$$+ \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|u|_{X_\sigma(T)}|(u-v)_{2x}|_{X_\sigma(T)} + \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}|u_{2x}|_{X_\sigma(T)}$$

$$+ \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|v|_{X_\sigma(T)}|(u-v)_{2x}|_{X_\sigma(T)} + \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}|v_{2x}|_{X_\sigma(T)}$$

$$+ \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|u_{2x}|_{X_\sigma(T)}|u-v|_{X_\sigma(T)} + \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|(u-v)_{2x}|_{X_\sigma(T)}|u|_{X_\sigma(T)}$$

$$+ \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|v_{2x}|_{X_\sigma(T)}|u-v|_{X_\sigma(T)} + \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|(u-v)_{2x}|_{X_\sigma(T)}|v|_{X_\sigma(T)}$$

$$= \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}\left[2|u|_{X_\sigma(T)} + 2|v|_{X_\sigma(T)} + 2|u_{2x}|_{X_\sigma(T)} + 2|v_{2x}|_{X_\sigma(T)}\right]$$

$$+ \frac{1}{2}C(\sigma,T)T^{\frac{1}{2}}|(u-v)_{2x}|_{X_\sigma(T)}\left[2|u|_{X_\sigma(T)} + 2|v|_{X_\sigma(T)}\right]$$

$$\leq C(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}\left[|u|_{X_\sigma(T)} + |v|_{X_\sigma(T)} + M|u|_{X_\sigma(T)} + M|v|_{X_\sigma(T)}\right] + C(\sigma,T)T^{\frac{1}{2}}M|u-v|_{X_\sigma(T)}\left[|u|_{X_\sigma(T)} + |v|_{X_\sigma(T)}\right].$$

Finally we have

$$|\mathscr{T}(u) - \mathscr{T}(v)|_{X_\sigma(T)} \leq C(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}\left[|u|_{X_\sigma(T)} + |v|_{X_\sigma(T)} + M|u|_{X_\sigma(T)} + M|v|_{X_\sigma(T)}\right] + C(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}\left[M|u|_{X_\sigma(T)} + M|v|_{X_\sigma(T)}\right]$$

$$= C(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}\left[|u|_{X_\sigma(T)}(2M+1) + |v|_{X_\sigma(T)}(2M+1)\right] = C(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}\left(|u|_{X_\sigma(T)} + |v|_{X_\sigma(T)}\right)(2M+1).$$

Since $|u|_{X_\sigma(T)} \leq 2R$ and $|v|_{X_\sigma(T)} \leq 2R$, we have

$$|\mathscr{T}(u) - \mathscr{T}(v)|_{X_\sigma(T)} \leq 4RC(\sigma,T)T^{\frac{1}{2}}|u-v|_{X_\sigma(T)}(2M+1). \tag{2.17}$$

From (2.16) we know that $4RC(\sigma,T)T^{\frac{1}{2}}(2M+1) \leq \frac{1}{\kappa}$, so, putting this into (2.17), we can write $|\mathscr{T}(u) - \mathscr{T}(v)|_{X_\sigma(T)} \leq \frac{1}{\kappa}|u-v|_{X_\sigma(T)}$. Hence, the mapping $\mathscr{T}|_{\mathscr{B}}$ is contraction if $\frac{1}{\kappa}|u-v|_{X_\sigma(T)} < |u-v|_{X_\sigma(T)}$, what is satisfied for any $\kappa > 1$. So, we have to choose R_0 and T such that

$$C(\sigma,T)|u_0|_{H^\sigma(\mathbb{R})} + |W_V|_{X_\sigma(T)} \leq R_0 \quad \text{and} \quad \kappa 4R_0 C(\sigma,T)T^{\frac{1}{2}}(1+2M) \leq 1 \quad \text{for some constant } \kappa > 1. \tag{2.18}$$

Remark 2.7. *In order to do this it is enough to take* $M := \sup\{M_u : u \in X_\sigma(T), |u|_{X_\sigma(T)} \le 4R\}$.

Proof. We estimated by M only terms $|u|_{X_\sigma(T)}$, $|v|_{X_\sigma(T)}$ and $|u - v|_{X_\sigma(T)}$. Since $|u|_{X_\sigma(T)} \le 2R$ and $|v|_{X_\sigma(T)} \le 2R$, so $|u - v|_{X_\sigma(T)} \le 4R$. □

Hence, the mapping $\mathscr{T}$ maps the ball $\mathscr{B}$ in $_2X_\sigma(T)$ centered at 0 with radius $2R$ into itself and, restricted to this ball, the mapping $\mathscr{T}$ is contraction. By Banach contraction theorem, the mapping $\mathscr{T}$ has fixed point in the set $_2X_\sigma(T)$, which is a unique solution to the equation (2.1) with initial condition (2.2). □

3. Near-identity transformation for KdV2

The famous Korteweg-de Vries equation [1] was first obtained in consideration of shallow water wave problem with the ideal fluid model. It is assumed that the fluid is inviscid and its motion is irrotational. Then the set of hydrodynamic (Euler's) equations with appropriate boundary conditions at the flat bottom and unknown surface is obtained. Scaling transformation to dimensionless variables introduces small parameters that allow us to apply perturbation approach. First order perturbation approach leads to KdV equation (below written in a fixed reference frame)

$$\eta_t + \eta_x + \frac{3}{2}\alpha\,\eta\,\eta_x + \frac{1}{6}\beta\,\eta_{3x} = 0. \tag{3.1}$$

More exact, second order perturbation approach gives the extended KdV equation [6] called by us KdV2 which has the following form

$$\eta_t + \eta_x + \frac{3}{2}\alpha\,\eta\,\eta_x + \frac{1}{6}\beta\,\eta_{3x} - \frac{3}{8}\alpha^2\eta^2\eta_x + \alpha\beta\left(\frac{23}{24}\eta_x\eta_{2x} + \frac{5}{12}\eta\,\eta_{3x}\right) + \frac{19}{360}\beta^2\eta_{5x} = 0. \tag{3.2}$$

In both equations (3.1) and (3.2) there appear parameters α, β, which should be small. Parameter $\alpha := \frac{A}{h}$ is the ratio of wave amplitude A to water depth h and determines nonlinear terms. Parameter $\beta := (\frac{h}{l})^2$, where l is an average wavelength describes the dispersion properties. When $\alpha \approx \beta \ll 1$ we have a classical shallow water problem. However, our recent paper [7] showed that exact solutions of KdV2 (3.2) occur when β is much less than α. Therefore for further considerations we can safely neglect in (3.2) the last term with fifth derivative. Transformation to a moving reference frame $x' = x - t$ and $t' = t$ yields KdV2 equation in the form

$$\eta_{t'} + \frac{3}{2}\alpha\,\eta\,\eta_{x'} + \frac{1}{6}\beta\,\eta_{3x'} - \frac{3}{8}\alpha^2\eta^2\eta_{x'} + \alpha\beta\left(\frac{23}{24}\eta_{x'}\eta_{2x'} + \frac{5}{12}\eta\,\eta_{3x'}\right) = 0. \tag{3.3}$$

In next steps we drop signs $'$ at x' and t', having in mind that (3.3) represents the KdV2 in a moving frame.

Kodama [16] showed that several nonlinear partial differential equations are *asymptotically equivalent*. This term means that solutions to these equations converge to the same solution when parameters $\alpha, \beta \to 0$. Kodama and several other authors [17, 19, 20] have shown that asymptotically equivalent equations are related to each other by near-identity transformation (NIT).

Let us introduce Near Identity Transformation (NIT for short) in the form used in Dullin et al. [17]

$$\eta = \eta' \pm \alpha a\eta'^2 \pm \beta b\eta'_{xx} + \cdots \tag{3.4}$$

[In the sequel we set the sign $+$. Then the inverse transformation, up to $O(\alpha^2)$ is $\eta' = \eta - \alpha a\eta^2 - \beta b\eta_{xx} + \cdots$]

NIT preserves the structure of the equation (3.3), at most altering some coefficients. Insertion (3.4) into (3.3) gives (up to 2nd order in α, β)

$$\eta'_t + \eta'_x + \alpha\left[\left(\frac{3}{2} + 2a\right)\eta'\eta'_x + 2a\eta'\eta'_t\right] + \beta\left[\left(\frac{1}{6} + b\right)\eta'_{3x} + b\eta'_{xxt}\right] \tag{3.5}$$

$$+ \alpha^2\left(-\frac{3}{8} + \frac{9}{2}a\right)\eta'^2\eta'_x + \alpha\beta\left\{\left[\left(\frac{23}{24} + a + \frac{3}{2}b\right)\eta'_x\eta'_{2x}\right] + \left[\left(\frac{5}{12} + \frac{1}{3}a + \frac{3}{2}b\right)\eta'\eta'_{3x}\right]\right\} + \beta^2\frac{1}{6}b\eta'_{5x} = 0.$$

Since terms with derivatives with respect to t appear with coefficients α and β, we can replace them by appropriate expressions obtained from (3.2) limited to first order (that is from KdV)

$$\eta'_t = -\eta'_x - \frac{3}{2}\alpha\eta'\eta'_x - \frac{1}{6}\beta\eta'_{3x} \tag{3.6}$$

and

$$\eta'_{xxt} = \partial_{xx}\left(-\eta'_x - \frac{3}{2}\alpha\eta'\eta'_x - \frac{1}{6}\beta\eta'_{3x}\right) = -\eta'_{3x} - \frac{3}{2}\alpha(3\eta'_x\eta'_{2x} + \eta'\eta'_{3x}) - \frac{1}{6}\beta\eta'_{5x}. \tag{3.7}$$

Then terms (3.6) and (3.7) cause the following changes

$$\alpha\,2a\eta'\eta'_t + \beta\,b\eta'_{xxt} = -2\alpha a\eta'\left(\eta'_x + \frac{3}{2}\alpha\eta'\eta'_x + \frac{1}{6}\beta\eta'_{3x}\right) - \beta b\left[\eta'_{3x} + \frac{3}{2}\alpha(3\eta'_x\eta'_{2x} + \eta'\eta'_{3x}) + \frac{1}{6}\beta\eta'_{5x}\right]$$

$$= -2\alpha a\,\eta'\eta'_x - 3\alpha^2 a\eta'^2\eta'_x - \beta b\eta'_{3x} - \alpha\beta\left[\frac{1}{2}b\,\eta'_x\eta'_{2x} + \left(\frac{1}{3}a + \frac{3}{2}b\right)\eta'\eta'_{3x}\right] - \frac{1}{6}\beta^2 b\eta'_{5x}. \tag{3.8}$$

Insertion of (3.8) into (3.5) yields

$$\eta'_t + \eta'_x + \frac{3}{2}\alpha\eta'\eta'_x + \frac{1}{6}\beta\eta'_{3x} + \alpha^2\left(-\frac{3}{8} + \frac{3}{2}a\right)\eta'^2\eta'_x + \alpha\beta\left[\left(\frac{23}{24} + a - 3b\right)\eta'_x\eta'_{2x} + \frac{5}{12}\eta'\eta'_{3x}\right] = 0. \tag{3.9}$$

Comparison of (3.9) with (3.3) shows that only two coefficients are altered, that at the term containing α^2, where $-\frac{3}{8} \to -\frac{3}{8} + \frac{3}{2}a$ and that with $\alpha\beta\eta'_x\eta'_{2x}$, where $\frac{23}{24} \to \frac{23}{24} + a - 3b$.

Equation (3.9) is asymptotically equivalent to (3.3). NIT gives us some freadom in choosing coefficients a, b. They can be chosen such that the most nonlinear term (with 3-rd order nonlinearity) is canceled and the final equations is integrable. The first goal is obtained if

$$-\frac{3}{8} + \frac{3}{2}a = 0 \qquad \Longrightarrow \qquad a = \frac{1}{4}.$$

Integrability is achieved when coefficient in front of the term with $\eta_x\eta_{2x}$ is twice the coefficient in front of the term with $\eta\eta_{3x}$. So, we can choose b such that

$$\frac{23}{24} + a - 3b = 2\frac{5}{12} \qquad \Longrightarrow \qquad b = \frac{1}{8}.$$

Then, applying to (3.3) NIT (3.4) with parameters $a = \frac{1}{4}$ and $b = \frac{1}{8}$ we obtain asymptotically equivalent integrable equation in the form

$$\eta'_t + \frac{3}{2}\alpha\eta'\eta'_x + \frac{1}{6}\beta\eta'_{3x} + \frac{5}{12}\alpha\beta\left(2\eta'_x\eta'_{2x} + \eta'\eta'_{3x}\right) = 0. \tag{3.10}$$

We will show that for (3.10) there exists Hamiltonian form

$$\eta'_t = \frac{\partial}{\partial x}\left(\frac{\delta\mathscr{H}}{\delta\eta'}\right), \tag{3.11}$$

where Hamiltonian $H = \int_{-\infty}^{\infty}\mathscr{H}\,dx$ has the density

$$\mathscr{H} = -\frac{1}{4}\alpha\eta'^3 + \frac{1}{12}\beta\eta'^2_x + \frac{5}{24}\alpha\beta\eta'\eta'^2_x.$$

Since $\mathscr{H} = \mathscr{H}(\eta', \eta'_x)$, then functional derivative is given by

$$\frac{\delta\mathscr{H}}{\delta\eta'} = \frac{\partial\mathscr{H}}{\partial\eta'} - \frac{\partial}{\partial x}\frac{\partial\mathscr{H}}{\partial\eta'_x} = -\frac{3}{4}\alpha\eta'^2 - \frac{1}{6}\beta\eta'_{2x} - \frac{5}{24}\alpha\beta\eta'^2_x - \frac{5}{12}\alpha\beta\eta'\eta'_{2x}. \tag{3.12}$$

Insertion of (3.12) into (3.11) gives

$$\eta'_t = -\frac{3}{2}\alpha\eta'\eta'_x - \frac{1}{6}\beta\eta'_{3x} - \frac{5}{12}\alpha\beta\left(2\eta'_x\eta'_{xx} + \eta'\eta'_{xxx}\right). \tag{3.13}$$

what coincides with (3.10).

It is worth to notice, that application of inverse NIT to (3.10) brings back the equation (3.3) (up to second order in α, β).

The existence of the Hamiltonian implies that there exist invariants of the equation (3.10). This is the first step towards obtaining a global mild solution according to approach due to de Bouard and Debussche [12].

Remark 3.1. *Equations (3.10) or (3.13), up to numerical coefficients, are the same as left hand side of stochastic equation (2.1). Then study of stochastic equation (2.1) is justified.*

4. Proof of Theorem 2.4

To make the paper self-contained, we recall the following results.

Theorem 4.1. *([12], Proposition A.1) Let $A = L^q_\omega(L^2_t)$ or $A = L^q(\Omega)$, with $1 < q < \infty$, and let u be an A-valued function of $x \in \mathbb{R}$. Assume that for some p, with $1 < p < \infty$ and some $\sigma > 0$*

$$u \in L^p_x(A), \quad D^\sigma u \in L^\infty_x(A);$$

then for any $\alpha \in [0, \sigma]$ $D^\alpha u \in L^{p_\alpha}_x$, with p_α defined by $\frac{1}{p_\alpha} = \frac{1}{p}\left(1 - \frac{\alpha}{\sigma}\right)$. Furthermore, there is a constant C such that

$$|D^\alpha|_{L^{p_\alpha}_x(A)} \leq C\,|u|^{1-\frac{\alpha}{\sigma}}_{L^{p_\alpha}_x(A)}\,|D^\sigma u|^{\frac{\alpha}{\sigma}}_{L^\infty_x(A)}.$$

Theorem 4.2. *([13], Lemma 2.1) Let $v_0 \in L^2(\mathbb{R})$. Then*

$$\int_{-\infty}^{\infty}\left|D^{\frac{\alpha}{2}}V^\alpha(t)v_0(x)\right|^2\,dt = c_\alpha\|v_0\|^2_2 \quad \text{for any} \quad x \in \mathbb{R}.$$

Theorem 4.3. *([13], Theorem 2.4) For any $(\theta, \beta) \in [0, 1] \times [0, \frac{\alpha-1}{2}]$*

$$\left(\int_{-\infty}^{\infty}\left\|D^{\theta\frac{\beta}{2}}U^\alpha(t)v_0\right\|^q_p\,dt\right)^{\frac{1}{q}} \leq c\,\|v_0\|_2$$

and

$$\left(\int_{-\infty}^{\infty}\left\|\int D^{\theta\frac{\beta}{2}}U^\alpha(t-s)f(\cdot, s)\,ds\right\|^q_p\,dt\right)^{\frac{1}{q}} \leq c\left(\int_{-\infty}^{\infty}\|f(\cdot, s)\|^{q'}_{p'}\,dt\right)^{\frac{1}{q'}},$$

where $(q, p) = (2(\alpha+1)/(\theta(\beta+1)), 2/(1-\theta))$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

Lemma 4.4. *Asume that $\widetilde{\sigma} > \sigma > \frac{3}{4}$ and $0 < \varepsilon < \inf\{\widetilde{\sigma}, 2\}$. Then*

$$D^{\widetilde{\sigma}-\varepsilon}\partial_x\left(\frac{\partial^2}{\partial x^2}W_V\right) \in L^2\left(\Omega; L_x^\infty(L_t^2)\right).$$

Proof. Let, as usually, $W_V := \int_0^t V(t-s)\Phi\, dW(s)$ and let $q = \frac{6}{\varepsilon}$. Estimate the expression $\left|D^{3+\widetilde{\sigma}}W_V\right|_{L_x^\infty(L_\omega^q(L_t^2))}$. We have

$$
\begin{aligned}
\left|D^{3+\widetilde{\sigma}}W_V\right|_{L_x^\infty(L_\omega^q(L_t^2))} &= \sup_{x\in\mathbb{R}}\mathbb{E}\left(\left(\int_0^T\left|\int_0^t D^{3+\widetilde{\sigma}}V(t-s)\Phi\, dW(s)\right|^2 dt\right)^{\frac{q}{2}}\right) \leq C\sup_{x\in\mathbb{R}}\int_0^T\mathbb{E}\left(\left(\left|\int_0^t D^{3+\widetilde{\sigma}}V(t-s)\Phi\, dW(s)\right|^2\right)^{\frac{q}{2}}\right)dt \\
&\leq C\sup_{x\in\mathbb{R}}\int_0^T\left(\int_0^t\sum_{i\in\mathbb{N}}\left|D^{3+\widetilde{\sigma}}V(t-s)\Phi e_i(s)\right|^2 ds\right)^{\frac{q}{2}}dt \leq C\int_0^T\left(\sum_{i\in\mathbb{N}}\sup_{x\in\mathbb{R}}\int_0^t\left|D^{3+\widetilde{\sigma}}V(t-s)\Phi e_i(s)\right|^2 ds\right)^{\frac{q}{2}}dt.
\end{aligned}
$$

(4.1)

Let us substitute in Theorem 4.2 $v_0 = D^{\widehat{\sigma}+\frac{5}{2}}\Phi e_i$ and $\alpha = 1$. Then we obtain

$$C\left|D^{\widehat{\sigma}+\frac{5}{2}}\Phi e_i\right|^2_{L^2} = \int_{-\infty}^\infty |D^{\frac{1}{2}}V(s)D^{\widehat{\sigma}+\frac{5}{2}}\Phi e_i|^2\, ds = \int_{-\infty}^\infty |D^{3+\widehat{\sigma}}V(s)\Phi e_i|^2\, ds \geq \int_0^t |D^{3+\widehat{\sigma}}V(t-s)\Phi e_i|^2\, ds.$$

Since Theorem 4.2 holds for all $x\in\mathbb{R}$ and $\left|D^{\widehat{\sigma}+\frac{5}{2}}\Phi e_i\right|_{L^2} \leq |\Phi e_i|_{H^{\widehat{\sigma}+\frac{5}{2}}}$, then

$$\sup_{x\in\mathbb{R}}\int_0^t |D^{3+\widehat{\sigma}}V(t-s)\Phi e_i|^2\, ds \leq C|\Phi e_i|^2_{H^{\widehat{\sigma}+\frac{5}{2}}}.$$

(4.2)

Insertion of (4.2) into (4.1), gives

$$\left|D^{3+\widetilde{\sigma}}W_V\right|^q_{L_x^\infty(L_\omega^q(L_t^2))} \leq C\int_0^T\left(\sum_{i\in\mathbb{N}}|\Phi e_i|^2_{H^{\widetilde{\sigma}+\frac{5}{2}}}\right)^{\frac{q}{2}}dt \leq C(T)|\Phi|^{0,\widehat{\sigma}+\frac{5}{2}}_2 \leq C(T)|\Phi|^q_{L_2^{0,\widetilde{\sigma}+\frac{5}{2}}}.$$

Let us estimate $\left|D^{\widetilde{\sigma}}W_V\right|^2_{L_x^2(L_\omega^q(L_t^2))}$. Basing on proof of Proposition 3.3 in de Bouard and Debussche [12] we have that

$$\left|D^{\widetilde{\sigma}}W_V\right|^2_{L_x^2(L_\omega^q(L_t^2))} \leq C|\Phi|^2_{L_2^{0,\widetilde{\sigma}}} \leq C|\Phi|^q_{L_2^{0,\widetilde{\sigma}+\frac{5}{2}}}.$$

Now, set in Theorem 4.1 $A = L_\omega^q(L_t^2)$, $p = 2$, $u = D^{\widetilde{\sigma}}W_V$ $\sigma = 3$ and $\alpha = 3 - \varepsilon$ for some $3 > \varepsilon > 0$. Then $D^{3-\varepsilon}D^{\widetilde{\sigma}}W_V = D^{3+\widetilde{\sigma}-\varepsilon}W_V \in L_x^{p\alpha}(L_\omega^q(L_t^2))$ and there exists a constant C, such that

$$\left|D^{3+\widetilde{\sigma}-\varepsilon}W_V\right|_{L_x^{p\alpha}(L_\omega^q(L_t^2))} \leq C\left|D^{\widetilde{\sigma}}W_V\right|^{1-\frac{3-\varepsilon}{3}}_{L^2(L_\omega^q(L_t^2))}\left|D^{3+\widetilde{\sigma}}W_V\right|^{\frac{3-\varepsilon}{3}}_{L_x^\infty(L_\omega^q(L_t^2))} = C\left|D^{\widetilde{\sigma}}W_V\right|^{\frac{\varepsilon}{3}}_{L^2(L_\omega^q(L_t^2))}\left|D^{3+\widetilde{\sigma}}W_V\right|^{1-\frac{\varepsilon}{3}}_{L_x^\infty(L_\omega^q(L_t^2))},$$

where

$$p_\alpha = \left(\frac{1}{2}\left(1 - \frac{3-\varepsilon}{3}\right)\right)^{-1} = \left(\frac{-\frac{\varepsilon}{3}}{2}\right)^{-1} = \frac{6}{\varepsilon} = q.$$

Then we have

$$\left|D^{3+\widetilde{\sigma}-\varepsilon}W_V\right|_{L_x^q(L_\omega^q(L_t^2))} \leq C\left|D^{\widetilde{\sigma}}W_V\right|^{\frac{2}{q}}_{L^2(L_\omega^q(L_t^2))}\left|D^{3+\widetilde{\sigma}}W_V\right|^{1-\frac{2}{q}}_{L_x^\infty(L_\omega^q(L_t^2))} \leq C|\Phi|_{L_2^{0,\widetilde{\sigma}+\frac{5}{2}}}.$$

(4.3)

Moreover, basing on the proof of Proposition 3.3 in de Bouard and Debussche [12],

$$|W_V|_{L_\omega^q(L_x^q(L_t^2))} \leq C|\Phi|_{L_2^{0,\widetilde{\sigma}}} \leq C|\Phi|_{L_2^{0,\widetilde{\sigma}+\frac{5}{2}}}.$$

(4.4)

Since $\left|D^{3+\widetilde{\sigma}-\varepsilon}W_V\right|_{L_\omega^q(L_x^q(L_t^2))} = \left|D^{3+\widetilde{\sigma}-\varepsilon}W_V\right|_{L_x^q(L_\omega^q(L_t^2))}$, then from (4.3) oraz (4.4) we obtain

$$|W_V|_{L_\omega^q(W_x^{3+\widetilde{\sigma}-\varepsilon,q}(L_t^2))} \leq C|\Phi|_{L_2^{0,\widetilde{\sigma}+\frac{5}{2}}}.$$

Because $q\varepsilon > 1$, then $W_x^{\varepsilon,q}(L_t^2) \subset L_x^\infty(L_t^2)$, therefore $D^{3+\widetilde{\sigma}-\varepsilon}W_V \in L_\omega^q(L_x^\infty(T_t^2))$. Moreover

$$\left|D^{3+\widetilde{\sigma}-\varepsilon}W_V\right|_{L_\omega^q(L_x^\infty(L_t^2))} \leq C|\Phi|_{L_2^{0,\widetilde{\sigma}+\frac{5}{2}}}.$$

Finally

$$D^{\widetilde{\sigma}-\varepsilon}\partial_x\left(\frac{\partial^2}{\partial x^2}W_V\right) = D^{\widetilde{\sigma}-\varepsilon}\partial_{3x}W_V = \int_0^t D^{\widetilde{\sigma}-\varepsilon}\partial_{3x}V(t-s)\Phi\, dW(s) = \int_0^t D^{3+\widetilde{\sigma}-\varepsilon}V(t-s)\mathscr{H}\Phi\, dW(s),$$

where $\mathscr{H}$ is the Hilbert transform, what finishes the proof. $\qquad\square$

Lemma 4.5.

$$\partial_x \left(\frac{\partial^2}{\partial x^2} W_V \right) \in L^2 \left(\Omega; L_t^4 \left(L_x^\infty \right) \right).$$

Proof. Let $\varepsilon = \widetilde{\sigma} - \frac{3}{4}$ and $q = 4 + \frac{12}{\varepsilon}$. Estimate $\left| D^{3+\varepsilon} W_V \right|_{L_t^4(L_x^\infty(L_\omega^q))}$.

We have

$$\left| D^{3+\varepsilon} W_V \right|_{L_t^4(L_x^\infty(L_\omega^q))}^4 = \int_0^T \sup_{x \in \mathbb{R}} \mathbb{E} \left(\left| \int_0^t D^{\widetilde{\sigma} + \frac{9}{4}} V(t-s) \Phi \, dW(s) \right|^q \right)^{\frac{4}{q}} dt \leq C \int_0^T \sup_{x \in \mathbb{R}} \left(\sum_{i \in \mathbb{N}} \int_0^t \left| D^{\widetilde{\sigma} + \frac{9}{4}} V(t-s) \Phi e_i \right|^2 ds \right)^2 dt$$

$$\leq C(T) \left(\sum_{i \in \mathbb{N}} \left(\int_0^T \sup_{x \in \mathbb{R}} \left| D^{\widetilde{\sigma} + \frac{9}{4}} V(t-s) \Phi e_i \right|^4 ds \right)^{\frac{1}{2}} \right)^2.$$

Substitute in Theorem 4.3 $\alpha = 2$, $\theta = 1$, $\beta = \frac{1}{2}$ (like in de Bouard and Debussche [12]). The result is $\int_0^T \sup_{x \in \mathbb{R}} \left| D^{\widetilde{\sigma} + \frac{9}{4}} V(t) \Phi e_i \right|^4 dt \leq$

$C \left| D^{\widetilde{\sigma} + 2} \Phi e_i \right|_{L_x^2}^4 \leq C \left| \Phi \right|_{L_2^{0,\widetilde{\sigma}+2}}^4$, where $L_2^{0,\widetilde{\sigma}+2} = L_2^0(L^2(\mathbb{R}); H^{\widetilde{\sigma}+2}(\mathbb{R}))$. This implies $\left| D^{3+\varepsilon} W_V \right|_{L_t^4(L_x^\infty(L_\omega^q))}^4 \leq C \left| \Phi \right|_{L_2^{0,\widetilde{\sigma}+2}}^4$, Moreover from the

proof of Proposition 3.4 in de Bouard and Debussche [12] we know that $\left| W_V \right|_{L_t^4(L_x^2(L_\omega^q))} \leq C \left| \Phi \right|_{L_2^{0,0}} \leq C \left| \Phi \right|_{L_2^{0,\widetilde{\sigma}}} \leq C \left| \Phi \right|_{L_2^{0,\widetilde{\sigma}+2}}$.

Now substitute in Theorem 4.1 $\sigma = 3 + \varepsilon$, $A = L_\omega^q = L_q(\Omega)$, $p = 2$, $u = W_V$, $\alpha = 3 + \frac{\varepsilon}{2}$. Then $p_\alpha = \left(\frac{1}{2} \left(1 - \frac{3 + \frac{\varepsilon}{2}}{3 + \varepsilon} \right) \right)^{-1} = 4 + \frac{12}{\varepsilon} = q$ and

$$\left| D^{3+\frac{\varepsilon}{2}} W_V \right|_{L_x^{p_\alpha}(L_\omega^q)} \leq C \left| W_V \right|_{L_x^{p_\alpha}(L_\omega^q)}^{1 - \frac{3+\frac{\varepsilon}{2}}{3+\varepsilon}} \left| D^{3+\varepsilon} W_V \right|_{L_x^\infty(L_\omega^q)}^{\frac{3+\frac{\varepsilon}{2}}{3+\varepsilon}} = C \left| W_V \right|_{L_x^{p_\alpha}(L_\omega^q)}^{\frac{\frac{\varepsilon}{2}}{3+\varepsilon}} \left| D^{3+\varepsilon} W_V \right|_{L_x^\infty(L_\omega^q)}^{\frac{3+\frac{\varepsilon}{2}}{3+\varepsilon}} = C \left| W_V \right|_{L_x^q(L_\omega^q)}^{\frac{2}{q}} \left| D^{3+\varepsilon} W_V \right|_{L_x^\infty(L_\omega^q)}^{1 - \frac{2}{q}}.$$

Since $q = 4 + \frac{12}{\varepsilon} \geq 4$, then

$$\left| D^{3+\frac{\varepsilon}{2}} W_V \right|_{L_\omega^4(L_t^4(L_x^q))} \leq \left| D^{3+\frac{\varepsilon}{2}} W_V \right|_{L_t^4(L_x^q(L_\omega^q))} \leq C \left| W_V \right|_{L_x^q(L_\omega^q)}^{\frac{2}{q}} \left| D^{3+\varepsilon} W_V \right|_{L_x^\infty(L_\omega^q)}^{1 - \frac{2}{q}} \leq C \left| \Phi \right|_{L_2^{0,\widetilde{\sigma}}} \leq C \left| \Phi \right|_{L_2^{0,\widetilde{\sigma}+2}}.$$

The proof of Proposition 3.4 in de Bouard and Debussche [12] implies that $\left| W_V \right| \leq C(T) \left| \Phi \right|_{L_2^{0,\widetilde{\sigma}}} \leq C(T) \left| \Phi \right|_{L_2^{0,\widetilde{\sigma}+2}}$, therefore

$\left| W_V \right|_{L_\omega^4 \left(L_t^4 \left(W_x^{3+\frac{\varepsilon}{2},q} \right) \right)} \leq C \left| \Phi \right|_{L_2^{0,\widetilde{\sigma}+2}}$ and, since $q \varepsilon / 2 > 1$, $\left| \partial_{3x} W_V \right|_{L_\omega^4(L_t^4(L_x^\infty))} \leq C \left| W_V \right|_{L_\omega^4 \left(L_t^4 \left(W_x^{3+\frac{\varepsilon}{2},q} \right) \right)} \leq C \left| \Phi \right|_{L_2^{0,\widetilde{\sigma}+2}}$. □

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