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A Relation Between Maclaurin Coefficients and Laplace Transform

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Abstract. In this paper, we study on a relation between Maclaurin coefficients and Laplace transform as follows:

$$f^{(n)}(0) = \frac{1}{(n+1)!} \lim_{r \to +0} \frac{d^{n+1}}{dr^{n+1}} L\{f\}\left(\frac{1}{r}\right),$$

where f is a complex valued function not necessarily analytic at point 0, L is the Laplace transform, $r = \frac{1}{s}$, s is the variable of the Laplace transform and $n \in \mathbb{N} \cup \{0\}$. Then, we give some examples of the formula.

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1. Introduction

Laplace transform is one of the most important operator in physics, mathematics etc (especially in differential equations theory) defined by the relation $L\{f\} = \int_0^\infty f(x) e^{-sx} dx$. It has some practical properties for complicated problems. These properties can be summarized as,

- It turns a differential equation into an algebraic equation,
- It turns a partial differential equation into an ordinary differential equation,
- It connects functions and generalized functions,

...

The most famous property of the Laplace transform is the derivative relation $L\left\{f^{(n)}\right\} = s^n L\{f\}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)$. In this paper, we use that relation and reveal the relation between Maclaurin coefficients and Laplace transform.

2. Preliminaries

Laplace operator is an invertible linear operator defined by

$$L\{f\}(s) = \int_0^\infty f(x)\, e^{-sx} dx, \tag{2.1}$$

where $f \in C[0,+\infty)$, $f = O(e^{ax})$, $a \in \mathbb{R}$ and $Re(s) > a$. For example, the Laplace transform of the function f defined by $f(x) = x^k e^{ax}$ is $\frac{k!}{(s-a)^{k+1}}$, i.e.;

$$L\{f\}(s) = \frac{k!}{(s-a)^{k+1}}, \tag{2.2}$$

where $k \in \mathbb{N} \cup \{0\}$, $a \in \mathbb{R}$.

The following relation is known as differentiation property of the Laplace transform

$$L\left\{f^{(n)}\right\}(s) = s^n L\{f\}(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0), \tag{2.3}$$

see [2], where $f \in C^{(n)}[0,+\infty)$.

Now, we remember Maclaurin coefficients and power series. If a complex valued function f is analytic at $x_0 = 0$, there exists a positive number R such that the power series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{2.4}$$

is valid in the disk $|x| < R$ on the plane $\mathbb{C}$. The series in (2.4) is uniformly convergent on the closed disk $|x| \leq r$ for each positive number $r < R$. In case of $R = \infty$, uniformly convergence is provided in all bounded subset of the plane $\mathbb{C}$. Note that f is said to be entire function when $R = \infty$, see [1].

3. Some Auxiliary Equalities

When f is an entire function, i.e., $f(x) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$, the relation

$$f^{(n)}(0) = \frac{1}{(n+1)!} \lim_{r \to +0} \frac{d^{n+1}}{dr^{n+1}} L\{f\}\left(\frac{1}{r}\right) \tag{3.1}$$

can be easily proved as follows:

$$L\{f\}(s) = \int_0^\infty \left(\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n e^{-sx} \right) dx = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} L\{x^n\}(s) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{s^{n+1}}.$$

By writing $s = \frac{1}{r}$, we have

$$L\{f\}\left(\frac{1}{r}\right) = \sum_{n=0}^{\infty} f^{(n)}(0)\, r^{n+1}.$$

Finally, we obtain (3.1) by differentiating both sides of the last equality $n+1$ times and taking $r \to +0$.

Under the condition that the function f is entire is so heavy. We now prove the validity of the relation (3.1) under light assumptions.

Lemma 3.1. *The following relation*

$$\frac{\partial^n}{\partial r^n}\left\{r^n e^{-\frac{x}{r}}\right\} = n! e^{-\frac{x}{r}} \sum_{k=0}^{n} \frac{x^k}{r^k k!} \tag{3.2}$$

is valid for $n \in \mathbb{N}$, $x \in [0, +\infty)$ and $r \in (0, +\infty)$.

Proof. We use the induction method. For $n = 1$, the relation is obvious. Assume that the relation (3.2) is provided for $n - 1$. We now prove the relation (3.2) for n:

$$\frac{\partial^n}{\partial r^n}\left\{r^n e^{-\frac{x}{r}}\right\} = \frac{\partial^n}{\partial r^n}\left\{r.r^{n-1} e^{-\frac{x}{r}}\right\} = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^k r}{\partial r^k} . \frac{\partial^{n-k}}{\partial r^{n-k}}\left\{r^{n-1} e^{-\frac{x}{r}}\right\}$$

$$= r\frac{\partial^n}{\partial r^n}\left\{r^{n-1} e^{-\frac{x}{r}}\right\} + n\frac{\partial^{n-1}}{\partial r^{n-1}}\left\{r^{n-1} e^{-\frac{x}{r}}\right\}.$$

Now, we use the induction hypothesis and the formula $\frac{\partial^n}{\partial r^n}\left\{r^{n-1} e^{-\frac{x}{r}}\right\} = \frac{x^n e^{-\frac{x}{r}}}{r^{n+1}}$ in Lemma 11 in [3]. Then, we obtain

$$\frac{\partial^n}{\partial r^n}\left\{r^n e^{-\frac{x}{r}}\right\} = r\frac{x^n e^{-\frac{x}{r}}}{r^{n+1}} + n\left(n-1!\right) e^{-\frac{x}{r}} \sum_{k=0}^{n-1} \frac{x^k}{r^k k!} = n! e^{-\frac{x}{r}} \sum_{k=0}^{n} \frac{x^k}{r^k k!}.$$

□

Lemma 3.2. *The relation*

$$\lim_{r\to+0} \frac{d^n}{dr^n}\left\{r^n L\left\{f\right\}\left(\frac{1}{r}\right)\right\} = 0 \tag{3.3}$$

is valid for each $n \in \mathbb{N}$, $a \in \mathbb{R}$, $f = O\left(e^{ax}\right)$.

Proof. There exists a positive number M such that the inequality $|f(x)| \leq Me^{ax}$ is provided for $x \in [0, +\infty)$. Also, the Laplace improper integral is uniformly convergent for $s \in [b, +\infty)$ with $a < b$, see [2]. Then, there exists a positive number c such that the improper integral

$$L\left\{f\right\}\left(\frac{1}{r}\right) = \int_0^\infty f(x)\, e^{-\frac{x}{r}} dx$$

is uniformly convergent for $r \in (0, c]$. By (2.2), we obtain

$$\begin{aligned}\left|\frac{d^n}{dr^n}\left\{r^n L\{f\}\left(\frac{1}{r}\right)\right\}\right| &= \left|\int_0^\infty f(x)\,\frac{\partial^n}{\partial r^n}\left\{r^n e^{-\frac{x}{r}}\right\}dx\right| \\ &\le \int_0^\infty |f(x)|\,\frac{\partial^n}{\partial r^n}\left\{r^n e^{-\frac{x}{r}}\right\}dx \le M\int_0^\infty e^{ax}\left(n!e^{-\frac{x}{r}}\sum_{k=0}^{n}\frac{x^k}{r^k k!}\right)dx \\ &= Mn!\sum_{k=0}^{n}\frac{1}{r^k k!}\int_0^\infty e^{\left(a-\frac{1}{r}\right)x}x^k dx = Mn!\sum_{k=0}^{n}\frac{s^k}{k!}\int_0^\infty e^{(a-s)x}x^k dx \\ &= Mn!\sum_{k=0}^{n}\frac{s^k}{k!}L\left\{e^{ax}x^k\right\}(s) = Mn!\sum_{k=0}^{n}\frac{s^k}{(s-a)^{k+1}}.\end{aligned}$$

Then

$$\lim_{r\to+0}\left|\frac{d^n}{dr^n}\left\{r^n L\{f\}\left(\frac{1}{r}\right)\right\}\right| \le Mn!\lim_{s\to+\infty}\sum_{k=0}^{n}\frac{s^k}{(s-a)^{k+1}} = 0,$$

where $s = \frac{1}{r}$. □

4. Main Results

Theorem 4.1. *Let* $m \in \mathbb{N}$, $f \in C^{(m)}[0,+\infty)$ *and* $f^{(m)} = O(e^{ax})$. *Then, the formula* (3.1) *is valid for* $0 \le n < m$, *i.e.,*

$$f^{(n)}(0) = \frac{1}{(n+1)!}\lim_{r\to+0}\frac{d^{n+1}}{dr^{n+1}}L\{f\}\left(\frac{1}{r}\right)$$

Proof. We use the formula (2.3) by writing $s = \frac{1}{r}$:

$$L\left\{f^{(n+1)}\right\}\left(\frac{1}{r}\right) = \frac{L\{f\}\left(\frac{1}{r}\right)}{r^{n+1}} - \sum_{k=0}^{n}\frac{f^{(k)}(0)}{r^{n-k}}.$$

Then,

$$r^{n+1}L\left\{f^{(n+1)}\right\}\left(\frac{1}{r}\right) = L\{f\}\left(\frac{1}{r}\right) - \sum_{k=0}^{n}f^{(k)}(0)\,r^{k+1}.$$

By applying the differential operator $\frac{d^{n+1}}{dr^{n+1}}$, we have

$$\frac{d^{n+1}}{dr^{n+1}}\left\{r^{n+1}L\left\{f^{(n+1)}\right\}\left(\frac{1}{r}\right)\right\} = \frac{d^{n+1}}{dr^{n+1}}\left\{L\{f\}\left(\frac{1}{r}\right)\right\} - (n+1)!f^{(n)}(0).$$

The formula (3.3) and the last relation complete the proof. □

We can use the formula (3.1) for the Maclaurin series expansion of an entire function. See following.

Example. Consider the function $f(x) = e^x$. Then, $L\{f\}(s) = \frac{1}{s-1}$ and so, $L\{f\}\left(\frac{1}{r}\right) = \frac{r}{1-r}$. The relation

$$\frac{d^{n+1}}{dr^{n+1}}\left\{\frac{r}{1-r}\right\} = \frac{(n+1)!}{(1-r)^{n+2}}$$

can be easily obtained by induction. Then, we have

$$f^{(n)}(0) = \frac{1}{(n+1)!}\lim_{r\to+0}\frac{d^{n+1}}{dr^{n+1}}L\{f\}\left(\frac{1}{r}\right) = \frac{1}{(n+1)!}\lim_{r\to+0}\frac{(n+1)!}{(1-r)^{n+2}} = 1.$$

by the formula (3.1). Consequently, the Maclaurin series expansion is

$$e^x = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

Example. We now show that there is no an entire function whose Laplace transform is $F(s) = e^{as}$. If a is not a negative real number, then $F(s)$ does not converge to zero, when $s \to +\infty$. Thus, $F(s)$ has not inverse Laplace transform when a is not a negative real number. Let a be a negative real number. Assume that an entire function f is the inverse Laplace transform of e^{as}. Then, by (3.1),

$$f^{(n)}(0) = \frac{1}{(n+1)!}\lim_{r\to+0}\frac{d^{n+1}}{dr^{n+1}}L\{f\}\left(\frac{1}{r}\right) = \frac{1}{(n+1)!}\lim_{r\to+0}\frac{d^{n+1}}{dr^{n+1}}e^{\frac{a}{r}} = 0$$

for each $n \in \mathbb{N} \cup \{0\}$. By the last relation, f is equal to the Maclaurin series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)\,x^n}{n!} = \sum_{n=0}^{\infty}\frac{0.x^n}{n!} = 0.$$

This is impossible because the Laplace transform of the zero function is zero. Indeed, the inverse Laplace transform of the function $F(s) = e^{as}$, $a < 0$, is the Dirac delta function $\delta(x+a)$ that is a generalized function.

5. Conclusions

In this paper, we investigate the connection between Laplace transform and Maclaurin coefficients. Inspired by the differentiation property of the Laplace transform, we obtain new formulas about Laplace transform and the derivatives of a function at the point 0. Even though a function is not analytic at zero, we obtain that the formulas we get is valid. Finally, we investigate whether the exponential function $F(s) = e^{as}$ can be image of a regular function or not.

References

[1] Ahlfors V.L.: Complex Analysis. McGraw-Hill Inc., New York (1979)

[2] Debnath L., Bhatta D.: Integral Transforms and Their Applications. CRC press, Boca Raton (2014)

[3] Widder V.D.: The Laplace Transform. Princeton University Press, Princeton (1941)

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