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On the quaternionic Bertrand curves of AW(k)-type in Euclidean space E^3

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Abstract

In this paper, We consider that the curvature conditions of AW(k)-type($1 \leq$k≤ 3) quaternionic curves in Euclidean space E^3 and investigates quaternionic Bertrand curves $\alpha : I \to Q$ with $k \neq 0$ and $r \neq 0$. Besides, we show that quaternionic Bertrand curves to be AW(2)-type and AW(3)-type quaternionic curves in E^3. But it is shown that there is no such a quaternionic Bertrand curve of AW(1)-type.

Keywords: AW(k)-type curve, General helix, Bertrand curves, Euclidean space, Quaternion algebra.

1 Introduction

The quaternion was introduced by Hamilton. His initial attempt to generalize the complex numbers by introducing a three-dimensional object failed in the sense that the algebra he constructed for these three-dimensional object did not have the desired properties. On the 16th October 1843 Hamilton discovered that the appropriate generalization is one in which the scalar(real) axis is left unchanged whereas the vector(imaginary) axis is supplemented by adding two further vector axis. Besides, there are three different types of quaternions, namely real, complex and dual quaternions. A real quaternion is defined as $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ is composed of four units $(1, e_1, e_2, e_3)$ where e_1, e_2, e_3 are orthogonal unit spatial vectors, $q_i\,(i = 0, 1, 2, 3)$ are real numbers and this quaternion may be written as a linear combination of a real part(scalar) and vectorial part(a spatial vector). Quaternions find uses in both theoretical and applied mathematics, in particular for calculations involving three-dimensional rotations such as in three-dimensional computer graphics and computer vision. They can be used alongside other methods, such as Euler angles and matrices, or as an alternative to them depending on the application. Furthermore, Baharathi and Nagaraj represented the curves by unit quaternions in E^3 and E^4 and called these curves as quaternionic curves[9]. They studied the differential geometry of space curves and introduced Frenet frames and formulae by using quaternions. After them, many mathematicians have studied on quaternionic curves. Karadag and Sivridag have defined and studied quaternionic inclined curves[10]. Çöken and Tuna have studied the same curves in semi-Euclidean space E^4_2[1]. Ata and Yaylı studied split quaternions[5]. Many interesting results on curves of AW(k)-type have been obtained by many mathematicians (see[2,8,11,15]). For example, in[2], Özgür and Gezgin studied a Bertrand curve of AW(k)-type and furthermore, they showed that there was no such Bertrand curve of AW(1)-type and was of AW(3)-type if and only if it was a right circular helix. In addition they studied weak AW(2)-type and AW(3)-type conical geodesic curves in E^3. Kızıltuğ and Yaylı[15,16] studies curves of AW(k)-type in three Galilean 3-space and given some interesting results. Besides, In 3-dimensional null cone and Lorentz space, the curves of AW(k)- type was investigated by Külahçi, Bektaş, Ergüt and Külahçi, Ergüt, respectively[11,12]. The purpose of the present paper is to provide AW(k)-type quaternionic curves in Euclidean 3-space and provides the properties of quaternionic Bertrand curves in Euclidean space.

Our paper is organized as follows. In Section 2, the basic notions and properties of a quaternionic curve are reviewed. In Section 3, we study quaternionic curves of AW(k)-type. We also study quaternionic Bertrand curves of AW(k)-type in Section 4.

2 Preliminaries

In this section, we give the basic elements of the theory of quaternions and quaternionic curves. A more complete elementary treatment of quaternions and quaternionic curves can be found in [9] and [10], respectively.

A real quaternion q is an expression of the form

$$q = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 \tag{1}$$

where a_i, $(1 \leq i \leq 4)$ are real numbers, and e_i, $(1 \leq i \leq 4)$ are quaternionic units which satisfy the non-commutative multiplication rules

$$\begin{aligned} e_i \times e_i &= -e_4, (1 \leq i \leq 3) \\ e_i \times e_j &= e_k = -e_j \times e_i, (1 \leq i, j \leq 3) \end{aligned} \tag{2}$$

where (i, j, k) is an even permutation of (123) in the Euclidean space. The algebra of the quaternions is denoted by Q and its natural basis is given by (e_1, e_2, e_3, e_4). A real quaternion can be given by the form

$$q = s_q + v_q \tag{3}$$

where $s_q = a_4$ is scalar part and $v_q = a_1e_1 + a_2e_2 + a_3e_3$ is vector part of q.

The conjugate of $q = s_q + v_q$ is defined by

$$\bar{q} = s_q - v_q. \tag{4}$$

This defines the symmetric real-valued, non-degenerate, bilinear form as follows:

$$h : Q \times Q \to \mathbb{R}, (p, q) \to h(p, q) = \frac{1}{2}(q \times \bar{p} + p \times \bar{q}) \tag{5}$$

which is called the quaternion inner product [9]. Then the norm of q is given by

$$\|q\|^2 = h(q, q) = q \times \bar{q} = \bar{q} \times q = a_1^2 + a_2^2 + a_3^2 + a_4^2. \tag{6}$$

If $\|q\| = 1$, then q is called unit quaternion. Then, inverse of the quaternion q is given by

$$q^{-1} \frac{\bar{q}}{\|q\|}. \tag{7}$$

Let $q = s_q + v_q = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$ and $p = s_p + v_p = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4$ two quaternions in Q. Then the quaternion product of q and p is given by

$$q \times p = s_q s_p - \langle v_q, v_p \rangle + s_q v_p + s_p v_q + v_q \wedge v_p \tag{8}$$

where $\langle , \rangle$ and $\wedge$ denote the inner product and vector product in Euclidean 3-space E^3.

q is called a spatial quaternion whenever $q + \bar{q} = 0$ and called a temporal quaternion whenever $q - \bar{q} = 0$.Then a general quaternion q can be given as

$$q=\frac{1}{2}\left(q+\bar{q}\right)+\frac{1}{2}\left(q-\bar{q}\right).$$

The quaternion $\frac{1}{2}\left(q-\bar{q}\right)$ is a spatial quaternion and called spatial part of q and the quaternion $\frac{1}{2}\left(q+\bar{q}\right)$ is a temporal quaternion and called temporal part of q[9].

The three-dimensional Euclidean space E^3 is identified with the space of spatial quaternions $\{q\in Q : q+\bar{q}=0\}$ in an obvious manner. Let $I=[0,1]$ be an interval in real line $\mathbb{R}$ and $s\in I$ be parameter along the regular curve

$$\alpha : I\subset\mathbb{R}\to Q, s\to\alpha\left(s\right)=\sum_{i=1}^{3}\alpha_i\left(s\right)e_i \tag{9}$$

chosen such that the tangent $\alpha^{'}(s)=t$ is unit, i.e., $\|t\|=1$ for all s. Then $\alpha\left(s\right)$ is called spatial quaternionic curve[9]. Since $\|t\|=1$, $\acute{t}\times\bar{t}+t\times\bar{\acute{t}}=0$ holds and it means that $t^{'}$ is orthogonal to t, and moreover $t^{'}\times\bar{t}$ is a spatial quaternion. Since $t^{'}$ is itself a spatial quaternion we define a spatial quaternion n_1 and non-negative scalar function $k=k(s)$ is called principal curvature. n_1 is orthogonal to t. Then by considering that $t^{'}\times\bar{t}$ is a spatial quaternion, there exists a unit spatial quaternion $n_2(s)=t(s)\times n_1(s)=-n_1(s)\times t(s)$. Then the set $\{t(s),n_1(s),n_2(s)\}$ is called Frenet frame along the quaternionic curve $\alpha(s)$, where $t(s)$ is unit tangent, $n_1(s)$ is unit principal normal and $n_2(s)$ is unit binormal of the curve $\alpha(s)$. The Frenet formulae of the quaternionic curve $\alpha(s)$ are

$$\begin{bmatrix} t^{'}(s)\\ n_1^{'}(s)\\ n_2^{'}(s)\end{bmatrix}=\begin{bmatrix}0 & k(s) & 0\\ -k(s) & 0 & r(s)\\ 0 & -r(s) & 0\end{bmatrix}\begin{bmatrix}t(s)\\ n_1(s)\\ n_2(s)\end{bmatrix} \tag{10}$$

where $k=k\left(s\right)$ is principal curvature and $r=r\left(s\right)$ is torsion of $\alpha\left(s\right)$. (For Details[9]).

Theorem 1 *[9]Let $\alpha : I\to Q$ be a real spatial quaternionic curve with arc length parameter s and nonzero curvatures $\{k(s),r(s)\}$. If α is general helix if and only if*

$$\frac{r(s)}{k(s)}\text{is constant.} \tag{11}$$

3 Quaternionic curves of AW(k)-type

Let $\alpha : I\to Q$ be an arc-lenght parametrized unit speed real spatial quaternionic curve in Euclidean 3-space. The curve α is called a Frenet curve of osculating order 3 if its derivatives $\alpha^{'}(s),\alpha^{''}(s),\alpha^{'''}(s),\alpha^{''''}(s)$ are linearly dependent and $\alpha^{'}(s),\alpha^{''}(s),\alpha^{'''}(s),\alpha^{''''}(s)$ are no longer linearly independent for all $s\in I$. To each Frenet curve of order 3 one can associate an orthonormal 3−frame $t(s),n_1(s),n_2(s)$ along α such that $(\alpha^{'}\left(s\right))=t$ called the Frenet frame and functions $k,r : I\to\mathbb{R}$ called the Frenet curvatures.

Proposition 2 *$\alpha : I\to Q$ be an arc-lenght parametrized unit speed real spatial quaternionic curve in Euclidean 3-space, then we have*

$$\begin{aligned}
\alpha^{'}\left(s\right) &= t(s)\\
\alpha^{''}\left(s\right) &= k(s)n_1(s)\\
\alpha^{'''}\left(s\right) &= -k^2(s)t(s)+k^{'}(s)n_1(s)+k(s)r(s)n_2(s)\\
\alpha^{''''}\left(s\right) &= (-3k(s)k^{'}(s))t(s)+(k^{''}(s)-k^3(s)-k(s)r^2(s))n_1(s)+(2k^{'}(s)r(s)+k(s)r^{'}(s))n_2(s).
\end{aligned}$$

Notation 3 *Let us write*

$$\begin{aligned} N_1(s) &= k(s)n_1(s) && (12)\\ N_2(s) &= k^{'}(s)n_1(s)+k(s)r(s)n_2(s) && (13)\\ N_3(s) &= (k^{''}(s)-k^3(s)-k(s)r^2(s))n_1(s)+(2k^{'}(s)r(s)+k(s)r^{'}(s))n_2(s) && (14)\end{aligned}$$

Remark 4 $\alpha^{'}(s), \alpha^{''}(s), \alpha^{'''}(s), \alpha^{''''}(s)$ *are linearly dependent if and only if* $N_1(s), N_2(s), N_3(s)$ *are linearly dependent.*

As the definition of Aw(k) type curves in[8], we have

Definition 5 *Real spatial quaternionic curves* (*of osculating order*3) *in Euclidean space are* (*see[8]*)
(i) *of type weak* $Aw(2)$ *if they satisfy*

$$N_3(s) = h(N_3(s), N_2^*(s))N_2^*(s), \tag{15}$$

(ii) *of type weak* $Aw(3)$ *if they satisfy*

$$N_3(s) = h(N_3(s), N_1^*(s))N_1^*(s) \tag{16}$$

where

$$N_1^*(s) = \frac{N_1(s)}{\|N_1(s)\|}, N_2^*(s) = \frac{N_2(s) - h(N_2(s), N_1^*(s))N_1^*(s)}{\|N_2(s) - h(N_2(s), N_1^*(s))N_1^*(s)\|}.$$

Proposition 6 *Let* α *be a real spatial quaternionic curve* (*of osculating order*3) *in Euclidean space. If* α *is of type weak* $Aw(2)$ *then*

$$k^3(s) - k^{''}(s) + k(s)r^2(s) = 0 \tag{17}$$

Proposition 7 *Let* α *be a real spatial quaternionic curve* (*of osculating order*3) *in Euclidean space. If* α *is of type weak* $Aw(3)$ *then*

$$2k^{'}(s)r(s) + k(s)r(s) = 0 \tag{18}$$

Definition 8 *Real spatial quaternionic curves* (*of osculating order*3) *in Euclidean space are*
(i)*of type* $Aw(1)$ *if they satisfy* $N_3\,(s) = 0$,
(ii) *of type* $Aw(2)$ *if they satisfy*

$$\|N_2(s)\|^2\, N_3(s) = h(N_3(s), N_2(s))N_2(s), \tag{19}$$

(iii) *of type* $Aw(3)$ *if they satisfy*

$$\|N_1(s)\|^2\, N_3(s) = h(N_3(s), N_1(s))N_1(s). \tag{20}$$

Theorem 9 *Let* α *be a real spatial quaternionic curve* (*of osculating order*3) *in Euclidean space. Then* α *is of type* $Aw(1)$ *if and only if*

$$k^3(s) - k^{''}(s) + k(s)r^2(s) = 0 \tag{21}$$

and

$$2k^{'}(s)r(s) + k(s)r(s) = 0 \tag{22}$$

Proof. Since α is a curve of type Aw(1), we have $N_3(s) = 0$. Then from Eq.(14), we have

$$(k^3(s) - k^{''}(s) + k(s)r^2(s))n_1(s) + ((2k^{'}(s)r(s) + k(s)r(s))n_2(s) = 0.$$

Furthermore, since $n_1(s)$ and $n_2(s)$ are linearly independent, we get

$$k^3(s) - k^{''}(s) + k(s)r^2(s) = 0 \text{ and } k^{'}(s)r(s) + k(s)r(s) = 0.$$

The converse statement is trivial. Hence our theorem is proved. ■

Theorem 10 *Let α be a real spatial quaternionic curve (of osculating order 3) in Euclidean space. Then α is of type $Aw(2)$ if and only if*

$$2(k^{'}(s))^2r(s)+k^{'}(s)k(s)r^{'}(s)-k^{''}(s)k(s)r(s)+k^4(s)r(s)+k^2(s)r^3(s)=0. \tag{23}$$

Proof. Suppose that α is a Frenet curve of order 3, then from (13) and (14), we can write

$$\begin{aligned} N_2(s) &= \gamma(s)n_1(s)+\beta(s)n_2(s), &(24)\\ N_3(s) &= \eta(s)n_1(s)+\delta(s)n_2(s), &(25)\end{aligned}$$

where γ, β, η and δ are differentiable functions. Since $N_2(s)$ and $N_3(s)$ are linearly dependent, coefficients determinant is equal to zero and hence one can write

$$\begin{vmatrix} \gamma(s) & \beta(s) \\ \eta(s) & \delta(s) \end{vmatrix} = 0. \tag{26}$$

Here,

$$\gamma(s)=k^{'}(s), \beta(s)=k(s)r(s)$$

and

$$\begin{aligned} \eta(s) &= k^{''}(s)-k^3(s)-k(s)r^2(s),\\ \delta(s) &= 2k^{'}(s)r(s)+k(s)r^{'}(s).\end{aligned}$$

Substituting these into (26), we obtain (23).

Conversely if the equation (23) holds it is easy to show that α is of type Aw(2). This completes the proof. ■

Corollary 11 *Let a real spatial quaternionic curve (of osculating order 3). If it is quaternionic cylindrical helix and α is of type $Aw(2)$ then*

$$3(k^{'}(s))^2-k^{''}(s)k(s)+k^4(s)\left(1+c^2\right)=0 \tag{27}$$

where $c=\frac{r(s)}{k(s)}$ is constant.

Theorem 12 *Let α be a quaternionic general helix in Euclidean 3−space. If α is of type $Aw(2)$, then*

$$\kappa(s)=\frac{1}{\sqrt{-As^2+Bs+C}} \text{ and } r(s)=\sqrt{A-1}\kappa(s) \tag{28}$$

where $A=1+c^2$, B and C are real constants.

Proof. Suppose that α is a general helix of type Aw(2). Then Eq.(27) holds. If we substitute $\kappa(s)=x$ in (27), we get

$$x\frac{d^2x}{ds^2}-3\left(\frac{dx}{ds}\right)^2=Ax^4, \;\; A=1+c^2. \tag{29}$$

Let us take $x=y^p$ and differentiating it twice we obtain

$$\begin{aligned} \frac{dx}{ds} &= py^{p-1}\frac{dy}{ds}, &(30)\\ \frac{d^2x}{ds^2} &= p(p-1)y^{p-2}\left(\frac{dy}{ds}\right)^2+py^{p-1}\frac{d^2y}{ds^2}. &(31)\end{aligned}$$

Now, the substitution of (30) and (31) into (29), we get

$$\begin{aligned} y^p\left[py^{p-1}\frac{d^2y}{ds^2}+p(p-1)y^{p-2}\left(\frac{dy}{ds}\right)^2\right]-3p^2y^{2p-2}\left(\frac{dy}{ds}\right)^2 &= Ay^{4p}, \\ py^{2p-1}\frac{d^2y}{ds^2}+p(p-1)y^{2p-2}\left(\frac{dy}{ds}\right)^2-3p^2y^{2p-2}\left(\frac{dy}{ds}\right)^2 &= Ay^{4p}. \end{aligned}$$

Putting $p(p-1)=3p^2$ (i.e. $p=-\frac{1}{2}$) into the last equation we get

$$py^{2p-1}\frac{d^2y}{ds^2}=Ay^{4p}.$$

So,

$$\frac{d^2y}{ds^2}=-2A.$$

Now, we solve this last equation. Since $\frac{dy}{ds}=-2As+B$, we get

$$y=-As^2+Bs+C.$$

Furthermore, use of $x=y^{\frac{-1}{2}}$ we obtain

$$x=(-As^2+Bs+C)^{\frac{1}{2}}.$$

Since $c=\frac{r(s)}{k(s)}$, we have the result. ■

Corollary 13 *Let α be a real spatial quaternionic curve (of osculating order 3) in Euclidean space. If α is of type $Aw(2)$, then α can not be a circular helix.*

Theorem 14 *Let α be a real spatial quaternionic curve (of osculating order 3) in Euclidean space. Then α is of type $Aw(3)$ if and only if*

$$r(s)=\frac{c}{k^2(s)} \tag{32}$$

Proof. Since α is a curve of type Aw(3), then (20) holds on α. Substituting (12) and (14) into (20), we get

$$2k'(s)r(s)+k(s)r(s)=0.$$

If we solve above differential equation, we get

$$r(s)=\frac{c}{k^2(s)}.$$

The converse statement is trivial. Hence our theorem is proved. ■

Corollary 15 *Let α be a general helix of osculating order 3. Then α is of type $Aw(3)$ if and only if α is a circular helix.*

Proof. Suppose that α is a general helix of type Aw(3). Combining (11) and (32) we find $k(s)$ and $r(s)$ are nonzero constants. Thus, α is circular helix. The converse statements is trivial. ■

4 AW(k)-type quaternionic Bertrand curves in E^3

This section characteries the curvatures of AW(k)-type real spatial quaternionic Bertrand curves in E^3. We provided some theorems and conclusion to show that there are Aw(k)-type($k = 1, 2, 3$) real spatial quaternionic Bertrand Curves in E^3.

Definition 16 *A curve $\alpha : I \to Q$ with $k(s) \neq 0$ is called a Bertrand curve if there exist a curve $\tilde{\alpha} : I \to Q$ such that the principal normal lines of α and $\tilde{\alpha}$ at $s \in I$ are equal. In this case $\tilde{\alpha}$ is called a Bertrand mate of α.*

The curve $\tilde{\alpha}$ is called a Bertrand mate of α and vice versa. A Frenet framed curve is said to be a Bertrand curve if it admits a Bertrand mate.

By definition, for a Bertrand mate $(\tilde{\alpha}, \alpha)$, there exists a functional relation $\tilde{s} = \tilde{s}(s)$ such that

$$\tilde{\lambda}(\tilde{s}(s)) = \lambda(s).$$

Let $(\tilde{\alpha}, \alpha)$ be Bertrand mate. Then we can write

$$\tilde{\alpha}(s) = \alpha(s) + \lambda(s) n_1(s), \quad s \in I \tag{33}$$

where λ is a smooth function on I and n_1 is the principal normal vector field of α.

Theorem 17 *Let $\alpha : I \to Q$ and $\tilde{\alpha} : I \to Q$ be a Bertrand curve couple with arc-length parameter s and $\tilde{s}$, respectively. Then corresponding points are a fixed distance apart for all $s \in I$.*

Proof. Let $(\tilde{\alpha}, \alpha)$ be Bertrand mate. From Definition (16) we can write

$$\tilde{\alpha}(s) = \alpha(s) + \lambda(s) n_1(s), \quad s \in I \tag{34}$$

By taking the derivative of Eq.(34) with respect to s and using Eq.(10), we obtain

$$\tilde{t}(s)\frac{d\tilde{s}}{ds} = (1 - \lambda k(s))t(s) + \dot{\lambda}(s) n_1(s) + (\lambda r(s)) n_2(s) \tag{35}$$

Since $\tilde{n}_1(s)$ and $n_1(s)$ are linearly dependent, we have $\langle \tilde{t}(s), n_1(s) \rangle = 0$. Then, we get

$$\dot{\lambda}(s) = 0.$$

This means that λ is a nonzero constant. On the other hand, from the distance function between two points, we have

$$d(\tilde{\alpha}(s), \alpha(s)) = \|\alpha(s) - \tilde{\alpha}(s)\| = \|\lambda(s) n_1(s)\| = |\lambda|.$$

Namely, $d(\tilde{\alpha}(s), \alpha(s)) =$ constant. This completes the proof. ■

Theorem 18 *Let $\alpha : I \to Q$ be unit speed real spatial quaternionic curve. If $\tilde{\alpha}$ is a Bertrand mate of α, then angle measurement of this curve between tangent vectors at corresponding points is constant.*

Proof. If $h(\tilde{t}(s), t(s))' = 0$, then the proof is complete.

$$\begin{aligned} h(\tilde{t}(s), t(s))' &= h((\tilde{t}(s))', t(s)) + h(\tilde{t}(s), (t(s))') \\ &= h(\tilde{k}(s)\tilde{n}_1(s), t(s)) + h(\tilde{t}(s), k(s) n_1(s)) \\ &= \tilde{k}(s) h(\tilde{n}_1(s), t(s)) + k(s) h(\tilde{t}(s), n_1(s)) \end{aligned} \tag{36}$$

Since $\tilde{n}_1(s)$is parallel to $n_1(s)$ and $n_1(s) \perp t(s)$, then

$$h(\tilde{n}_1(s), t(s)) = 0. \tag{37}$$

Since $\tilde{n}_1(s)$ is parallel to $n_1(s)$ and $\tilde{t}(s) \perp \tilde{n}_1(s)$, then

$$h(\tilde{t}(s), n_1(s)) = 0. \tag{38}$$

Substituting (37) and (38) in (36), we have

$$h(\tilde{t}(s), t(s))^{'} = 0.$$

Hence, the proof is completed. ■

Proposition 19 *Let $\alpha : I \to Q$ be unit speed real spatial quaternionic curve. For $k(s) \neq 0$. α is a Bertrand curve if and only if there exists a linear relation*

$$\lambda k(s) + \mu r(s) = 1 \tag{39}$$

where λ, μ are non-zero constants and $k(s)$ and $r(s)$ are the curvature functions of α.

Proof. Denote the Darboux frames of $\alpha(s)$ and $\tilde{\alpha}(s)$ by $\{t(s), n_1(s), n_2(s)\}$ and $\left\{\tilde{t}(s), \tilde{n}_1(s), \tilde{n}_2(s)\right\}$, respectively. Let angle between $t(s)$ and $\tilde{t}(s)$, which is tangent vector of $\tilde{\alpha}(s)$ be θ. As $(\tilde{n}_1(s), n_1(s))$ is a linearly dependent set, we can write

$$\tilde{t}(\tilde{s}) = \cos\theta t(s) + \sin\theta n_2(s). \tag{40}$$

If we differentiate the Eq.(40) and consider $(\tilde{n}_1(s), n_1(s))$ is a linearly dependent set we can easily see that is θ a constant function.

Since $\alpha(s)$ and $\tilde{\alpha}(s)$ are Bertrand curve mate we have

$$\tilde{\alpha}(s) = \alpha(s) + \lambda n_1(s) \tag{41}$$

If Differentiating (41) with respect to s , we get

$$\tilde{t}(\tilde{s})\frac{d\tilde{s}}{ds} = (1 - \lambda k(s))t(s) + (\lambda r(s))n_2(s) \tag{42}$$

Thus, from (40) and (42) we have

$$\frac{(1 - \lambda k(s))}{\cos\theta} = \frac{\lambda r(s)}{\sin\theta} \tag{43}$$

From Above equation, we obtain

$$\lambda k(s) + \cot\theta\lambda r(s) = 1$$

If take $\cot\theta\lambda = \mu$, we get

$$\lambda k(s) + \mu r(s) = 1$$

■

Corollary 20 *Suppose that $k(s) \neq 0$ and $r(s) \neq 0$. Then α is a real spatial quaternionic Bertrand curve if and only if there exist a nonzero real number λ such that*

$$\lambda(k(s)r^{'}(s) - k^{'}(s)r(s)) - r^{'}(s) = 0 \tag{44}$$

Proof. By the proposition (19), α is a Bertrand curve if and only if there exist real numbers $\lambda \neq 0$ and μ such that $\lambda k(s) + \mu r(s) = 1$. This is equivalent to the condition that there exists a real number $\lambda \neq 0$ such that $\frac{1-\lambda k(s)}{r(s)}$ is constant. Differentiating both sides of the last equality, we get

$$\lambda(k(s)r^{'}(s) - k^{'}(s)r(s)) - r^{'}(s) = 0.$$

■

Proposition 21 *Let $\alpha : I \to Q$ be a real spatial quaternionic Bertrand curve with $k(s) \neq 0$ and $r(s) \neq 0$. Then α is of type $Aw(2)$ if and only if there is a non zero rael number λ such that*

$$k^{''}(s)k^2(s)r^2(s) - k^5(s)r^2(s) - k^3(s)r^4(s) - \lambda(k^{'}(s))^2k^2(s)r^{'}(s) - (k^{'}(s))^3r(s)(\lambda + 2) = 0. \tag{45}$$

Proof. Since α is of type Aw(2), Eq.(23) holds and since α is a real spatial quaternionic Bertrand curve, Eq.(44) holds. If both of these equations are considered, (45) is obtained. ■

Theorem 22 *Let $\alpha : I \to Q$ be a real spatial quaternionic Bertrand curve with $k(s) \neq 0$ and $r(s) \neq 0$. Then α is of type $Aw(3)$ if and only if α is a right circular helix.*

Proof. Now suppose that $\alpha : I \to Q$ is a quaternionic Bertrand curve of Aw(3) $-$type with $k(s) \neq 0$ and $r(s) \neq 0$. Then the Eqs.(32) and (44) hold on α. Differentiaitng (32), we have

$$r^{'}(s) = \frac{-2ck^{'}(s)}{k^3(s)} \tag{46}$$

Substituting (32) and (46) in (44), we get

$$k(s) = \frac{2}{3\lambda} = \text{const.} \tag{47}$$

If substituting (47) in (32), the following equation is obtained

$$r(s) = \frac{9\lambda^2 c}{4} = \text{const.}$$

Since $k\,(s)$ and $r\,(s)$ are nonzero constants, α is a right circular helix. ■

Theorem 23 *Let $\alpha : I \to Q$ be a real spatial quaternionic Bertrand curves with $k(s) \neq 0$ and $r(s) \neq 0$. If α is of weak type $Aw(2)$, then following equation hold*

$$\lambda k(s)r(s)\left(k^2(s) + r^2(s)\right) - \lambda r^{''}(s)\left(\lambda k(s) - 1\right) = 0 \tag{48}$$

Proof. Since α is of type weak AW(2), Eq.(17) holds and since α is a quaternionic Bertrand curve, Eq.(44) holds. Arranging Eq.(17), we have

$$k^{''}(s) = k^3(s) + k(s)r^2(s) \tag{49}$$

Differentiating (44), we get

$$\lambda(k(s)r^{''}(s) - k^{''}(s)r(s)) - r^{''}(s) = 0. \tag{50}$$

If Eq.(44) is substituted in (50), then (48) is obtained. This completes the proof. ■

Theorem 24 *There is not a real spatial quaternionic Bertrand curve $\alpha : I \to Q$ with $k(s) \neq 0$ and $r(s) \neq 0$ of type $AW(1)$.*

Proof. Since α is of type AW(1), Eq.(21) and (22) holds and since α is a quaternionic Bertrand curve, Eq.(44) holds. If we solve this differential equation (22), we get

$$r(s) = \frac{c}{k^2(s)}. \tag{51}$$

Differentiating (51), we get

$$r^{'}(s) = \frac{-2ck^{'}(s)}{k^3(s)}. \tag{52}$$

So substituting (51) and (52) in (44) and making necessary arrangements, we get

$$\lambda^2 = \frac{-2c^2}{72} \tag{53}$$

whic given us $\lambda^2 < 0$.Since λ is real number, this isnt possible. So $\alpha : I \to Q$ isnt a AW(1) −type Bertrand curve. This completes proof. ■

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