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The Influence of Several Major Irreversibilities on the Performance Characteristics of an n-Stage Combined Heat Pump System

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Abstract

A universal cycle model of an n-stage combined heat pump system, which includes the irreversibility of finite-rate heat transfer across finite temperature differences, the heat leak loss between the external heat reservoirs, and the irreversibilities inside the working fluid, is established and used to investigate the influence of these irreversibilities on the performance of the system. The coefficient of performance is taken as an objective function for optimization. A general optimum relation is derived for a given specific heating load and a given total heat-transfer area of heat exchangers. The general performance characteristic curves of the system are obtained. The maximum coefficient of performance with non-zero specific heating load is determined. Some key design variables, such as the specific heating load, the temperature ratios of the working fluid in the isothermal processes, the distribution of the heat-transfer areas of heat exchangers, and the total power input of the system, are optimized. The optimal performance of an arbitrary-stage irreversible, endoreversible, and reversible combined heat pump system can be directly derived from the results in this paper.

Key words: heat pump, combined cycle, irreversibility, coefficient of performance, specific heating load, optimization.

1. Introduction

A large quantity of heat energy can be extracted from the low-level heat sources such as the ambient air, waste heat, geothermal energy, solar energy, and so on, but the temperatures of these heat sources are usually too low to be practical for most direct applications. One way of exploiting these heat sources is to use heat pumps. The single-stage heat pump is adequate for most applications and has been used widely. However, some industrial applications require moderately high temperature and the temperature range they involve may be too large for a single-stage heat pump to be practical. For these special cases, one has to perform the heat pumping process in stages, that is, to use multi-stage heat pump cycles which operate in series. Thus, it is of real significance to investigate the optimal performance of n-stage combined heat pump systems.

According to the theory of classical thermodynamics, the performance of a multi-stage reversible combined heat pump cycle operating between the heated space at temperature T_p and the heat sink at temperature T_c is identical with that of a single-stage reversible heat pump cycle operating in the same temperature range. Its coefficient of performance is given by

$$\psi_c = \frac{T_p}{T_p - T_c} \qquad (1)$$

However, the results obtained from the theory of finite-time thermodynamics show clearly that when the irreversible losses in the heat pump systems are considered, the performance of single-stage and multi-stage heat pump systems are, in general, different from each other (Blanchard, 1980; Yan and Chen, 1992; Wu, 1993; Chen and Wu, 1995a,b). Although many authors have investigated the influence of some irreversibilities on the performance of a single-stage heat pump system in detail and obtained a lot of significant results (Blanchard, 1980; Yan and Chen, 1992; Wu, 1993; Chen, 1994; Cheng and Chen, 1995; Chen, et al., 1997), the performance characteristics of an n-stage combined heat pump system affected simultaneously by multi-irreversibilities are rarely investigated.

In this paper, we will comprehensively analyze the influence of several major irreversibilities on the performance of an n-stage combined heat pump system.

2. A Universal Cycle Model

In real heat pump systems, there exist a lot of irreversibilities such as the irreversibility of finite-rate heat transfer across finite temperature differences, the heat leak loss between the external heat reservoirs, the irreversibilities inside the working fluid, and so on. When these irreversibilities are taken into account, the schematic diagram of an n-stage combined heat pump system operating between two heat reservoirs at temperatures T_p and T_c may be shown as *Figure 1*. Each cycle in the system is irreversible and combined through the heat exchangers between two adjacent cycles. It may be assumed that the heat exchangers in the system are well insulated such that the amount of heat q_i released from the working fluid of the *i*th cycle per unit time is absorbed totally by the working fluid of the (i-1)th cycle. The working fluids in respective cycles flowcontinuously such that the combined heat pump system operates at a steady-state. Since there is no mixing taking place in the heat exchangers, the working fluid with more desirable characteristics in different temperature ranges can be used in each cycle. Such combined systems may not only operate efficiently in a large temperature range but also improve the performance of heat pump systems.

In *Figure 1*, q_L is the rate of heat leak (Bejan, 1989) from the heated space at temperature T_p to the heat sink at temperature T_c, q_i and q_{i+1} (i=1, 2, ..., n) are, respectively, the rates of heat transfer from and into the working fluid of the *i*th cycle in the system, T_{2i} and T_{2i+1} (i=1, 2, ..., n) are, respectively, the temperatures of the working fluid in the high- and low-temperature isothermal processes of the *i*th cycle, T_1=T_p, T_{2n+2}=T_c, and P is the total power input required by the combined heat pump system.

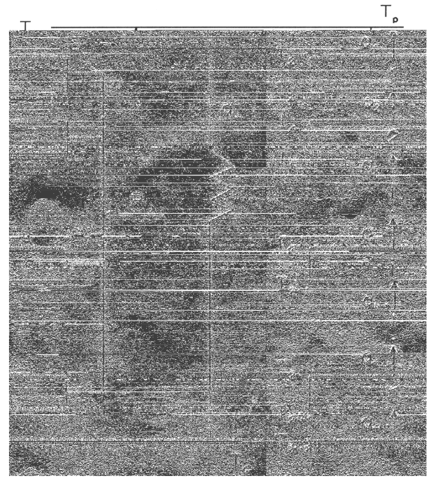

Figure 1. The schematic diagram of an n-stage irreversible combined heat pump system.

The performance of an irreversible cycle is directly dependent on heat-transfer. It is often assumed that convective and conductive heat transfers obey a linear law (Newton's law of cooling) (Andresen, 1983; De Vos, 1992), so that heat-transfer equations may be written as

$$q_i = U_i A_i (T_{2i} - T_{2i-1}) \quad (i=1,2,\ldots,n+1) \qquad (2)$$

and

$$q_L = k_L (T_p - T_c) \qquad (3)$$

where k_L is the heat leak coefficient (or thermal conductance) between the heated space and the heat sink, U_i and A_i (i=1, 2, ..., n+1) are, respectively, the overall heat-transfer coefficient and area of the *i*th heat exchanger. For an n-stage combined heat pump system, there are (n+1) heat exchangers, so the total heat-transfer area of heat exchangers in the system equals

$$A = \sum_{i=1}^{n+1} A_i \qquad (4)$$

Owing to the effect of the heat leak loss between the external heat reservoirs, the net amounts of heat q_p (heating load) and q_c transferred to the heated space and from the heat sink per unit time are given by

$$q_p = q_1 - q_L \qquad (5)$$

and

$$q_c = q_{n+1} - q_L \qquad (6)$$

respectively.

The irreversibilities inside the working fluid result primarily from friction, pressure drops, internal heat transfer, and so on. It is very difficult to describe accurately the influence of the internal irreversibilities. However, there have been several approximate ways (Yan and Chen, 1992; El-Wakil, 1962; Howe, 1982; Wu and Kinag, 1992; Ibrahim, et al., 1991, 1992) to account for the internal irreversibilities. In order to derive an analytical solution for an internally irreversible cycle similar to the solution obtained for the internally reversible one, we assume that each cycle in the system consists of two irreversible isothermal and two irreversible adiabatic processes and introduce a parameter (Wu and Kinag, 1992; Ibrahim, et al., 1991, 1992)

$$I_i = \frac{\dfrac{q_i}{T_{2i}}}{\dfrac{q_{i+1}}{T_{2i+1}}} \geq 1 \qquad (i=1,2,\ldots,n) \tag{7}$$

to describe the irreversibilities inside the working fluid in the *i*th cycle. In the following analysis, I_i is assumed to be constant. It is clearly seen from Eq. (7) that $I_i=1$ when the *i*th cycle is internally reversible and $I_i>1$ when the *i*th cycle is internally irreversible. The total irreversibility parameter of the working fluids of the whole system may be expressed by

$$I = \prod_{i=1}^{n} I_i \tag{8}$$

The above model is an important cycle model. It includes not only the irreversibility of finite-rate heat transfer across finite temperature differences but also the heat leak loss between the external heat reservoirs and the irreversibilities due to the internal dissipation of the working fluids. It is more general than those adopted in the literature, so that the optimal performance of an arbitrary-stage irreversible, endoreversible, and reversible combined heat pump system (Blanchard, 1980; Yan and Chen, 1992; Wu, 1993; Chen and Wu, 1995a,b; Chen, 1994; Cheng and Chen, 1995; Chen, et al., 1997) may be directly derived from the cycle model.

3. A General Optimum Relation

The coefficient of performance ψ, specific heating load π (heating load per unit total heat exchanger area (Wu, 1993)), and total power input P are three important performance parameters of an n-stage irreversible combined heat pump system. In the investigation relative to heat pump systems, one should first calculate the three performance parameters and derive the relations between them.

Using the above equations, one can obtain

$$\psi = \frac{q_p}{q_p - q_c} = \frac{1-\dfrac{q_L}{q_1}}{1-\dfrac{q_{n+1}}{q_1}} = \frac{1-\dfrac{q_L}{q_1}}{1-\dfrac{T_3}{I_1 T_2}\dfrac{T_5}{I_2 T_4}\cdots\dfrac{T_{2n+1}}{I_n T_{2n}}} = \frac{1-\dfrac{q_L}{q_1}}{1-\dfrac{T_c}{IT_p}\prod_{i=1}^{n+1}\dfrac{1}{x_i}} \tag{9}$$

$$\pi = \frac{q_p}{A} = \frac{q_1}{A} - \frac{q_L}{A} = \frac{q_1}{\sum_{i=1}^{n+1}\dfrac{q_i}{U_i(T_{2i}-T_{2i-1})}} - b$$
$$= T_p \Bigg/ \left[\frac{1}{U_1(x_1-1)} + \frac{1}{U_2(x_2-1)I_1x_1} + \frac{1}{U_3(x_3-1)I_1x_1I_2x_2} + \cdots + \frac{1}{U_{n+1}(x_{n+1}-1)\prod_{i=1}^{n} I_i x_i}\right] - b \tag{10}$$

and

$$P = q_1 - q_{n+1} = q_1\left(1-\frac{q_{n+1}}{q_1}\right) = A(\pi+b)\left(1-\frac{T_c}{IT_p\prod_{i=1}^{n+1}x_i}\right) \tag{11}$$

where $b=q_L/A$ and $x_i=T_{2i}/T_{2i-1}$ (i=1, 2, ..., n+1) is the temperature ratio of the working fluids in the heat exchange processes.

For the sake of convenience, let

$$U_i^* = \frac{U_i}{\prod_{j=i}^{n+1} I_j}, \quad (i=1,2,\ldots,n+1) \tag{12a}$$

$$T_{2i-1}^* = T_{2i-1}\prod_{j=i}^{n+1} I_j, \quad (i=1,2,\ldots,n+1) \tag{12b}$$

$$T_{2i}^* = T_{2i}\prod_{j=i}^{n+1} I_j, \quad (i=1,2,\ldots,n+1) \tag{12c}$$

where $I_{n+1}=1$ is stipulated because there only are n cycles in the combined system. The introduction of the parameters U^*_i, T^*_{2i-1} and T^*_{2i} will simplify the related calculations. For example, Eqs. (2), (9), and (10) may be rewritten as

$$q_i = U_i^* A_i \left(T_{2i}^* - T_{2i-1}^*\right) \text{ , (i=1,2,…,n+1)} \quad (13)$$

$$\psi = \frac{1 - \frac{q_L}{q_1}}{1 - \frac{T_c}{T_p^*} \prod_{i=1}^{n+1} \frac{1}{x_i}} \quad (14)$$

and

$$\pi = T_p^* \Big/ \left[\frac{1}{U_1^*(x_1 - 1)} + \frac{1}{U_2^*(x_2 - 1)x_1} + \frac{1}{U_3^*(x_3 - 1)x_1 x_2} + \cdots \frac{1}{U_{n+1}^*(x_{n+1} - 1)\prod_{i=1}^{n} x_i} \right] - b \quad (15)$$

respectively.

Using Eqs. (10) and (15) and eliminating q_1 and x_{n+1} in Eq. (14), one can obtain the relation between the coefficient of performance and the specific heating load as

$$\psi = \frac{1 - \frac{b}{\pi + b}}{1 - \frac{T_c}{T_p^*} \frac{1}{\prod_{i=1}^{n} x_i + \frac{1}{U_{n+1}^* D}}} \quad (16)$$

where

$$D = \frac{T_p^*}{\pi + b} - \frac{1}{U_1^*(x_1 - 1)} - \frac{1}{U_2^*(x_2 - 1)x_1} - \frac{1}{U_3^*(x_3 - 1)x_1 x_2} - \cdots - \frac{1}{U_n^*(x_n - 1)x_1 x_2 \cdots x_{n-1}} \quad (17)$$

For a given specific heating load π and a given total heat-transfer area A, using Eq. (16) and the extremal conditions

$$\left(\frac{\partial \psi}{\partial x_i} \right)_{\pi, A} = 0 \text{ , (i=1,2,…,n)} \quad (18)$$

one finds that the optimal relation between the coefficient of performance and the specific heating load is given by (a detailed derivation is given in appendix A)

$$\psi = \frac{\left(1 - \frac{b}{\pi + b}\right)\left(1 + \frac{\pi + b}{U^* T_p^*}\right)}{1 - \frac{T_c}{T_p^*} + \frac{\pi + b}{U^* T_p^*}} \quad (19)$$

where

$$U^* = \frac{1}{\left(\sum_{i=1}^{n+1} \frac{1}{\sqrt{U_i^*}} \right)^2} \quad (20)$$

is the equivalent overall heat-transfer coefficient of the combined heat pump system. Equation (19) is a general optimum relation of an n-stage irreversible combined heat pump system. It may be used to derive other optimal relations of an n-stage combined heat pump system. For example, using Eqs. (19), (9), and (11), we obtain the optimal relation between the total power input and the specific heating load as

$$\psi = \frac{A(\pi + b)\left(1 - \frac{T_c}{T_p^*} + \frac{\pi + b}{U^* T_p^*}\right)}{1 + \frac{\pi + b}{U^* T_p^*}} \quad (21)$$

For some special cases, the optimal performance of an arbitrary-stage combined heat pump system may be derived from Eq. (19).

(i) When $I_i=1$ (i=1,2,...,n), there does not exist the irreversibilities inside the working fluid and the cycle is internally reversible.

$$U^* = \frac{1}{\left(\sum_{i=1}^{n+1} \frac{1}{\sqrt{U_i}} \right)^2} = U \quad (22)$$

In such a case, Eq. (19) may be used to discuss the optimal performance of an n-stage endoeversible combined heat pump system affected by finite-rate heat transfer and heat leak loss as long as we substitute U and T_p for U^* and T_p^* in Eq. (19).

(ii) When $U_i \to \infty$ (i=2,..., n), the irreversibility of heat transfer between two adjacent cycles in the combined system is negligible. The equivalent overall heat-transfer coefficient of the combined system may be simplified as

$$U^* = \frac{1}{\left(1\big/\sqrt{U_1/I} + 1\big/\sqrt{U_{n+1}}\right)^2} \quad (23)$$

which is identical to that of a single-stage irreversible heat pump system having the same irreversibility factor I, so that Eq. (19) may be used to discuss the optimal performance a single-stage irreversible heat pump system (Chen, 1994).

(iii) When k_L=0, Eq. (19) may be written as

$$\psi = \frac{1+\left(1+\frac{\pi}{U^*T_p^*}\right)}{1-\frac{T_c}{T_p^*}+\frac{\pi}{U^*T_p^*}} \tag{24}$$

which shows that the coefficient of performance is a monotonically decreasing function of the specific heating load.

(iv) When k_L=0 and I_i=1 (i=1,2,...,n), then U^*=U and T_p^*=T_p. The relation between ψ and π is simplified as

$$\psi = \frac{1+\frac{\pi}{UT_p}}{1-\frac{T_c}{T_p}+\frac{\pi}{UT_p}} \tag{25}$$

Equation (25) has been used to discuss the optimal performance of an n-stage endoreversible combined heat pump systems (Chen and Wu, 1995b).

(v) When $U_i \rightarrow \infty$ (i=1,2,..., n+1), then $U \rightarrow \infty$ and the irreversibility of heat transfer in the system is negligible. The relation between the coefficient of performance ψ and the heating load q_p,

$$\psi = \left(1-\frac{q_L}{q_p+q_L}\right)\frac{T_p^*}{T_p^*-T_c} \tag{26}$$

may be derived directly from Eq. (19).

(vi) When k_L=0, I_i=1 (i=1,2,...,n), and $U_i \rightarrow \infty$ (i=1,2,..., n+1), there does not exist any irreversibility in the whole cycle system and the coefficient of performance is given by Eq. (1), which can be derived from Eq. (19). In such a case, the heating load

$$q_p = P\psi_c \tag{27}$$

may be an arbitrary value, which is determined by the total power input.

Similarly, starting from Eq. (19), we can also discuss the special cases that there exist only the irreversibilities inside the working fluid or the heat leak loss between the external heat reservoirs for an n-stage combined heat pump system.

4. General Performance Characteristic Curves

Using Eqs. (19) and (21) and defining the dimensionless specific heating load $\pi^*=\pi/(U^*T_p^*)$ and the dimensionless specific power input $P^*=P/(U^*AT_p^*)$, we can generate easily the ψ-π^*, P^*-π^*, and ψ-P^* characteristic curves of an n-stage combined heat pump system, as shown in *Figures* 2-4, respectively. In these figures, curves a [I=1, $b/(U^*T_p^*)$=0], b [I=1.1, $b/(U^*T_p^*)$=0], c [I=1, $b/(U^*T_p^*)$=0.05], and d [I=1.1, $b/(U^*T_p^*)$=0.05] are presented for T_p/T_c=1.5.

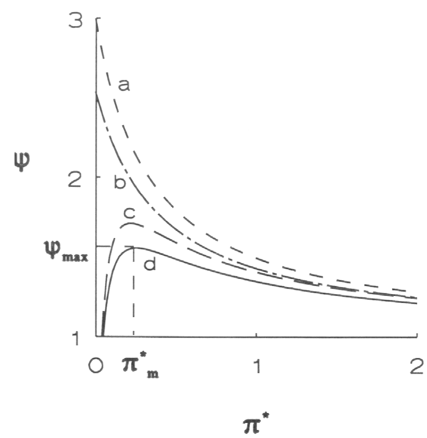

Figure 2. The coefficient of performance versus dimensionless specific heating load. Curves a [I=1, $b/(U^*T_p^*)$=0], b [I=1.1, $b/(U^*T_p^*)$=0], c [I=1, $b/(U^*T_p^*)$=0.05], and d [I=1.1, $b/(U^*T_p^*)$=0.05] are presented for T_p/T_c=1.5.

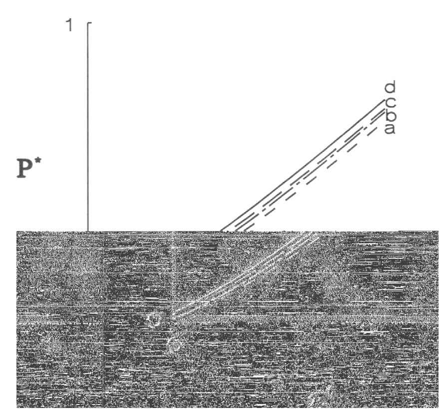

Figure 3. The dimensionless specific power input versus dimensionless specific heating load. The values of I, $b/(U^*T_p^*)$, and T_p/T_c are the same as those used in Figure 2.

The curve d in *Figure* 2 or 4 shows clearly that when the various irreversibilities mentioned above are taken into account, the coefficient of performance of an n-stage combined heat pump system is not a monotonic function of the specific heating load or the total power input. There exists a maximum for the coefficient of performance.

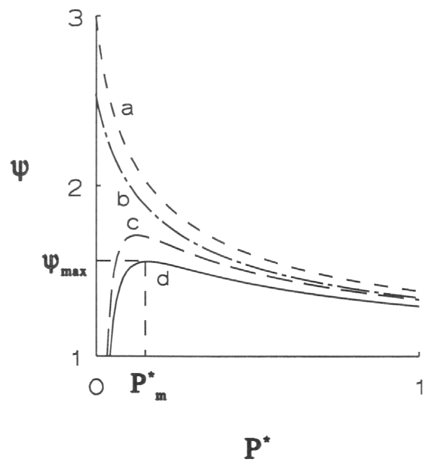

Figure 4. The coefficient of performance versus dimensionless specific power input. The values of I, $b/(U^*T_p^*)$, and T_p/T_c are the same as those used in Figure 2.

5. Maximum Coefficient of Performance

From Eq. (19) and its extremal condition

$$\frac{d\psi}{d\pi} = 0 \tag{28}$$

one can prove that when

$$\pi^* = \frac{b}{U^*T_p^*}\left[1-\frac{T_c}{T_p^*}+\frac{b}{U^*T_p^*} + \sqrt{\left[1+\left(1-\frac{T_c}{T_p^*}\right)\frac{U^*T_p^*}{b}\right]\frac{T_c}{T_p^*}}\right] \Bigg/ \left[\frac{T_c}{T_p^*}-\frac{b}{U^*T_p^*}\right] \equiv \pi_m^* \tag{29}$$

the coefficient of performance attains its maximum, i.e.,

$$\psi_{max} = \left[\sqrt{\frac{b}{U^*T_p^*}\left(1-\frac{T_c}{T_p^*}+\frac{b}{U^*T_h^*}\right)} + \sqrt{\frac{T_c}{T_p^*}}\right]^2 \Bigg/ \left[\sqrt{\frac{b}{U^*T_p^*}} + \sqrt{\left(1-\frac{T_c}{T_p^*}+\frac{b}{U^*T_p^*}\right)\frac{T_c}{T_p^*}}\right]^2 \tag{30}$$

In such a case, the power input of the system is

$$P_m^* = \left(\frac{b}{U^*T_p^*}+\pi_m^*\right) \times \frac{1-\frac{T_c}{T_p^*}+\frac{b}{U^*T_p^*}+\pi_m^*}{1+\frac{b}{U^*T_p^*}+\pi_m^*} \tag{31}$$

It is seen from the curve d in *Figure* 2 that when $\psi<\psi_{max}$, there are two different π^* for a given coefficient of performance ψ, where one is larger than π^*_m and the other is smaller than π^*_m. In the region of $\pi^*<\pi^*_m$, the coefficient of performance decreases as the specific heating load decreases. The region is not optimal for an n-stage combined heat pump system. In the region of $\pi^*>\pi^*_m$, the coefficient of performance increases as the specific heating load decreases, and vice versa. One always wants to obtain the specific heating load as large as possible for the same coefficient of performance. Thus, the optimal region should be

$$\pi^* \geq \pi_m^* \tag{32}$$

This shows that ψ_{max} and π^*_m are two important performance parameters of n-stage combined heat pump systems. They determine the upper bound on the coefficient of performance with non-zero specific heating load and the allowable value of the lower bound of the specific heating load, respectively. According to Eq. (32), the power input should be

$$P \geq P_m \tag{33}$$

It is also seen from the curve d in Fig. 2 that when the specific heating load is very large, the coefficient of performance will approximate to 1. It is thus clear that it is unsuitable for a heat pump to be operated with a very large specific heating load.

6. Optimal Combined Conditions

It is well known that only when each cycle in the system is combined optimally can the whole system operate in the optimum working states. Thus, the parameters in the system can not be chosen arbitrarily and must satisfy certain conditions.

Using Eqs. (A5), (7), and (13), we obtain a concise relation for the optimal distribution of the heat-transfer areas as

$$\sqrt{U_i^*}A_i = \sqrt{U_{i+1}^*}A_{i+1} \quad (i=1,2,\ldots,n) \tag{34}$$

Solving Eqs. (34) and (4) gives the optimal relations between the heat-transfer area A_i of the *i*th heat exchanger and the total heat-transfer area A as

$$A_i = \frac{A}{\sum_{j=1}^{n+1} \sqrt{U_i^* / U_j^*}}$$

$$(n=1,2,\ldots,n+1) \qquad (35)$$

Equations (34) and (35) show clearly that the optimal distribution of the heat-transfer areas is, in general, dependent on the overall heat-transfer coefficients of heat exchangers and the irreversibility parameters I_i inside the working fluids, but independent of the temperatures of the external heat reservoirs and the heat leak loss between the external heat reservoirs. It is of interest to note that Eqs. (34) and (35) are the same as the optimal distribution of the heat-transfer areas of heat exchangers in an n-stage combined refrigeration system.

Now, we continue to determine the optimal ratios of the temperatures of the working fluids in the isothermal processes. From Eqs. (A5), (A7), (A8), (14), and (16), we obtain

$$x_1 x_2 \cdots x_i = = 1 + \frac{\pi + b}{T_p^*} \sum_{k=1}^{i} \frac{1}{\sqrt{U_k^*}} \sum_{j=1}^{n+1} \frac{1}{\sqrt{U_j^*}} \qquad (36)$$

$$(i=1,2,\ldots,n)$$

and

$$x_{n+1} = 1 + \frac{1}{\sqrt{U_{n+1}^*} \sum_{i=1}^{n} \left(\frac{1}{\sqrt{U_i^*}} + \frac{U^* T_p^*}{\pi + b} \sum_{i}^{n+1} \frac{1}{\sqrt{U_i^*}} \right)} \qquad (37)$$

When the combined system operates in the state of the maximum coefficient of performance, the optimal ratios of the temperatures of the working fluids in the isothermal processes are determined by Eqs. (36), (37), and (29).

7. Conclusions

The cycle model established in this paper can capture the principal irreversibility sources of some real heat pump systems. Owing to the introduction of the parameter I_i describing the irreversibilities inside the working fluid, the analytical solution can be derived directly from the cycle model. The key design variables are optimized. The upper bound on the coefficient of performance of heat pump systems is given. The optimal region of the specific heating load and the reasonable range of the power input are determined. Several special cases are discussed in detail. The results obtained here can be used to analyse the comprehensive or respective influence of the irreversibility of finite-rate heat transfer across finite temperature differences, the heat leak loss between the external heat reservoirs, and the irreversibilities inside the working fluid on the performance of an n-stage combined heat pump system. It is important that some common characteristics of heat pump systems are revealed so that the optimal performance of an arbitrary-stage irreversible, endoreversible, or reversible combined heat pump system may be derived directly from the results obtained in this paper. It is more important that the establishment of this universal cycle model may promote the further investigation into the performance of irreversible heat pump systems.

Acknowledgments

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Appendix A

From Eqs. (16) and (18), we obtain

$$U_{n+1}^* D^2 x_2 x_3 \cdots x_n - \left[\frac{1}{U_1^*(x_1-1)^2} + \frac{1}{U_2^*(x_2-1)x_1^2} + \cdots + \frac{1}{U_n^*(x_n-1)x_1^2 x_2 \cdots x_{n-1}} \right] = 0 \qquad (A1)$$

$$U_{n+1}^* D^2 x_1 x_3 \cdots x_n - \left[\frac{1}{U_2^*(x_2-1)^2 x_1} + \frac{1}{U_3^*(x_3-1)x_1 x_2^2} + \cdots + \cdots + \frac{1}{U_n^*(x_n-1)x_1 x_2^2 \cdots x_{n-1}} \right] = 0 \qquad (A2)$$

. . . .

$$U_{n+1}^* D^2 x_1 x2 \cdots x_{i-1} x_{i+1} \cdots x_n - \left[\frac{1}{U_i^*(x_i-1)^2 x_1 x_2 \cdots x_{i-1}} + \cdots + \frac{1}{U_n^*(x_n-1)x_1 x_2 \cdots x_{i-1}} \times \frac{1}{x_i^2 x_{i+1} \cdots x_{n-1}} \right] = 0 \qquad (A3)$$

. . . .

$$U_{n+1}^* D^2 x_1 x_2 \cdots x_{n-1} - \frac{1}{U_n^*(x_n - 1)^2} \times \frac{1}{x_1 x_2 \cdots x_{n-1}} = 0 \qquad (A4)$$

Solving Eqs. (A1)-(A4), we have

$$\sqrt{U_1^*}(x_1 - 1) = \sqrt{U_2^*}(x_2 - 1)x_1 = \sqrt{U_3^*}(x_3 - 1)x_1 x_2 = \cdots = \sqrt{U_i^*}(x_i - 1)x_1 x_2 \cdots x_{i-1} = \cdots = \sqrt{U_n^*}(x_n - 1)x_1 x_2 \cdots x_{n-1} = \frac{1}{\sqrt{U_{n+1}^*}D} \qquad (A5)$$

Using Eq. (A5), D may be simplified as

$$D = \frac{T_p^*}{\pi + b} - \frac{1}{\sqrt{U_1^*}(x_1 - 1)} \sum_{i=1}^{n} \frac{1}{\sqrt{U_i^*}} \qquad (A6)$$

Solving Eqs. (A5) and (A6) yields

$$\prod_{i=1}^{n} x_i = 1 + \frac{\pi + b}{T_p^*} \left(\sum_{j=1}^{n+1} \frac{1}{\sqrt{U_j^*}} \right) \left(\sum_{k=1}^{n} \frac{1}{\sqrt{U_k^*}} \right) \qquad (A7)$$

and

$$\frac{1}{DU_{n+1}^*} = \frac{\pi + b}{\sqrt{U_{n+1}^*} T_p^*} \times \sum_{i=1}^{n+1} \frac{1}{\sqrt{U_i^*}} \qquad (A8)$$

Substituting Eqs. (A7) and (A8) into Eq. (16), we obtain Eq. (19).

Nomenclature

A	Total heat-transfer area (m^2)
A_i	Heat-transfer area the *i*th heat exchanger (m^2)
b	$=q_L/A$ (W/m^2)
D	A function defined by Eq. (17) ($K(m^2)/W$)
I	Total internal irreversibility parameter
I_i	Internal irreversibility parameter of the *i*th cycle
k_L	Heat leak coefficient between heated space and heat sink (W/K)
P	Power input (W)
P^*	Dimensionless specific power input
P_m	Power input at maximum coefficient of performance (W)
P^*m	Dimensionless specific power input at maximum coefficient of performance
q_c	Net amount of heat transferred from heat sink per unit time (W)
q_i	Rate of heat transfer from the *i*th cycle to the (i-1)th cycle (W)
q_L	Rate of heat leak from the heated space to the heat sink (W)
q_p	Net amount of heat transferred to heated space per unit time (heating load) (W)
T_{2i}	Temperature of working fluid in high-temperature isothermal process of the *i*th cycle (K)
T_{2i+1}	Temperature of working fluid in low-temperature isothermal process of the *i*th cycle (K)
T_c	Temperature of heat sink (K)
T_p	Temperature of heated space (K)
T^*k	Equivalent temperature defined by Eq. (12) (K).
T^*p	$=IT_p$ (K)
U_i	Overall heat-transfer coefficient of the *i*th heat exchanger ($WK^{-1}m^{-2}$)
U^*	Equivalent overall heat-transfer coefficient of the system ($WK^{-1}m^{-2}$)
U^*i	Equivalent overall heat-transfer coefficient defined by Eq. (12) ($WK^{-1}m^{-2}$)
x_i	$=T_{2i}/T_{2i+1}$
ψ	Coefficient of performance
ψ_c	Coefficient of performance of a reversible Carnot heat pump
ψ_{max}	Maximum coefficient of performance
π	Specific heating load (W/m^2)
π^*	Dimensionless specific heating load
π_m	Specific heating load at maximum coefficient of performance (W/m^2)
$\pi^*{}_m$	Dimensionless specific heating load at maximum coefficient of performance

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