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Analytic Solutions to Power-Law Graded Hyperbolic Rotating Discs Subjected to Different Boundary Conditions

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Abstract

The exact elastic response of a convergent/divergent hyperbolic rotating disc made of a power-law graded material is studied under different boundary conditions. Soundness of the formulas derived is verified with the literature. A parametric study is performed to investigate elastic responses of those discs under four boundary conditions such as a stationary disc subjected to internal/external pressures, a rotating disc: both surfaces may expand freely, a rotating disc mounted a rigid shaft: outer surface either may freely expand or contains a rigid casing.

Keywords: Exact solution, Variable thickness, Functionally graded, Rotating disk, Hyperbolic profile.

1. Introduction

Analytical and numerical studies on functionally graded discs have gained a momentum since 1990s. There are numerous studies on stationary/rotating discs with constant/variable thickness and made of an isotropic and non-homogeneous material in the available literature. Some of those studies were performed analytically [1-17]. For some types of those material grading patterns such as a simple power rule, or an exponential variation or a linear function it is possible to obtain a closed form solution to the problem. For instance if the grading function is chosen as a familiar simple power function [1-2, 6, 10, 17], as in the present study, the differential equations turn into Euler-Cauchy types with constant coefficients. For linearly varying properties, the solution is obtained in terms of hyper-geometric functions. If the material grading is performed by an exponential function [3, 5, 8, 12] then Whittaker / Kumer functions or Frobenius series are employed in the solution. Apart from those, unlike general use of polar/cylindrical coordinates, elliptic cylindrical coordinate system may also be used to get closed form solutions in the formulation [14]. For the two-dimensional analytical solution of such problems separation of variables technique with complex Fourier series may even implemented in the solution procedure as Jabbari et al. [15] did. For arbitrary grading rules and for cylindric monoclinic materials, the analytical solution may also be achieved by using iterative power series [16]. However, in general, for other types of grading rules it is required to use a numerical solution techniques.
In the literature, especially analytical studies on such structures subjected to the only the inner pressure are relatively large. As a result the scholars, by and large, are obliged to verify their advanced results with the available literature which covers merely the formulas for stationary/rotating uniform disks subjected to an internal pressure. The number of studies considering the continuously variation of the thickness of the disc, simultaneous effects of both the inner and

outer pressures together with the rotation at a constant angular velocity under different boundary conditions are scarce. In the present study those effects are all considered in the formulation.

Horgan and Chan [1-2] gave explicit solutions for rotating discs of constant density and thickness. Horgan and Chan [1] investigated the effects of material inhomogeneity on the response of linearly elastic isotropic hollow circular cylinders or disks under uniform internal or external pressure. The special case of a body with Young's modulus depending on the radial coordinate only, and with constant Poisson's ratio, was examined. It was shown that the stress response of the inhomogeneous cylinder (or disk) is significantly different from that of the homogeneous body. For example, the maximum hoop stress does not, in general, occur on the inner surface in contrast with the situation for the homogeneous material. Zenkour [3] studied analytically exponentially graded rotating annular discs with constant thickness. Closed form solutions incorporates Whittaker's functions for exponential variation of both elasticity modulus and density. Eraslan and Akış [4] used two variants of a parabolic function for disks made of functionally graded materials. Zenkour [5] extended his study for such discs with rigid casing. Bayat et al. [6], based on the power-law distribution, gave both analytical and semi-analytical elastic solutions for axisymmetric rotating hollow discs with parabolic and hyperbolic thickness profiles. This semi-analytic solution was obtained by dividing the disc with varying thickness into sub-domains with uniform thickness. Peng and Li [7] studied a thermoelastic problem of a circular annulus made of functionally graded materials with an arbitrary gradient. Their analysis involving a Fredholm integral equation neither requires a special form of the gradient of material properties nor demands partitioning the entire structure into a multilayered homogeneous structure. Zenkour and Massat [8] used the modified Runge-Kutta algorithm in their numerical analysis while the hyper-geometric and Kummer's functions were employed in their analytical study. Çallıoğlu et al. [10] performed an exact stress analysis of annular rotating discs made of functionally graded materials by assuming that both elasticity modulus and material density vary radially as a function of a simple power rule with the same inhomogeneity parameter. Çallıoğlu et al. [10] gave analytical formulas for uniform rotating discs subjected to the boundary condition such as expansions are free at both surfaces under rotation. Ghorbani [11] used a time domain semi-analytical solution to study thermoelastic creep behavior of functionally graded rotating axisymmetric disks with variable thickness. In analytical solution Ghorbani [11] divided the disk into some virtual sub-domains. General solution of equilibrium equations in each sub-domain were obtained by imposing the continuity conditions at the interface of the adjacent sub-domains together with global conditions. Nejad et al. [12-13] gave a closed-form analytical solution in terms of hyper-geometric functions to elastic analysis of exponentially functionally graded stationary discs subjected to internal and external pressures. Khorshidvand and Khalili [14] studied thermal and mechanical stresses as an analytical solution of the Navier equation for symmetric thick hollow cylinder made of exponentially functionally graded material which is rotating around its axis is presented in elliptic-cylinder coordinate system. For a particular case they showed that, with the help of the elliptical coordinate system, Navier's equations is converted to non-homogeneous ordinary differential equation with constant coefficients since elliptic cylinder is converted to circular cylinder and exponential-law is converted to power- law along the radial direction. Recently, Yıldırım and Boğa [17] presented closed-form formulas for a power-law graded rotating uniform discs under different boundary conditions.

A little number of exact studies presenting the external pressure and rotation effects together with the internal pressure on the elastic behavior of the variable thickness disc made of a functionally graded material (FGM) under different boundary conditions may exist in the literature. As far as the author knows the closed form formulas which consider the rotating disks mounted to a shaft and have

continuously varying thickness profile might exist in the available literature. This motivated to the author to accomplish this study.
In this work the exact elastic response of a rotating disk made of a nonhomogeneous material is studied by extending of the study in Reference [17] to the rotating disk having a continuously varying hyperbolic thickness profile. Both convergent-hyperbolic and divergent-hyperbolic disk profiles together with uniform profile are all studied. Power-law grading is used for material gradation pattern. As stated above, as a special application of a simple power material grading rule the coefficients of the governing equation become constants. This allows ones to get closed-form solutions to the problem having Euler-Cauchy type of differential equations. The present formulation comprises both continuosly variations of elasticity modulus and material density including continuously variation of the thickness of the disc except variation of Poisson's ratios. Contrary to the literature all effects affecting the elastic behavior of the disk with varying thickness such as internal and external pressures including rotation at a constant angular velocity are all studied under four physical boundary conditions and presented in compact forms.

2. Governing equation and its solution

The governing equation which is in the form of a second degree nonhomogeneous differential equations with variable coefficients are obtained by exploiting the strain-displacement relations, constitutive equations and equilibrium equations for an axisymmetric plane-stress case. For isotropic functionally graded materials and axisymmetric loading, the linearly elastic rotating disc problem is reduced to the solution of a second order nonhomogeneous differential equation with variable coefficients as follows:

$$\frac{d^2}{dr^2}u_r(r)+\frac{d}{dr}u_r(r)\left(\frac{\frac{d}{dr}E(r)}{E(r)}+\frac{h(r)+r\frac{d}{dr}h(r)}{rh(r)}\right)+u_r(r)\left(\frac{\nu\frac{d}{dr}E(r)}{rE(r)}-\frac{1}{r^2}+\frac{\nu\frac{d}{dr}h(r)}{rh(r)}\right)=\frac{\omega^2 r\rho(r)(1-\nu^2)}{E(r)} \quad (1)$$

where r is the radial coordinate, $u_r(r)$ is the radial displacement, ν is the Poisson's ratio, $h(r)$ defines the thickness profile, ω is the constant angular velocity, *E(r)* is the Young's modulus, and $\rho(r)$ is the density. Eq. (1) in terms of the radial displacement called Navier equation comprises both the inhomogeneity of the materials and the rotation as a body force. This type of problems are also referred to as boundary value problems. The complete solution is obtained by adding the homogeneous and particular solutions.
For a special case, by introducing a simple power material grading rule, such a second degree homogeneous/nonhomogeneous differential equation takes the form of Euler-Cauchy differential equation with constant coefficients, which allows a closed-form solution. For any other material grading pattern which causes variable coefficients in the equation, in general, an appropriate numerical technique is required. Any second-order non-homogeneous differential equation is called as an Euler Cauchy equation if it can be written in the following form

$$\mu_1 x^2 y''(x)+\mu_2 x y'(x)+\mu_3 y(x)=f(x) \quad (2)$$

where μ_1, μ_2 and μ_3 are real/complex-valued constants. Assuming a homogeneous solution is in the form of $y(x)=x^{\alpha}$, a general solution of Euler-Cauchy equation is given by

$$y(x)=A_1x^{\alpha_1}+A_2x^{\alpha_2}+y_p(x) \quad (3)$$

where $y_p(x)$ is the particular solution which may be found by general technique "variation of parameters" and α_i are roots of the characteristic equation. After solution of Eq. (1) for u_r, Hooke's law

$$\sigma_r(r) = \frac{E(r)}{1-\nu^2}\left[\varepsilon_r(r) + \nu\varepsilon_\theta(r)\right] \tag{4}$$

$$\sigma_\theta(r) = \frac{E(r)}{1-\nu^2}\left[\varepsilon_\theta(r) + \nu\varepsilon_r(r)\right]$$

together with strain-stress relations

$$\varepsilon_r(r) = u_r'(r) \quad ; \quad \varepsilon_\theta(r) = \frac{u_r(r)}{r} \tag{5}$$

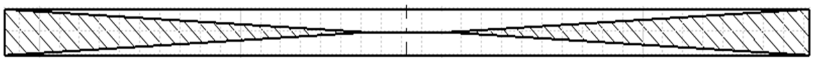
are used to calculate both the radial and hoop stresses, σ_r and σ_θ. In Eq. (5) ε_r and ε_θ represent the radial and tangential strains, respectively.

3. Disc geometry and material gradient

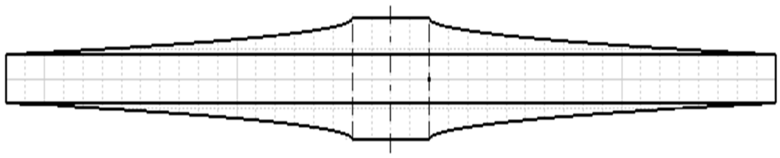
The disk whose inner radius is denoted by a and outer radius is denoted by b is assumed to be symmetric with respect to the mid plane, and its profile vary radially continuously in an hyperbolically form given by

$$h(r) = h_b\left(\frac{r}{b}\right)^m \tag{6}$$

where h_b is the thickness of the disc at the outer surface. From the above function a uniform disc profile is obtained with $m = 0$, a convergent hyperbolic dick profile is obtained with $m < 0$ and for $m > 0$ a divergent hyperbolic disc profile is reached (Fig. 1)



Divergent profile



Convergent profile

Figure 1: Hyperbolic disk profiles

A metal-ceramic pair is mixed radially provided that it complies with a simple power material grading rule in order to create a new isotropic but no longer homogeneous material having more attractive material properties especially for the heat resistance. Suppose that Material-a is located at the inner surface and Material-b is located at the outer surface. Between inner and outer surfaces material properties vary by obeying either

$$E(r) = E_a\left(\frac{r}{a}\right)^\beta \quad ; \quad \rho(r) = \rho_a\left(\frac{r}{a}\right)^q \tag{7a}$$

or

$$E(r) = E_b \left(\frac{r}{b}\right)^{\beta}; \;\; \rho(r) = \rho_b \left(\frac{r}{b}\right)^{q} \tag{7b}$$

rules. In Eq. (7) inhomogeneity parameters are defined as follows

$$\beta = \frac{ln(\frac{E_a}{E_b})}{ln(\frac{a}{b})} = \frac{ln(\frac{E_b}{E_a})}{ln(\frac{b}{a})} \quad ; \quad q = \frac{ln(\frac{\rho_a}{\rho_b})}{ln(\frac{a}{b})} = \frac{ln(\frac{\rho_b}{\rho_a})}{ln(\frac{b}{a})} \tag{8}$$

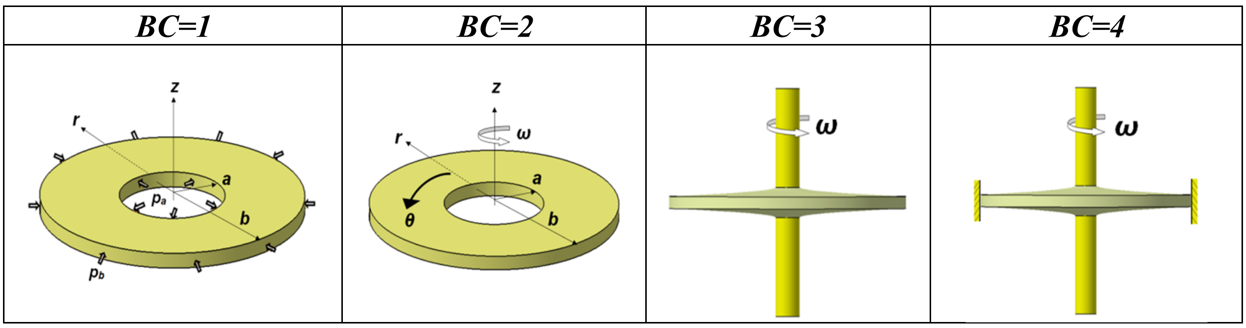
4. Closed-form elastic solutions for FGM discs

In the below the internal/external uniform pressure effect and rotation effect are studied separately. For small deformations, the superposition principle will be hold to estimate the overall behavior of the structure under whole loadings. It may be noted that those equations presented here cannot be used directly for cylindrical vessels. Because the exact stiffness for a disk geometry under plane stress assumption were used in the solution to get simpler equations under for all boundary conditions shown in Fig. 2. Apart from those boundary conditions BC=3 and BC=4 (Fig. 2) together with varying thickness assumption do not matter for pressurized cylinders. However, at the final stage of the substitution if the resulting formula involves the terms E or ν the analogs between the plane-strain and the plain stress equations may be employed. For instance, if one replace formally ν with $\frac{\nu}{1-\nu}$, and E with $\frac{E}{1-\nu^2}$ he may get the results for the plain-strain case from the plane stress solutions. As it is known ν should be replaced formally with $\frac{\nu}{1+\nu}$, and E is to be replaced with $\frac{E(1+2\nu)}{(1+\nu)^2}$ to get the plane stress results from the plain strain solutions. In this study since the functionally graded material is composed of two materials and Poisson's ratio is assumed to be unchanged along the radial direction, it may be functional to use the arithmetic mean of the numerical values of Poisson's ratios in calculations. In application of the analogs, m=0 should be taken in Eq. (6) to reach the equations for the plain-strain case. For simplicity in what follows, the author confines attention to the plane stress problem. It may be notable that if Eq. (7a) is employed then the followings

$$E^* = \frac{E_a}{a^{\beta}}; \;\; \rho^* = \frac{\rho_a}{a^{\beta}} \tag{9a}$$

if Eq. (7b) is adopted then the followings should be used in the following formulas.

$$E^* = \frac{E_b}{b^{\beta}}\,; \;\; \rho^* = \frac{\rho_b}{b^{\beta}} \tag{9b}$$



$\sigma_r(a) = -p_a$ $\sigma_r(b) = -p_b$	$\sigma_r(a) = 0$ $\sigma_r(b) = 0$	$u_r(a) = 0$ $\sigma_r(b) = 0$	$u_r(a) = 0$ $u_r(b) = 0$
Internal/external pressures at both surfaces ($\omega = 0$)	Expansions are free at both surfaces ($\omega \neq 0$)	Rigid shaft at the center Free expansion at the outer ($\omega \neq 0$)	Rigid shaft at the center Rigid casing at the outer ($\omega \neq 0$)

Figure 2: Boundary conditions considered in the present study

4.1. Internal/External uniform pressure effects

The differential equation given by Eq. (10) is solved for pressurized disk with varying sections under the boundary condition BC-1 (Fig. 2), $\sigma_r(a) = -p_a$ and $\sigma_r(b) = -p_b$. Where, p_a is the internal uniform pressure and p_b is the external uniform pressure.

$$\frac{(-1+m\nu+\beta\nu)u_r}{r^2} + \frac{(1+m+\beta)u_r'}{r} + u_r''=0 \tag{10}$$

By introducing the following

$$\xi = \sqrt{(4 + (m + \beta)(m + \beta - 4\nu))} \tag{11}$$

homogeneous solution takes the form of

$$u_r = r^{\frac{1}{2}(-m-\beta-\xi)}(C_1 + C_2 r^{\xi}) \tag{12}$$

By employing Eqs. (4) and (5) both the radial and hoop stress are simplified as follows

$$\sigma_r = -\frac{1}{2}C_{11}r^{\frac{1}{2}(-2-m-\beta-\xi)}(C_2 r^{\xi}(m + \beta - 2\nu - \xi) + C_1(m + \beta - 2\nu + \xi))$$

$$\sigma_\theta = \frac{1}{2}r^{\frac{1}{2}(-2-m-\beta-\xi)}\left(C_2 C_{11} r^{\xi}\left(2 - \nu(m + \beta - \xi)\right) - C_1 C_{11}\left(-2 + \nu(m + \beta + \xi)\right)\right) \tag{13}$$

In Eqs. (12) and (13)

$$C_1 = \frac{2a^{\frac{\xi-\beta}{2}}b^{\frac{\xi-\beta}{2}}(a^{\frac{m}{2}+1}b^{\frac{\beta+\xi}{2}}p_a - a^{\frac{\beta+\xi}{2}}b^{\frac{m}{2}+1}p_b)(\nu^2-1)}{(a^{\xi}-b^{\xi})E^*(m+\beta-2\nu+\xi)} \tag{14}$$

$$C_2 = \frac{2a^{-\beta/2}b^{-\beta/2}(a^{\beta/2}b^{\frac{1}{2}(m+\xi+2)}p_b - a^{\frac{1}{2}(m+\xi+2)}b^{\beta/2}p_a)(\nu^2 - 1)}{(a^{\xi} - b^{\xi})E^*(m + \beta - 2\nu - \xi)}$$

and

$$C_{11} = \frac{E^* r^{\beta}}{1-\nu^2} \tag{15}$$

After substituting Eq. (14) into the solution in Eqs. (12) and (13) we will have the followings closed formed solutions for the radial displacement and radial stress. Since the explicit form of the hoop stress occupies more volume it is not presented here.

$$u_{r=}\frac{2a^{\frac{1}{2}(m-\beta+\xi+2)}p_a r^{\frac{1}{2}(-m-\beta-\xi)}(\nu^2-1)(b^{\xi}(m+\beta-2\nu-\xi)-r^{\xi}(m+\beta-2\nu+\xi))}{(a^{\xi}-b^{\xi})E^{*}((m+\beta-2\nu)^2-\xi^2)}$$

$$-\frac{2b^{\frac{1}{2}(m-\beta+\xi+2)}p_b r^{\frac{1}{2}(-m-\beta-\xi)}(\nu^2-1)(a^{\xi}(m+\beta-2\nu-\xi)-r^{\xi}(m+\beta-2\nu+\xi))}{(a^{\xi}-b^{\xi})E^{*}((m+\beta-2\nu)^2-\xi^2)} \quad (16a)$$

$$\sigma_{r=}\frac{a^{\frac{1}{2}(m-\beta+\xi+2)}p_a r^{\frac{1}{2}(-m+\beta-\xi-2)}(b^{\xi}-r^{\xi})}{a^{\xi}-b^{\xi}}+\frac{b^{\frac{1}{2}(m-\beta+\xi+2)}p_b r^{\frac{1}{2}(-m+\beta-\xi-2)}(a^{\xi}-r^{\xi})}{b^{\xi}-a^{\xi}} \quad (16b)$$

As stated above, since Eq. (16b) does not incorporate the terms related to either Young's modulus or Poisson's ratio at the final stage of substitution, it may be directly used for cylindrical structures. However for the radial displacement in Eq. (16a) it is necessary to use

$$\nu=\frac{\nu}{1-\nu};\quad E^{*}=\frac{E^{*}}{1-\nu^2} \quad (17)$$

to transform Eq. (16) from the plane stress to the plane strain.

4.2. Rotation effect

Let's consider just the effect of rotation, ($p_a=p_b=0$) by the following nonhomogeneous equation

$$\frac{(-1+m\nu+\beta\nu)u_r}{r^2}+\frac{(1+m+\beta)u_r'}{r}+u_r''=-\frac{r^{1+q-\beta}(1-\nu^2)\rho^{*}\omega^2}{E^{*}} \quad (18)$$

The general solution of Eq. (18) is written in terms of unknown coefficients C_1 and C_2 as follows

$$u_r=r^{\frac{1}{2}(-m-\beta-\xi)}(C_1+C_2 r^{\xi})+r^{3+q-\beta}\Omega \quad (19)$$

Substitution Eq. (19) into Eq. (4) together with Eq. (5) renders

$$\sigma_r=-\frac{1}{2}C_{11}r^{\frac{1}{2}(-2-m-2\beta-\xi)}(C_2 r^{\frac{\beta}{2}+\xi}(m+\beta-2\nu-\xi)+C_1 r^{\beta/2}(m+\beta-2\nu+\xi)$$

$$-2r^{\frac{1}{2}(6+m+2q+\xi)}(3+q-\beta+\nu)\Omega) \quad (20a)$$

$$\sigma_\theta=-\frac{1}{2}C_{11}r^{\frac{1}{2}(-2-m-2\beta-\xi)}(C_2 r^{\frac{\beta}{2}+\xi}(-2+\nu(m+\beta-\xi))$$

$$+C_1 r^{\beta/2}(-2+\nu(m+\beta+\xi))-2r^{\frac{1}{2}(6+m+2q+\xi)}(1+(3+q-\beta)\nu)\Omega) \quad (20b)$$

where

$$\Omega=\frac{(-1+\nu^2)\rho^{*}\omega^2}{E^{*}(8+q(6+q-\beta)-3\beta+\beta\nu+m(3+q-\beta+\nu))} \quad (21)$$

The unknowns in Eqs. (19) and (20) should be determined from the boundary conditions. For the boundary condition BC-2 (Fig. 2), $\sigma_r(a)=0$ and $\sigma_r(b)=0$, they are attained as follows

$$C_1 = \frac{2a^{\frac{\xi-\beta}{2}} b^{\frac{\xi-\beta}{2}} (a^{\frac{m}{2}+q+3} b^{\frac{\beta+\xi}{2}} - a^{\frac{\beta+\xi}{2}} b^{\frac{m}{2}+q+3})(-q+\beta-\nu-3)\Omega}{(a^{\xi}-b^{\xi})(m+\beta-2\nu+\xi)}$$

$$C_2 = \frac{2a^{-\beta/2} b^{-\beta/2} (a^{\frac{m+2q+\xi+6}{2}} b^{\beta/2} - a^{\beta/2} b^{\frac{m+2q+\xi+6}{2}})(q-\beta+\nu+3)\Omega}{(a^{\xi}-b^{\xi})(m+\beta-2\nu-\xi)} \tag{22}$$

For BC-3 (Fig. 2), $u_r(a)=0$ and $\sigma_r(b)=0$, C_1 and C_2 are determined as follows

$$C_1 = \frac{a^{-\beta/2} b^{-\beta/2} (\Omega b^{\frac{\beta}{2}+\xi} a^{\frac{m+2q+\xi+6}{2}} (\beta-2\nu+m-\xi) - 2\Omega a^{\frac{\beta}{2}+\xi} (\beta-\nu-q-3) b^{\frac{m+2q+\xi+6}{2}})}{a^{\xi}(\beta-2\nu+m+\xi) + b^{\xi}(-\beta+2\nu-m+\xi)}$$

$$C_2 = -\frac{\Omega a^{-\beta/2} b^{-\beta/2} (b^{\beta/2} a^{\frac{m+2q+\xi+6}{2}} (\beta-2\nu+m+\xi) - 2a^{\beta/2} (\beta-\nu-q-3) b^{\frac{m+2q+\xi+6}{2}})}{a^{\xi}(\beta-2\nu+m+\xi) + b^{\xi}(-\beta+2\nu-m+\xi)} \tag{23}$$

And finally, for BC-4 (Fig. 2), $u_r(a)=0$ and $u_r(b)=0$, those constants are got as follows

$$C_1 = -\frac{a^{\xi} b^{\frac{1}{2}(m+2q-\beta+\xi+6)}\Omega - a^{\frac{1}{2}(m+2q-\beta+\xi+6)} b^{\xi}\Omega}{a^{\xi}-b^{\xi}} \tag{24}$$

$$C_2 = \frac{(b^{\frac{1}{2}(m+2q-\beta+\xi+6)} - a^{\frac{1}{2}(m+2q-\beta+\xi+6)})\Omega}{a^{\xi}-b^{\xi}}$$

5. Verification of the present formulas

The present formulas are applied to uniform discs made of an isotropic and homogeneous material and results are presented in Table 1 in a concise manner. From the table, it is shown that, for BC=1 and BC=2, present formulas coincides with the formulas given for uniform discs made of an isotropic and homogeneous material [18-19].

Now, we may consider isotropic and inhomogeneous materials to test the present formulas. Let's consider Eq. (16b). At the final stage of substitution any term related to Young's modulus and Poisson's ratio is not observed. This means that there is no term to be transformed from the plane stress to the plane strain. Hence it may also be used directly for cylindrical vessels. In Eq. (16b), the first term represents the inner pressure effect on the radial stress and the second term represents the external pressure effect. For uniform disks, with m=0, Eq. (16b) becomes

$$\sigma_{r(m=0)} = \frac{a^{\frac{1}{2}(-\beta+\xi+2)} p_a r^{\frac{1}{2}(\beta-\xi-2)} (b^{\xi}-r^{\xi})}{a^{\xi}-b^{\xi}} + \frac{b^{\frac{1}{2}(-\beta+\xi+2)} p_b r^{\frac{1}{2}(\beta-\xi-2)} (a^{\xi}-r^{\xi})}{b^{\xi}-a^{\xi}} \tag{25}$$

where

$$\xi = \sqrt{4+\beta^2-4\beta\nu} \tag{26}$$

Now, Eq. (25) is available for both uniform discs and cylinders subjected to the boundary condition BC=1 (Fig. 2). Horgan and Chan [1] proposed formulas for linear elastic response of uniform cylinders or discs made of a power-graded material and subjected to BC=1. Horgan and Chan’s [1] equation for radial stress is rewritten here by using the present notation

$$\sigma_{r-HORGAN} = -\frac{a^{\frac{-\beta}{2}} b^{\frac{-\beta}{2}} r^{\frac{1}{2}(-2-\xi+\beta)}}{b^{\xi}-a^{\xi}}\left(-a^{\xi+\frac{\beta}{2}} b^{1+\frac{\xi}{2}} p_b + a^{\frac{\beta}{2}} b^{1+\frac{\xi}{2}} p_b r^{\xi} + b^{\frac{\beta}{2}} a^{1+\frac{\xi}{2}} p_a\left(b^{\xi}-r^{\xi}\right)\right) \quad (27)$$

or in the form of

$$\sigma_{r-HORGAN} = \frac{a^{1+\frac{\xi}{2}-\frac{\beta}{2}} r^{\frac{1}{2}(-2-\xi+\beta)}}{a^{\xi}-b^{\xi}} p_a\left(b^{\xi}-r^{\xi}\right) + \frac{p_b b^{1+\frac{\xi}{2}-\frac{\beta}{2}} r^{\frac{1}{2}(-2-\xi+\beta)}}{b^{\xi}-a^{\xi}}\left(-r^{\xi}+a^{\xi}\right) \quad (28)$$

Table 1: Formulas for uniform discs made of an isotropic and homogeneous material
$\left(C_{11} = \frac{E}{1-\nu^2}, \beta = q = m = 0, \xi = 2, \Omega = \frac{(-1+\nu^2)\rho\omega^2}{8E}\right)$

BC=1 (*)	$u_r = -\frac{a^2 p_a(b^2(\nu+1)-(\nu-1)r^2)}{Er(a^2-b^2)} + \frac{b^2 p_b(a^2(\nu+1)-(\nu-1)r^2)}{Er(a^2-b^2)}$ $\sigma_r = \frac{a^2 p_a(b^2-r^2)}{r^2(a^2-b^2)} + \frac{b^2 p_b(a-r)(a+r)}{r^2(b^2-a^2)}$ $\sigma_\theta = -\frac{a^2 p_a(b^2+r^2)}{r^2(a^2-b^2)} + \frac{b^2 p_b(a^2+r^2)}{r^2(a^2-b^2)}$
BC=2 (**)	$u_r = \frac{\rho\omega^2(a^2(\nu+3)(b^2(\nu+1)-(\nu-1)r^2)-(\nu-1)r^2(b^2(\nu+3)-(\nu+1)r^2))}{8Er}$ $\sigma_r = \frac{\rho\omega^2(\nu+3)(a^2-r^2)(r^2-b^2)}{8r^2}$ $\sigma_\theta = \frac{\rho\omega^2(a^2(\nu+3)(b^2+r^2)+r^2(b^2(\nu+3)-(3\nu+1)r^2))}{8r^2}$
BC=3	$u_r = \frac{\Omega(a-r)(a+r)(a^2(b^2(\nu+1)-(\nu-1)r^2)+b^2((\nu+1)r^2-b^2(\nu+3)))}{a^2(\nu-1)r-b^2(\nu+1)r}$ $\sigma_r = \frac{E\Omega(r^2-b^2)(a^4(\nu^2-1)-a^2(\nu-1)(\nu+3)(b^2+r^2)+b^2(\nu+1)(\nu+3)r^2)}{(\nu^2-1)r^2(a^2(\nu-1)-b^2(\nu+1))}$ $\sigma_\theta = \frac{E\Omega\left(a^4(\nu^2-1)(b^2+r^2)-a^2(\nu-1)(b^4(\nu+3)+(3\nu+1)r^4)-b^2(\nu+1)r^2(b^2(\nu+3)-(3\nu+1)r^2)\right)}{(\nu^2-1)r^2\left(a^2(\nu-1)-b^2(\nu+1)\right)}$
BC=4	$u_r = \frac{\Omega(r^2-a^2)(r^2-b^2)}{r}$ $\sigma_r = \frac{E\Omega(a^2((\nu+1)r^2-b^2(\nu-1))+r^2(b^2(\nu+1)-(\nu+3)r^2))}{(\nu^2-1)r^2}$ $\sigma_\theta = \frac{E\Omega(a^2(b^2(\nu-1)+(\nu+1)r^2)+r^2(b^2(\nu+1)-(3\nu+1)r^2))}{(\nu^2-1)r^2}$

(*) Roark's formulas [18]
(**) [19].

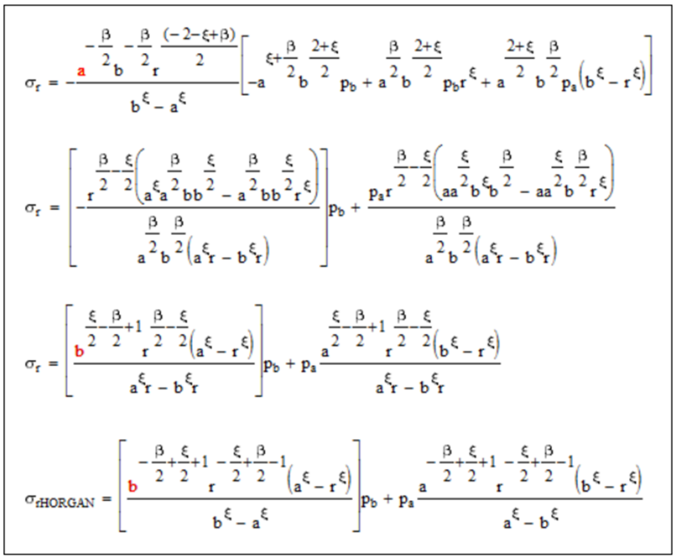


Figure 3: Verification of Eq. (16b) with Horgan and Chan's [1] formula.

Formulas (25) and (27-28) exactly equal to each other. Verification process is shown in Fig. 3. Çallıoğlu et al. [10] studied the elastic response of power-graded uniform rotating disks subjected to BC=2 (Fig. 2). They assumed that both the Young's modulus and density change with the same inhomogeneity index as follows ($E = E_b$, $\rho = \rho_b$)

$$E(r) = E\left(\frac{r}{b}\right)^{\beta} \; ; \; \rho(r) = \rho\left(\frac{r}{b}\right)^{\beta} \tag{29}$$

From Eqs. (19-21) for the radial stress with m=0 , $q = \beta$, and with the followings

$$\xi = \sqrt{4+\beta^2-4\beta\nu} \quad ; \quad \Omega = \frac{(-1+\nu^2)\rho_b\omega^2}{E_b(8+3\beta+\beta\nu)} \quad ; \quad C_{11} = \frac{E_b r^{\beta}}{b^{\beta}(1-\nu^2)} \tag{30}$$

we may get the following from Eq. (20a)

$$\sigma_{r-(m=0,q=\beta)} = \frac{1}{2}C_{11}r^{\frac{1}{2}(-2-\beta-\xi)}\left(-C_1(\beta-2\nu+\xi)+C_2r^{\xi}(-\beta+2\nu+\xi)+2r^{\frac{1}{2}(6+\beta+\xi)}(3+\nu)\Omega\right) \tag{31}$$

where (from Eq. (22))

$$C_1 = \frac{2(\nu+3)\Omega a^{\xi} b^{\frac{1}{2}(\beta+\xi+6)} - 2(\nu+3)\Omega b^{\xi} a^{\frac{1}{2}(\beta+\xi+6)}}{(a^{\xi}-b^{\xi})(\beta-2\nu+\xi)}$$

$$C_2 = \frac{2(\nu+3)\Omega\left(a^{\frac{1}{2}(\beta+\xi+6)}-b^{\frac{1}{2}(\beta+\xi+6)}\right)}{(a^{\xi}-b^{\xi})(\beta-2\nu-\xi)} \tag{32}$$

$$C = -\rho_b \omega^2 \frac{(3+\nu)}{b^{\beta}\left[\frac{(-1+\nu^2)\rho_b\omega^2}{E_b\Omega}\right]}$$

$$C_1 = \frac{C\left(a^{\frac{\beta+\xi+6}{2}} - b^{\frac{\beta+\xi+6}{2}}\right)}{b^{\xi} - a^{\xi}}$$

$$C_2 = \frac{C\left(a^{\xi} b^{\frac{\beta+\xi+6}{2}} - b^{\xi} a^{\frac{\beta+\xi+6}{2}}\right)}{b^{\xi} - a^{\xi}}$$

$$\sigma_r = C_1 r^{\frac{\beta+\xi-2}{2}} + C_2 r^{\frac{\beta-\xi-2}{2}} + C r^{\beta+2}$$

$$\sigma_r = -\frac{E_b \Omega r^{\frac{\beta}{2}-\frac{\xi}{2}-1}(\nu+3)\left(a^{\frac{\xi}{2}+\frac{\beta}{2}+3} b^{\xi} - a^{\xi} b^{\frac{\xi}{2}+\frac{\beta}{2}+3} - a^{\frac{\xi}{2}+\frac{\beta}{2}+3} r^{\xi} + a^{\xi} r^{\frac{\xi}{2}+\frac{\beta}{2}+3} + b^{\frac{\xi}{2}+\frac{\beta}{2}+3} r^{\xi} - b^{\xi} r^{\frac{\xi}{2}+\frac{\beta}{2}+3}\right)}{b^{\beta}(a^{\xi} - b^{\xi})(\nu^2 - 1)}$$

$$\sigma_{r.ÇALLIOGLU} = -\frac{E_b r^{\frac{\beta}{2}-\frac{\xi}{2}-1}\Omega(\nu+3)\left[(b^{\xi} - r^{\xi}) a^{\frac{\xi}{2}+\frac{\beta}{2}+3} + \left(r^{\frac{\xi}{2}+\frac{\beta}{2}+3} - b^{\frac{1}{2}\xi+\frac{1}{2}\beta+3}\right) a^{\xi} + b^{\frac{\xi}{2}+\frac{\beta}{2}+3} r^{\xi} - b^{\xi} r^{\frac{\xi}{2}+\frac{\beta}{2}+3}\right]}{b^{\beta}(a^{\xi} - b^{\xi})(\nu^2 - 1)}$$

Figure 4: Verification of Eq. (20a) with Eqs. (21-22) with Çallıoğlu et. al.'s [10] formula.

After substituting Eq. (32) into Eq. (31) the radial stress takes the following form

$$\sigma_{r-(m=0,q=\beta)} = \frac{E_b(\nu+3)\Omega r^{\frac{1}{2}(\beta-\xi-2)}\left(a^{\frac{1}{2}(\beta+\xi+6)}(r^{\xi}-b^{\xi}) + a^{\xi}\left(b^{\frac{1}{2}(\beta+\xi+6)} - r^{\frac{1}{2}(\beta+\xi+6)}\right) - r^{\xi} b^{\frac{1}{2}(\beta+\xi+6)} + b^{\xi} r^{\frac{1}{2}(\beta+\xi+6)}\right)}{b^{\beta}(\nu^2-1)(a^{\xi}-b^{\xi})} \quad (33)$$

Eqs. (33) coincides exactly with Çallıoğlu et. al.'s [10] solution. Verification is shown in Fig. 4.

6. Numerical examples

The geometrical properties of the disc are: $a = 0.025\, m$, $b = 0.25\, m$, $h_b = 0.025m$ (h_b is the thickness at the outer radius of the disc). For the convergent disc profile $m = -1$; for the divergent disc profile $m = 1$; for the uniform-thickness disc profile $m = 0$ are employed. Material properties of metal and ceramic are presented in Table 2. Numerical results are presented in Figs 5-8.

Fig. 5 shows the elastic response of the disk having different thickness profile to the internal and external pressures. For disks subjected to the equal internal and external pressures, maximum hoop stress is observed at inner surface of divergent disc. Uniform disc profile offers minimum hoop stress at the inner surface. For equal internal and external pressures, again, divergent hyperbolic profile gives the maximum radial stress in the vicinity of the inner surface. Maximum radial displacement observed for divergent hyperbolic profiles and equal internal and external pressures at the outer surface of the disc.

Fig. 6 shows the variation of the radial displacement with angular velocity, boundary conditions, and disc profiles. BC=2 and BC=3 present the maximum radial displacement at the outer surface for divergent hyperbolic disc profile at $\omega = 150\, rad/s$. For all disc profiles and angular velocities, BC=4 offers maximum radial displacement in the vicinity of the mid surface.

Fig. 7 shows the variation of the radial stress with angular velocity, boundary conditions, and disc profiles. Maximum radial stress occurs at the vicinity of the inner surface for the divergent disc profile and BC=2. For BC=3 maximum radial stress is located at the vicinity of the inner surface of the convergent disc profile while at the inner surface for uniform and divergent disk profiles. Its magnitude becomes 4 times higher than uniform profile at $\omega = 150\,rad/s$. For the boundary condition BC=4 maximum radial stress is observed at the inner surface of the divergent hyperbolic disc while it is located at the outer surface for convergent disc profile.

Fig. 8 shows the variation of the hoop stress with angular velocity, boundary conditions, and disc profiles. For BC=2, maximum hoop stress occurs at the inner surface for all disc profiles while for BC=3 and BC=4 it is observed at the vicinity of the inner surface. Again, divergent disc profiles renders the maximum magnitude of the hoop stress.

In general when just rotation is considered, radial stresses becomes dominant than the hoop stresses. Increasing constant angular velocity also increases the effects of all the elastic responses.

Table 2: Material properties

		E (GPa)	*ρ (kg/m³)*	*ν*
METAL	Nickel (Ni)	199.5	8900	0.3
CERAMIC	Zirconium Oxide (ZrO_2)	116.4	3657	0.3

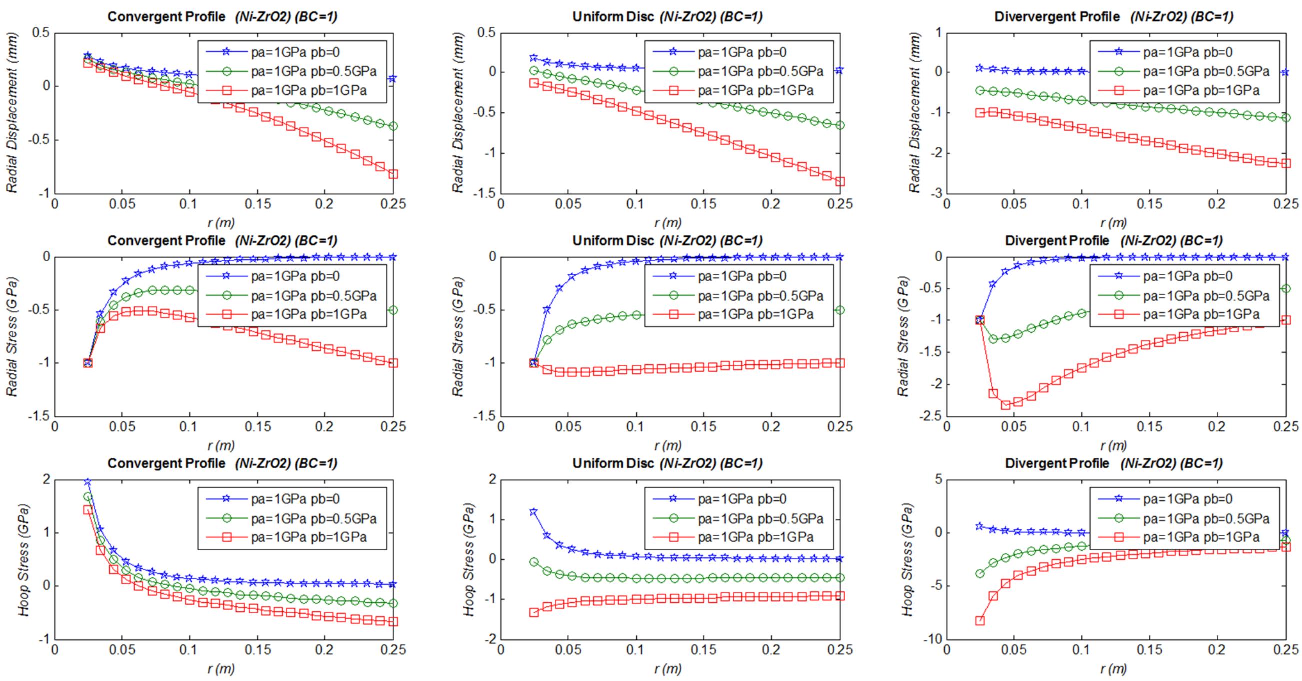


Figure 5: Elastic response of the disk having different disc profiles to the internal and external pressures

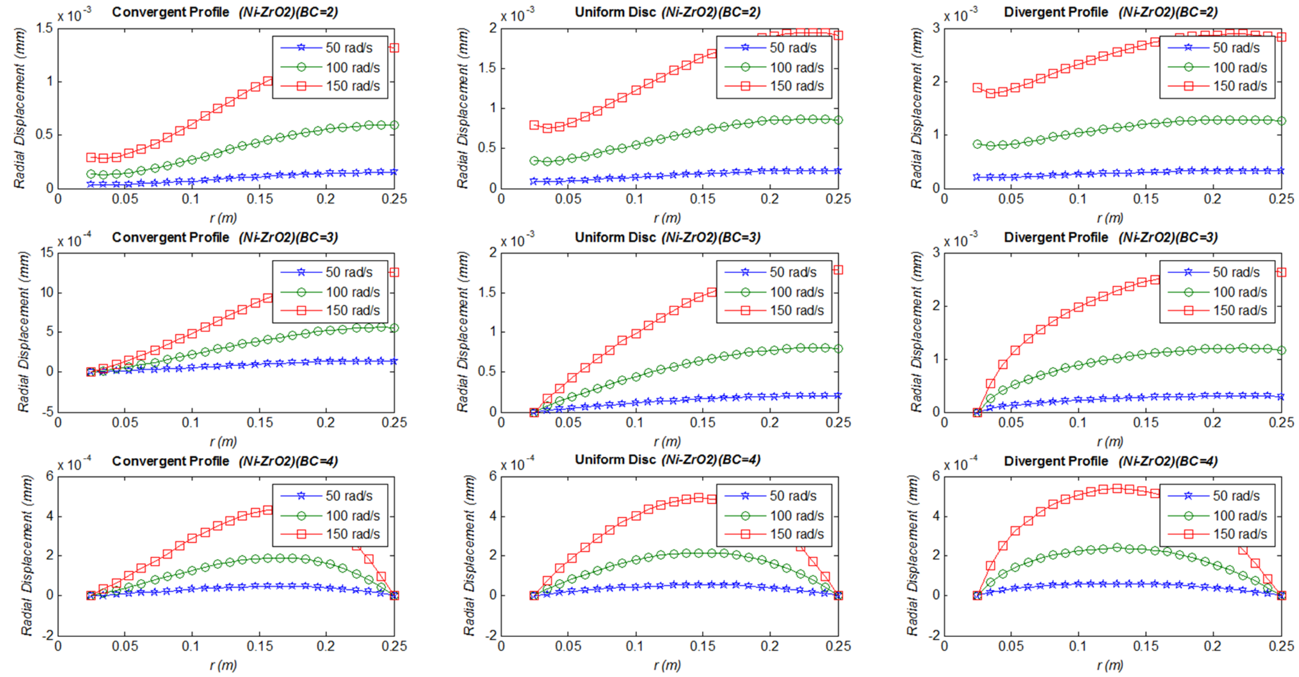


Figure 6: Variation of the radial displacement with angular velocity, boundary conditions, and disc profiles

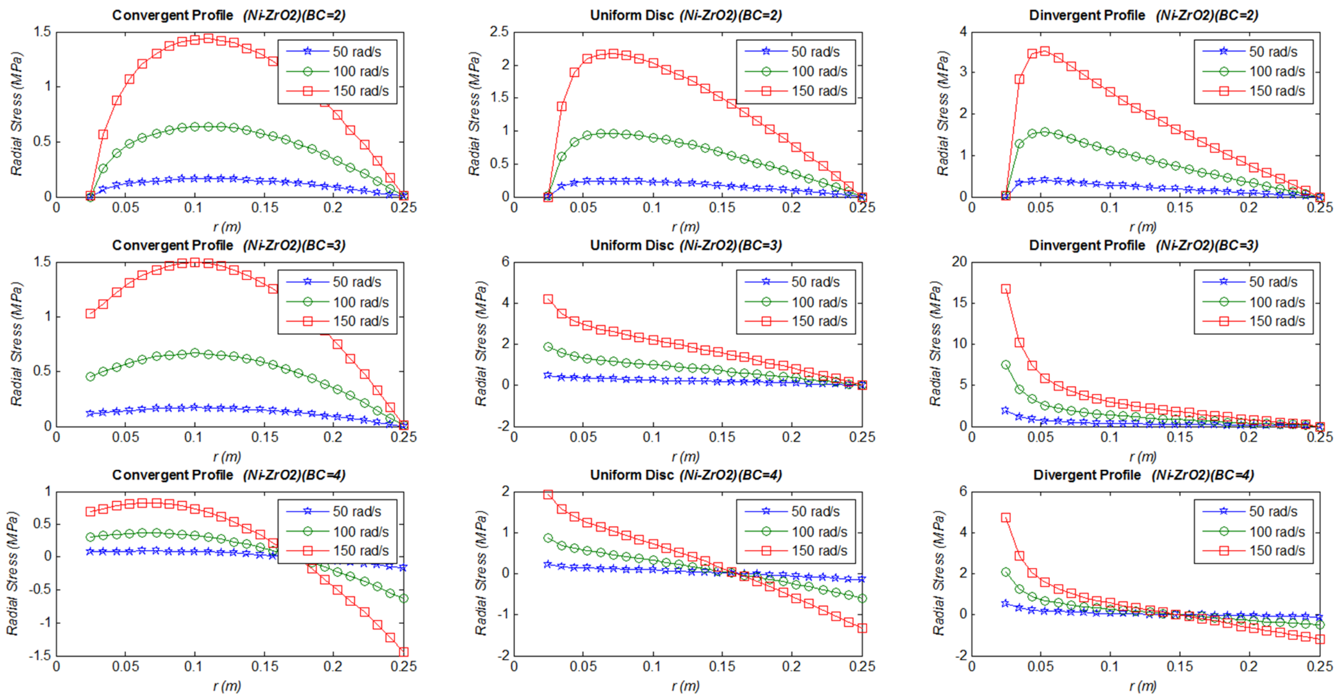


Figure 7: Variation of the radial stress with angular velocity, boundary conditions, and disc profiles

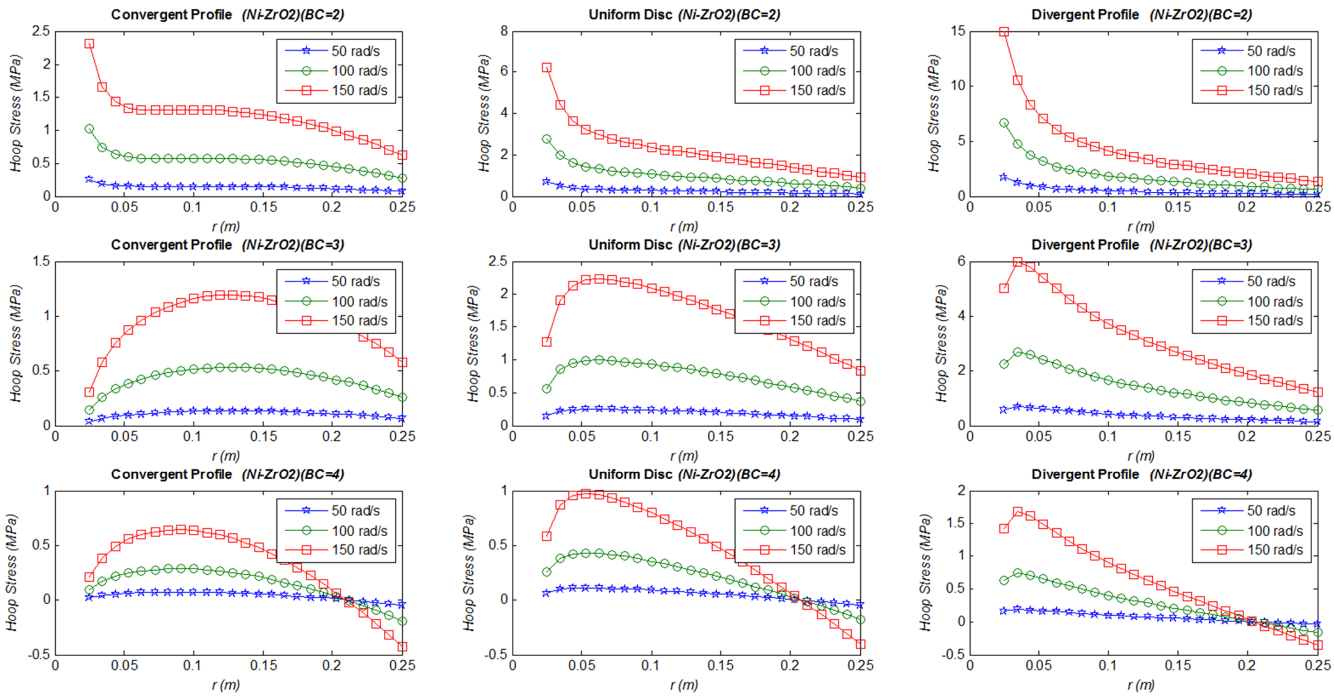


Figure 8: Variation of the hoop stress with angular velocity, boundary conditions, and disc profiles

7. Conclusions

In this work the exact elastic response of a continuously varying hyperbolic rotating disk made of a nonhomogeneous material is studied analytically by considering all effects affecting the elastic behavior of the disk such as internal and external pressures including rotation at a constant angular velocity under four physical boundary conditions. A parametric study is also performed to see the variation of the radial displacement, radial and hoop stress with the internal/external uniform pressures, angular velocity for convergent/divergent hyperbolic disk profiles and uniform disks.

As it is observed from the literature that existing formulas for the elastic responses of a rotating disc made of a functionally graded material comprise uniform thickness and boundary conditions such as either *BC=1* or *BC=2*. In the derivation of some exact formulas inhomogeneity constants are generally taken as equal to each other. For two physical materials to be arranged, this assumption does never hold. For other boundary conditions, namely for *BC=3* and *BC=4*, existing formulas for discs with varying sections and made of even an isotropic and homogeneous material are also scare. Due to those reasons, the present study offers comprehensive analytical formulas. The closed-form expressions offered in this study may be directly used safely in some of engineering applications including material tailoring and optimization problems of such discs. The present study may also be very helpful to the scholars in the quick verification process of their valuable results.

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