Ikonion Journal of Mathematics
https://dergipark.org.tr/tr/pub/ikjm
Research Article
Open Access
https://doi.org/10.54286/ikjm.1227629
ISSN: 2687-6531

A New Hybrid Block Method via Combined Hermite Polynomials and Exponential Functions as Basis Function

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Keywords:
Basis function, Collocation, Exponential function, Hermite polynomials, Hybrid block method, Interpolation, Ordinary Differential Equations.

Abstract — The development and implementation of a hybrid block method of order nine for first-order initial value problems (IVPs) of ordinary differential equations (ODEs) that are stiff or oscillatory in nature are presented in this paper. The hybrid block method was created using continuous collocation and interpolation techniques by combining Hermite polynomials and exponential functions as the basis function to produce a continuous implicit linear multistep method (LMM). The method's properties were studied and proven to be consistent, convergent, and zero-stable with an A-stable region of absolute stability, making it a suitable approach for stiff and oscillatory ODEs. The application of a combined basis in the generation of LMMs is an approach that should be widely adopted. The technique shows that continuous LMMs can be derived from a combination of any polynomials and exponential functions through an interpolation and collocation approach. On two sampled stiff and oscillatory problems, the new integrator was tested. The numerical findings demonstrate that our hybrid block integrator is computationally efficient and outperforms previous methods of similar derivations in stability and accuracy of results.

Subject Classification (2020): 65L04, 65L05, 65L6, 65L20.

1. Introduction

We investigate a numerical solution to first-order initial value problems (IVPs) of the ordinary differential equations (ODEs) that may exhibit stiffness or oscillatory behaviour given by

$$y' = f\big(t, y(t)\big), \qquad y(t_0) = y_0, \quad \forall \ \mathrm{a} \le \mathrm{t} \le \mathrm{b}, \tag{1.0}$$

where t_0 is the initial point, y_0 is the solution at t_0, and f is assumed to be continuous and satisfy the Lipchitz theorem for the existence and uniqueness of the solution.

The problem (1.0) frequently arises in studying dynamic systems and electrical networks [4]. According to [10], equation (1.0) is used to simulate population growth, particle trajectory, simple harmonic motion, beam deflection, and other phenomena. Notably, mixture models, the basic Susceptible, Infection, and Recovery (SIR) models, and other related models may all be formulated as problems of the form (1.0).

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Article History: Received: 09.01.2023 — Accepted: 11.07.2023 — Published: 12.09.2023

The solutions of nonlinear, stiff, and oscillatory problems of ODEs such as (1.0) are often highly unstable [11].

Definition 1.1 [8]. A differential equation is considered to be stiff if $\mathrm{Re}(\xi_j) < 0,\ j = 1,2,\dots,m,$ here ξ is the Eigenvalue of the given differential equation.

Definition 1.2 [13]. A differential equation with at least one oscillating solution is said to be oscillatory. If a nontrivial solution (function) of an ODE converges not to a finite limit (or diverges), it is said to be oscillating. (i.e. if the function has an infinity of results).

To deal with this class of problems, researchers have historically focused on developing efficient, stable, and high-order linear multistep methods (LMMs). Because LMMs do not start on their own, they require initial values from one-step methods like Euler's method and the Runge-Kutta family of methods. Ref. [11] gives the k-step generalized LMM as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \qquad \alpha_0 + \beta_0 \neq 0, \qquad \alpha_k = 1, \tag{1.1}$$

where α_j *and* β_j are uniquely determined, $h =$ step length, such that $t_{k+n} - t_k = nh$.

According to [11], existing LMMs for solving ODEs may be derived using approaches such as Taylor's series, numerical integration, determining the order of the LMM, and the interpolation approach, all of which are major discrete schemes constrained by assuming the order of convergence.

Ref. [1] and [11] reported that several researchers have shifted to employing the continuous collocation and interpolation process, resulting in the emergence of continuous LMMs of the form

$$y(t) = \sum_{j=0}^{k} \alpha_j(t) y_{n+j} + h \sum_{j=0}^{k} \beta_j(t) f_{n+j}, \tag{1.2}$$

where $\alpha_j(t)$ and $\beta_j(t)$ are continuous functions of t that should be differentiable at least once.

The continuous collocation and interpolation approach is a milestone in numerical analysis and computation for it is widely used. In this study consequently, we will derive continuous LMM and implement it in block form to eliminate its non-self-starting drawback.

Scholars have used continuous collocation technique to derive LMMs using a variety of single basis functions, including power series, Lagrange polynomials, Chebychev polynomials, Legendre polynomials, Hermite polynomials, and exponential functions among others.

It is established that the efficiency of these methods depends mainly on the basis functions chosen and the problem to be solved [2], [9], and [11]. Consequently, in search of a method with better efficiency and stability properties, [13] introduced a combined basis function for the derivation of LMM for the problem (1.0) of the form

$$y(t) = \sum_{j=0}^{r+n-1} a_j t^j + a_{r+n} \sum_{j=0}^{r+n} \frac{\alpha^j t^j}{j!}, \tag{1.3}$$

this combines power series and exponential functions. We improve upon this in terms of the methodology of the derivation, and the order and stability of the method.

In this paper, therefore, we proposed a different combined basis function, which is Hermite polynomials and exponential functions for the derivation of LMM to generate a higher order and efficiently stable hybrid block method for the solution of problem (1.0).

2. Methodology

The collocation procedure for continuous LMM in equation (1.2) intended for ODEs such as equation (1.0) is in general based on a basic idea: identify a function of a defined form that exactly satisfies the differential equation at a given set of points. The approximation function must also meet some additional conditions placed by the nature of the problem under consideration.

In this study, we concatenate Probabilist's Hermite polynomials and exponential functions to be an approximate solution to the problem (1.0) in the form

$$y(t) = \sum_{r=0}^{k} a_r H_r(t) + \sum_{r=k+1}^{m} \sum_{j=0}^{r} a_r \frac{\beta^j t^j}{j!}, \quad m = i + c\,. \tag{1.4}$$

Equation (1.4) is called the *basis function* and is continuously differentiable. where c denotes the number of collocation points, i is the number of interpolation points and $\beta \in \mathbb{R}$.

The coefficients $a_r \in \mathbb{R}$, $r = 0, 1, \dots, m$ of the series (1.4), are determined over the interval of integration $[a, b]$, for $a = t_0 < t_1 < \cdots < t_N = b$, with a constant step size h given by $h = t_{n+1} - t_n$; $n = 0, 1, \dots, N-1$. $H_r(t)$ are the Probabilist's Hermite polynomials generated by the formula

$$H_n(t) = (-1)^n e^{\left(\frac{t^2}{2}\right)} \frac{d^n}{dt^n} e^{\left(-\frac{t^2}{2}\right)} = \left(1 - \frac{d}{dt}\right)^n \cdot 1, \tag{1.5}$$

and whose recursive relation is

$$H_{n+1}(t) = tH_n(t) - H_n'(t)\,. \tag{1.6}$$

The first ten probabilist's Hermite polynomials are:

$$H_0 = 1, \qquad H_1 = t, \qquad H_2 = t^2 - 1, \qquad H_3 = t^3 - 3t, \qquad H_4 = t^4 - 6t^2 + 3$$

$$H_5 = t^5 - 10t^3 + 15t, \quad H_6 = t^6 - 15t^4 + 45t^2 - 15, \quad H_7 = t^7 - 21t^5 + 105t^3 - 105t$$

$$H_8 = t^8 - 28t^6 + 210t^4 - 420t^2 - 105, \qquad H_9 = t^9 - 36t^7 + 378t^5 - 1260t^3 - 945t$$

Now, obtaining the first derivative of (1.4) we have

$$y'(t) = \sum_{r=1}^{k} a_r H_r'(t) + \sum_{r=k+1}^{m} \sum_{j=1}^{r} a_r \frac{\beta^j t^{j-1}}{(j-1)!}, \quad m = i + c. \tag{1.7}$$

Interpolating equation (1.4) at $t = t_n$ and collocating equation (1.7) at $t = t_{n+c}$, $c \in \mathbb{R}$; a system of nonlinear equations is produced, which is compactly expressed in the form

$$\left.\begin{aligned} y_n &= \sum_{r=0}^{k} a_r H_r(t_n) + \sum_{r=k+1}^{m}\sum_{j=0}^{r} a_r \frac{\beta^j t_n^j}{j!}, \\ f_{n+c} &= \sum_{r=1}^{k} a_r H_r'(t_{n+c}) + \sum_{r=k+1}^{m}\sum_{j=1}^{r} a_r \frac{\beta^j t_{n+c}^{j-1}}{(j-1)!}\ . \end{aligned}\right\} \quad (1.8)$$

The unknown constants a_r in equation (1.8) are determined using standard methods like Gaussian elimination or matrices inversion method and substituted into equation (1.4). Thus, applying the transformation $x = \frac{t-t_n}{h}$ and algebraic manipulation on equation (1.4), a hybrid continuous LMM of the form in equation (1.2) is obtained for different values of m, and it is implemented in block form.

3. Derivation of Hybrid block Method

The approximate solution to the problem (1.0) is the equation (1.4) where $m = 9$, i.e.

$$y(t) = a_0 + a_1 t + a_2(t^2-1) + a_3(t^3-3t) + a_4(t^4-6t^2+3) + a_5(t^5-10t^3+15t) + a_6\sum_{j=0}^{6}\frac{\beta^j t^j}{j!}$$

$$+a_7\sum_{j=0}^{7}\frac{\beta^j t^j}{j!} + a_8\sum_{j=0}^{8}\frac{\beta^j t^j}{j!} + a_9\sum_{j=0}^{9}\frac{\beta^j t^j}{j!}\ . \quad (1.9)$$

Taking the first derivative of equation (1.9) and substituting in equation (1.0) gives

$$f(t,y) = a_1 + 2a_2 t + 3a_3(t^2-1) + 4a_4(t^3-3t) + 5a_5(t^4-6t^2+3) + a_6\sum_{j=1}^{6}\frac{\beta^j t^{j-1}}{(j-1)!}$$

$$+a_7\sum_{j=1}^{7}\frac{\beta^j t^{j-1}}{(j-1)!} + a_8\sum_{j=1}^{8}\frac{\beta^j t^{j-1}}{(j-1)!} + a_9\sum_{j=1}^{9}\frac{\beta^j t^{j-1}}{(j-1)!}\ . \quad (1.10)$$

Now, interpolating equation (1.9) at point $t_{n+i},\ i = 0$ and collocating equation (1.10) at point $t_{n+c},\ c = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}$ and 1, the following nonlinear system of equations is obtained

$$B \cdot A = U, \quad (1.11)$$

where

$$A = (a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)^T,$$

$$U = \left(y_n,\ f_n,\ f_{n+\frac{1}{8}},\ f_{n+\frac{1}{4}},\ f_{n+\frac{3}{8}},\ f_{n+\frac{1}{2}}, f_{n+\frac{5}{8}},\ f_{n+\frac{3}{4}}, f_{n+\frac{7}{8}},\ f_{n+1}\right)^T \text{ and}$$

$$
B=\begin{bmatrix}
1 & t_n & (t_n^2-1) & (t_n^3-3t_n) & (t_n^4-6t_n^2+3) & (t_n^5-10t_n^3+15t_n) & \sum_{j=0}^{6}\frac{\beta^j t_n^j}{j!} & \sum_{j=0}^{7}\frac{\beta^j t_n^j}{j!} & \sum_{j=0}^{8}\frac{\beta^j t_n^j}{j!} & \sum_{j=0}^{9}\frac{\beta^j t_n^j}{j!}\\
0 & 1 & 2t_n & 3(t_n^2-1) & 4(t_n^3-3t_n) & 5(t_n^4-6t_n^2+3) & \sum_{j=1}^{6}\frac{\beta^j t_n^{j-1}}{(j-1)!} & \sum_{j=1}^{7}\frac{\beta^j t_n^{j-1}}{(j-1)!} & \sum_{j=1}^{8}\frac{\beta^j t_n^{j-1}}{(j-1)!} & \sum_{j=1}^{9}\frac{\beta^j t_n^{j-1}}{(j-1)!}\\
0 & 1 & 2t_{n+\frac{1}{8}} & 3\left(t_{n+\frac{1}{8}}^2-1\right) & 4\left(t_{n+\frac{1}{8}}^3-3t_{n+\frac{1}{8}}\right) & 5(t_{n+\frac{1}{8}}^4-6t_{n+\frac{1}{8}}^2+3) & \sum_{j=1}^{6}\frac{\beta^j t_{n+1/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{7}\frac{\beta^j t_{n+1/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{8}\frac{\beta^j t_{n+1/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{9}\frac{\beta^j t_{n+1/8}^{j-1}}{(j-1)!}\\
0 & 1 & 2t_{n+\frac{1}{4}} & 3\left(t_{n+\frac{1}{4}}^2-1\right) & 4\left(t_{n+\frac{1}{4}}^3-3t_{n+\frac{1}{4}}\right) & 5(t_{n+\frac{1}{4}}^4-6t_{n+\frac{1}{4}}^2+3) & \sum_{j=1}^{6}\frac{\beta^j t_{n+1/4}^{j-1}}{(j-1)!} & \sum_{j=1}^{7}\frac{\beta^j t_{n+1/4}^{j-1}}{(j-1)!} & \sum_{j=1}^{8}\frac{\beta^j t_{n+1/4}^{j-1}}{(j-1)!} & \sum_{j=1}^{9}\frac{\beta^j t_{n+1/4}^{j-1}}{(j-1)!}\\
0 & 1 & 2t_{n+\frac{3}{8}} & 3\left(t_{n+\frac{3}{8}}^2-1\right) & 4\left(t_{n+\frac{3}{8}}^3-3t_{n+\frac{3}{8}}\right) & 5(t_{n+\frac{3}{8}}^4-6t_{n+\frac{3}{8}}^2+3) & \sum_{j=1}^{6}\frac{\beta^j t_{n+3/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{7}\frac{\beta^j t_{n+3/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{8}\frac{\beta^j t_{n+3/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{9}\frac{\beta^j t_{n+3/8}^{j-1}}{(j-1)!}\\
0 & 1 & 2t_{n+\frac{1}{2}} & 3\left(t_{n+\frac{1}{2}}^2-1\right) & 4\left(t_{n+\frac{1}{2}}^3-3t_{n+\frac{1}{2}}\right) & 5(t_{n+\frac{1}{2}}^4-6t_{n+\frac{1}{2}}^2+3) & \sum_{j=1}^{6}\frac{\beta^j t_{n+1/2}^{j-1}}{(j-1)!} & \sum_{j=1}^{7}\frac{\beta^j t_{n+1/2}^{j-1}}{(j-1)!} & \sum_{j=1}^{8}\frac{\beta^j t_{n+1/2}^{j-1}}{(j-1)!} & \sum_{j=1}^{9}\frac{\beta^j t_{n+1/2}^{j-1}}{(j-1)!}\\
0 & 1 & 2t_{n+\frac{5}{8}} & 3\left(t_{n+\frac{5}{8}}^2-1\right) & 4\left(t_{n+\frac{5}{8}}^3-3t_{n+\frac{5}{8}}\right) & 5(t_{n+\frac{5}{8}}^4-6t_{n+\frac{5}{8}}^2+3) & \sum_{j=1}^{6}\frac{\beta^j t_{n+5/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{7}\frac{\beta^j t_{n+5/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{8}\frac{\beta^j t_{n+5/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{9}\frac{\beta^j t_{n+5/8}^{j-1}}{(j-1)!}\\
0 & 1 & 2t_{n+\frac{3}{4}} & 3\left(t_{n+\frac{3}{4}}^2-1\right) & 4\left(t_{n+\frac{3}{4}}^3-3t_{n+\frac{3}{4}}\right) & 5(t_{n+\frac{3}{4}}^4-6t_{n+\frac{3}{4}}^2+3) & \sum_{j=1}^{6}\frac{\beta^j t_{n+3/4}^{j-1}}{(j-1)!} & \sum_{j=1}^{7}\frac{\beta^j t_{n+3/4}^{j-1}}{(j-1)!} & \sum_{j=1}^{8}\frac{\beta^j t_{n+3/4}^{j-1}}{(j-1)!} & \sum_{j=1}^{9}\frac{\beta^j t_{n+3/4}^{j-1}}{(j-1)!}\\
0 & 1 & 2t_{n+\frac{7}{8}} & 3\left(t_{n+\frac{7}{8}}^2-1\right) & 4\left(t_{n+\frac{7}{8}}^3-3t_{n+\frac{7}{8}}\right) & 5(t_{n+\frac{7}{8}}^4-6t_{n+\frac{7}{8}}^2+3) & \sum_{j=1}^{6}\frac{\beta^j t_{n+7/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{7}\frac{\beta^j t_{n+7/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{8}\frac{\beta^j t_{n+7/8}^{j-1}}{(j-1)!} & \sum_{j=1}^{9}\frac{\beta^j t_{n+7/8}^{j-1}}{(j-1)!}\\
0 & 1 & 2t_{n+1} & 3(t_{n+1}^2-1) & 4(t_{n+1}^3-3t_{n+1}) & 5(t_{n+1}^4-6t_{n+1}^2+3) & \sum_{j=1}^{6}\frac{\beta^j t_{n+1}^{j-1}}{(j-1)!} & \sum_{j=1}^{7}\frac{\beta^j t_{n+1}^{j-1}}{(j-1)!} & \sum_{j=1}^{8}\frac{\beta^j t_{n+1}^{j-1}}{(j-1)!} & \sum_{j=1}^{9}\frac{\beta^j t_{n+1}^{j-1}}{(j-1)!}
\end{bmatrix}. \quad (1.12)
$$

Solving equation (1.11) in maple soft, using the matrix inversion method, the value of the unknown column vector A is obtained. The value of the vector A is then substituted in equation (1.9) to give a continuous implicit scheme. Thus, applying the transformation $x = \frac{(t-t_n)}{h}$, and algebraic manipulation for all values of $\beta \in \mathbb{R}$ we have a continuous implicit hybrid LMM of the form in equation (1.2) given as

$$
y(x) = \alpha_0(x)y_n + h\left[\sum_{j=0}^{1}\beta_j(x)f_{n+j}\right], \quad (1.13)
$$

where $j = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}$ and 1, $f_{n+j} = f\big(t_n + jh, y(t_n + jh)\big)$, while $\alpha_0(x)$ *and* $\beta_j(x)$ represent continuous coefficients which are obtained as follow

$$
\left.\begin{aligned}
&\alpha_0 = 1\\
&\beta_0 = \frac{xh}{28350}(1310720x^8 - 6635520x^7 + 14376960x^6 - 17418240x^5 + 12930624x^4 - 6055560x^3 + 1771860x^2 - 308205x + 28350)\\
&\beta_{\frac{1}{8}} = -\frac{32x^2h}{14175}(163840x^7 - 806400x^6 + 1681920x^5 - 1932000x^4 + 1326528x^3 - 549675x^2 + 129870x - 14175)\\
&\beta_{\frac{1}{4}} = \frac{8x^2h}{14175}(2293760x^7 - 10967040x^6 + 22026240x^5 - 24057600x^4 + 15411312x^3 - 5781195x^2 + 1173690x - 99225)\\
&\beta_{\frac{3}{8}} = -\frac{32x^2h}{14175}(1146880x^7 - 5322240x^6 + 10298880x^5 - 10735200x^4 + 6483456x^3 - 2259495x^2 + 420630x - 33075)\\
&\beta_{\frac{1}{2}} = \frac{2x^2h}{2835}(4587520x^7 - 20643840x^6 + 38522880x^5 - 38492160x^4 + 22161888x^3 - 7343280x^2 + 1305990x - 99225)\\
&\beta_{\frac{5}{8}} = -\frac{32x^2h}{14175}(1146880x^7 - 4999680x^6 + 9008640x^5 - 8672160x^4 + 4810176x^3 - 1540665x^2 + 266490x - 19845)\\
&\beta_{\frac{3}{4}} = \frac{8x^2h}{14175}(2293760x^7 - 9676800x^6 + 16865280x^5 - 15724800x^4 + 8476272x^3 - 2650725x^2 + 450030x - 33075)\\
&\beta_{\frac{7}{8}} = -\frac{32x^2h}{14175}(163840x^7 - 668160x^6 + 1128960x^5 - 1024800x^4 + 540288x^3 - 166005x^2 + 27810x - 2025)\\
&\beta_1 = \frac{x^2h}{28350}(1310720x^7 - 5160960x^6 + 8478720x^5 - 7526400x^4 + 3898944x^3 - 1181880x^2 + 196020x - 14175)
\end{aligned}\right\} \quad (1.14)
$$

When equation (1.13) is evaluated at $t = \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}, 1$ and implemented in block form, it yields a discrete hybrid block method of the type

$$
A^{(0)}\boldsymbol{Y}_m = E\boldsymbol{y}_n + hD\boldsymbol{f}(\boldsymbol{y}_n) + hB\boldsymbol{F}(\boldsymbol{Y}_m), \quad (1.15)
$$

Where

$$\boldsymbol{Y}_m = \left[y_{n+\frac{1}{8}}, y_{n+\frac{1}{4}}, y_{n+\frac{3}{8}}, y_{n+\frac{1}{2}}, y_{n+\frac{5}{8}}, y_{n+\frac{3}{4}}, y_{n+\frac{7}{8}}, y_{n+1}\right]^T,$$

$$\boldsymbol{y}_n = \left[y_{n-\frac{7}{8}}, y_{n-\frac{3}{4}}, y_{n-\frac{5}{8}}, y_{n-\frac{1}{2}}, y_{n-\frac{3}{8}}, y_{n-\frac{1}{4}}, y_{n-\frac{1}{8}}, y_n\right]^T,$$

$$\boldsymbol{F}(\boldsymbol{Y}_m) = \left[f_{n+\frac{1}{8}}, f_{n+\frac{1}{4}}, f_{n+\frac{3}{8}}, f_{n+\frac{1}{2}}, f_{n+\frac{5}{8}}, f_{n+\frac{3}{4}}, f_{n+\frac{7}{8}}, f_{n+1}\right]^T,$$

$$\boldsymbol{f}(\boldsymbol{y}_n) = \left[f_{n-\frac{7}{8}}, f_{n-\frac{3}{4}}, f_{n-\frac{5}{8}}, f_{n-\frac{1}{2}}, f_{n-\frac{3}{8}}, f_{n-\frac{1}{4}}, f_{n-\frac{1}{8}}, f_n\right]^T,$$

$$\mathrm{A}^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathrm{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathrm{D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1070017}{29030400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32377}{907200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{12881}{358400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4063}{113400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{41705}{1161216} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{401}{11200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{149527}{4147200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{989}{28350} \end{bmatrix},$$

$$B=\begin{bmatrix}
\frac{2233547}{14515200} & -\frac{2302297}{14515200} & \frac{2797679}{14515200} & -\frac{31457}{181440} & \frac{1573169}{14515200} & -\frac{645607}{14515200} & \frac{156437}{14515200} & -\frac{33953}{29030400}\\
\frac{22823}{113400} & -\frac{21247}{453600} & \frac{15011}{113400} & -\frac{2903}{22680} & \frac{9341}{113400} & -\frac{15577}{453600} & \frac{953}{113400} & -\frac{119}{129600}\\
\frac{35451}{179200} & \frac{1719}{179200} & \frac{39967}{179200} & -\frac{351}{2240} & \frac{17217}{179200} & -\frac{7031}{179200} & \frac{243}{25600} & -\frac{369}{358400}\\
\frac{2822}{14175} & \frac{61}{28350} & \frac{4094}{14175} & -\frac{227}{2835} & \frac{1154}{14175} & -\frac{989}{28350} & \frac{122}{14175} & -\frac{107}{113400}\\
\frac{115075}{580608} & \frac{3775}{580608} & \frac{159175}{580608} & -\frac{125}{36288} & \frac{85465}{580608} & -\frac{24575}{580608} & \frac{5725}{580608} & -\frac{175}{165888}\\
\frac{279}{1400} & \frac{9}{5600} & \frac{403}{1400} & -\frac{9}{280} & \frac{333}{1400} & \frac{79}{5600} & \frac{9}{1400} & -\frac{9}{11200}\\
\frac{408317}{2073600} & \frac{24353}{2073600} & \frac{542969}{2073600} & \frac{343}{25920} & \frac{368039}{2073600} & \frac{261023}{2073600} & \frac{111587}{2073600} & -\frac{8183}{4147200}\\
\frac{2944}{14175} & -\frac{464}{14175} & \frac{5248}{14175} & -\frac{454}{2835} & \frac{5248}{14175} & -\frac{464}{14175} & \frac{2944}{14175} & \frac{989}{28350}
\end{bmatrix}.$$

4. Analysis of the Method

4.1 . Zero Stability of the Method

Definition 4.1: [3] if the roots $r_n,\ n=1,2,\dots,k$ of the characteristics polynomial $P(r)$ given by $P(r)=\left|(rA^{(0)}-E)\right|$ satisfies $|r_n|\le 1$ and every root satisfying $|r_n|\le 1$ has a multiplicity not greater than the order of the differential equation, then the Block Integrator (1.15) is said to be zero-stable, Furthermore, as $h\to 0, P(r)=r^{\alpha-\mu}(r-1)^{\mu}$ where μ is the order of the differential equation, α is the order of the matrices $A^{(0)}$and E (see also [7]).

Thus, for our block integrator, we have

$$P(r)=\left| r\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}\right|=0. \qquad (1.16)$$

$P(r)=(r-1)r^7=0, => r_1=r_2=\cdots=r_7=0, r_8=1.$

Hence, our block integrator is zero-stable.

4.2. Order and Error Constant

Using the approach described in [6] and [13]. In equation (1.15), we define the linear difference operator connected with the new hybrid block method as

$$L\{y(t),h\}=A^{(0)}\boldsymbol{Y}_m-E\boldsymbol{y}_n-h[D\boldsymbol{f}(\boldsymbol{y}_n)+B\boldsymbol{F}(\boldsymbol{Y}_m)] \qquad (1.17)$$

We assume $y(t)$ has higher derivatives, as such when the Taylor series is used to expand equation (1.17) and the coefficients of h are compared, the result is

$$L\{y(t),h\} = c_0 y(t) + c_1 h y'(t) + c_2 h^2 y''(t) + c_3 h^3 y'''(t) + \cdots + c_p h^p y^p(t) + \ c_{p+1} h^{(p+1)} y^{(p+1)}(t) + \cdots, \quad (1.18)$$

where

$$c_p = \frac{1}{p!}\left(\sum_{j=1}^{k} j^p \alpha_j - p\sum_{j=1}^{k} j^{p-1}\beta_j\right), p = 0, 1, 2, 3, \ldots, n. \quad (1.19)$$

Definition 4.2. According to [6], if $c_0 = c_1 = c_2 = c_3 = \cdots = c_p = 0$, $c_{p+1} \neq 0$, then the linear difference operator and the corresponding continuous LMM are considered to be of the order p. The c_{p+1} is termed the error constant and the local truncation error is defined by

$$T_{n+k} = c_{p+1} h^{(p+1)} y^{(p+1)}(t_n) + O\left(h^{(p+2)}\right) \quad (1.20)$$

Thus from equation (1.15), we have that

$$L\{y(t),h\} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{8}}\\ y_{n+\frac{1}{4}}\\ y_{n+\frac{3}{8}}\\ y_{n+\frac{1}{2}}\\ y_{n+\frac{5}{8}}\\ y_{n+\frac{3}{4}}\\ y_{n+\frac{7}{8}}\\ y_{n+1} \end{bmatrix} - \begin{bmatrix} 1\\1\\1\\1\\1\\1\\1\\1 \end{bmatrix} [y_n] - h \begin{bmatrix} \frac{2233547}{14515200} & -\frac{2302297}{14515200} & \frac{2797679}{14515200} & -\frac{31457}{181440} & \frac{1573169}{14515200} & -\frac{645607}{14515200} & \frac{156437}{14515200} & -\frac{33953}{29030400} \\ \frac{22823}{113400} & -\frac{21247}{453600} & \frac{15011}{113400} & -\frac{2903}{22680} & \frac{9341}{113400} & -\frac{15577}{453600} & \frac{953}{113400} & -\frac{119}{129600} \\ \frac{35451}{179200} & \frac{1719}{179200} & \frac{39967}{179200} & -\frac{351}{2240} & \frac{17217}{179200} & -\frac{7031}{179200} & \frac{243}{25600} & -\frac{369}{358400} \\ \frac{2822}{14175} & \frac{61}{28350} & \frac{4094}{14175} & -\frac{227}{2835} & \frac{1154}{14175} & -\frac{989}{28350} & \frac{122}{14175} & -\frac{107}{113400} \\ \frac{115075}{580608} & \frac{3775}{580608} & \frac{159175}{580608} & -\frac{125}{36288} & \frac{85465}{580608} & -\frac{24575}{580608} & \frac{5725}{580608} & -\frac{175}{165888} \\ \frac{279}{1400} & \frac{9}{5600} & \frac{403}{1400} & -\frac{9}{280} & \frac{333}{1400} & \frac{79}{5600} & \frac{9}{1400} & -\frac{9}{11200} \\ \frac{408317}{2073600} & \frac{24353}{2073600} & \frac{542969}{2073600} & \frac{343}{25920} & \frac{368039}{2073600} & \frac{261023}{2073600} & \frac{111587}{2073600} & -\frac{8183}{4147200} \\ \frac{2944}{14175} & -\frac{464}{14175} & \frac{5248}{14175} & -\frac{454}{2835} & \frac{5248}{14175} & -\frac{464}{14175} & \frac{2944}{14175} & \frac{989}{28350} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{8}}\\ f_{n+\frac{1}{4}}\\ f_{n+\frac{3}{8}}\\ f_{n+\frac{1}{2}}\\ f_{n+\frac{5}{8}}\\ f_{n+\frac{3}{4}}\\ f_{n+\frac{7}{8}}\\ f_{n+1} \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix} \quad (1.21)$$

Expanding equation (1.21) in the Taylor series and evaluating the coefficients using equation (1.19) we have

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_8 = \ c_9 = 0.$$

Therefore the hybrid block method has an order of nine (9) and an error constant as:

$$c_{10} = [\ 7.3505E - 12, \quad 5.9871E - 12, \quad 6.4964E - 12, \quad 6.1760E - 12, \quad 6.4964E - 12, \quad 5.9871E - 12, \quad 7.3505E - 12]^T$$

Region of Absolute Stability of the Method

Definition 4.2 [14]: A region of absolute stability is one in which $r = \lambda h$ in the complex z plane.

For all initial conditions, it is well-defined as the values for which the numerical solutions of $y' = -\lambda y$ satisfy $y_i \to 0$ as $i \to \infty$.

To establish the region of absolute stability of our block integrator, the boundary locus approach is used. This is accomplished by substituting the test equation

$$y' = -\lambda y,$$

into the block formula in equation (1.15). This gives

$$\boldsymbol{A}^{(0)}\boldsymbol{Y}_m(r) = \boldsymbol{E}\boldsymbol{y}_n(r) - h\lambda\boldsymbol{D}\boldsymbol{y}_n(r) - h\lambda\boldsymbol{B}\boldsymbol{Y}_m(r). \tag{1.22}$$

Given that, $\bar{h} = \lambda h$ and $r = e^{i\theta}$, thus we have

$$\bar{h}(r) = -\left(\frac{\boldsymbol{A}^{(0)}\boldsymbol{Y}_m(r) - \boldsymbol{E}\boldsymbol{y}_n(r)}{\boldsymbol{D}\boldsymbol{y}_n(r) + \boldsymbol{B}\boldsymbol{Y}_m(r)}\right), \tag{1.23}$$

which is the characteristics/stability polynomial. Using equation (1.23), we obtain the stability polynomial for our block method as:

$$\begin{aligned}\bar{h}(r) = {} & \left(\frac{1}{150994944}r^8 - \frac{1}{150994944}r^7\right)h^8 + \left(-\frac{761}{2642411520}r^8 - \frac{761}{2642411520}r^7\right)h^7 \\ & + \left(\frac{29531}{3963617280}r^8 - \frac{29531}{3963617280}r^7\right)h^6 + \left(-\frac{89}{655360}r^8 - \frac{89}{655360}r^7\right)h^5 \\ & + \left(\frac{1069}{589824}r^8 - \frac{1069}{589824}r^7\right)h^4 + \left(-\frac{9}{512}r^8 - \frac{9}{512}r^7\right)h^3 + \left(\frac{91}{768}r^8 - \frac{91}{768}r^7\right)h^2 \\ & + \left(-\frac{1}{2}r^8 - \frac{1}{2}r^7\right)h + r^8 - r^7.\end{aligned}$$

This gives us the absolute stability region shown in Figure 4.1 below.

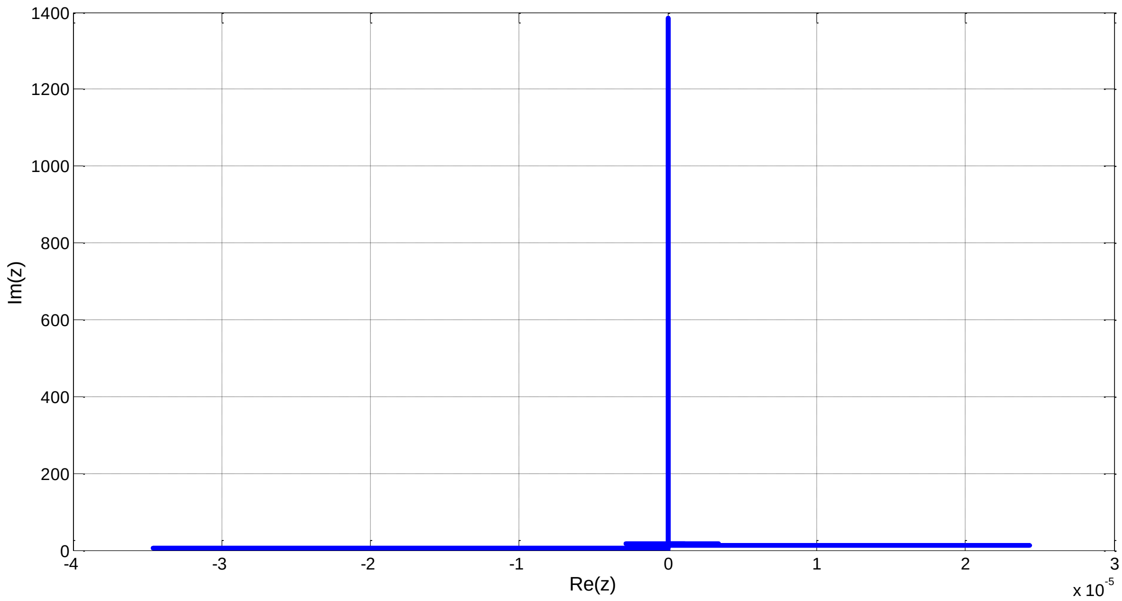

Figure 4.1: Showing the Absolute Stability Region of the Block Method

According to Figure 4.1, the new hybrid block method is effective in handling stiff problems since its RAS (Region of the Absolute Stability) is unbounded [6]. A numerical scheme is considered A-stable if its region of absolute stability R covers the entire complex plane $\mathbb{C}$, which is defined as, $i.e.$ $R = \{Z \in \mathbb{C}/\ Re(Z) < 0\}$ [7]. This confirms that the hybrid block method is an A-stable method.

4.2 Consistency of the Method

If a block method has an order greater than one, it is considered to be consistent [7]. The foregoing analysis shows that our block integrator is consistent.

4.3 Convergence of the Method

An LMM is considered convergent if and only if it satisfies both the requirements of consistency and zero stability [5]. Hence our block integrator is convergent.

5. Numerical Implementations

We compare the results of our method to those obtained by similar methods on some of the most difficult stiff and oscillatory problems in the literature.

The following notations are used in the results tables.

ERROR: The absolute value difference between the exact solution and the computed numerical result is an error. I.e.

i. $\boldsymbol{ERROR = |Exact\ solution - Numerical\ result|}$.

ii. $y_{computed}$ = Numerical result **using the new hybrid block method.**

iii. y_{exact} = Exact solution.

Example 5.1: Consider the stiff first-order ODE in [12].

$$y'(t) = \frac{y(1-y)}{2y-1}, \qquad y(0) = \frac{5}{6}, 0 \leq t \leq 1, \tag{1.24}$$

with the analytical solution $y(t) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{5}{35}e^{-t}}$.

Example 5.2: The Prothero-Robinson Oscillatory ODE

We also study the Prothero-Robinson Oscillatory problem solved by [13].

$$y' = L(y - \sin t) + \cos t, \qquad L = -1, \qquad y(0) = 0 \tag{1.25}$$

with the analytical solution $y(t) = \sin t$.

The results obtained at different values of time t, are shown in figures 5.1-5.2, and the absolute error in tables 5.1-5.2.

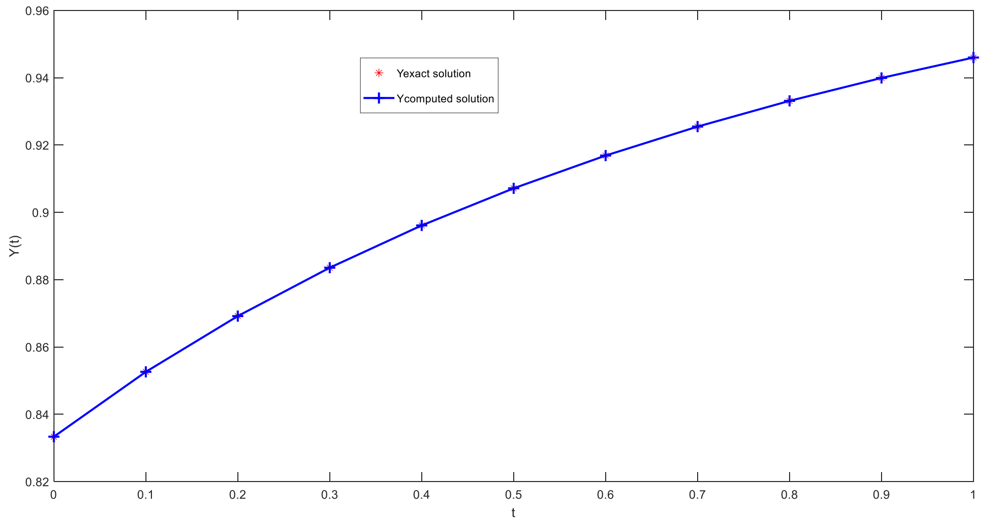

Figure 5.1: Showing the results of Example 5.1 using both analytical and numerical approaches.

h	y_{Exact}	$y_{computed}$	ERROR in New Method	ERROR in [12]
10^{-1}	0.85260195175848715618	0.85260195175848714034	1.584E-17	5.63131E-5
10^{-2}	0.83539987872083210020	0.83539987872083210018	2.0E-20	6.83365E-8
10^{-3}	0.83354149753621050416	0.83354149753621050415	1.0E-20	7.00620E-11
10^{-4}	0.83335416497409883587	0.83335416497409883586	1.0E-20	7.03881E-14
10^{-5}	0.83333541664973972384	0.83333541664973972383	1.0E-20	7.24374E-19

Table 5.1: Results and Absolute Errors of Example 5.1.

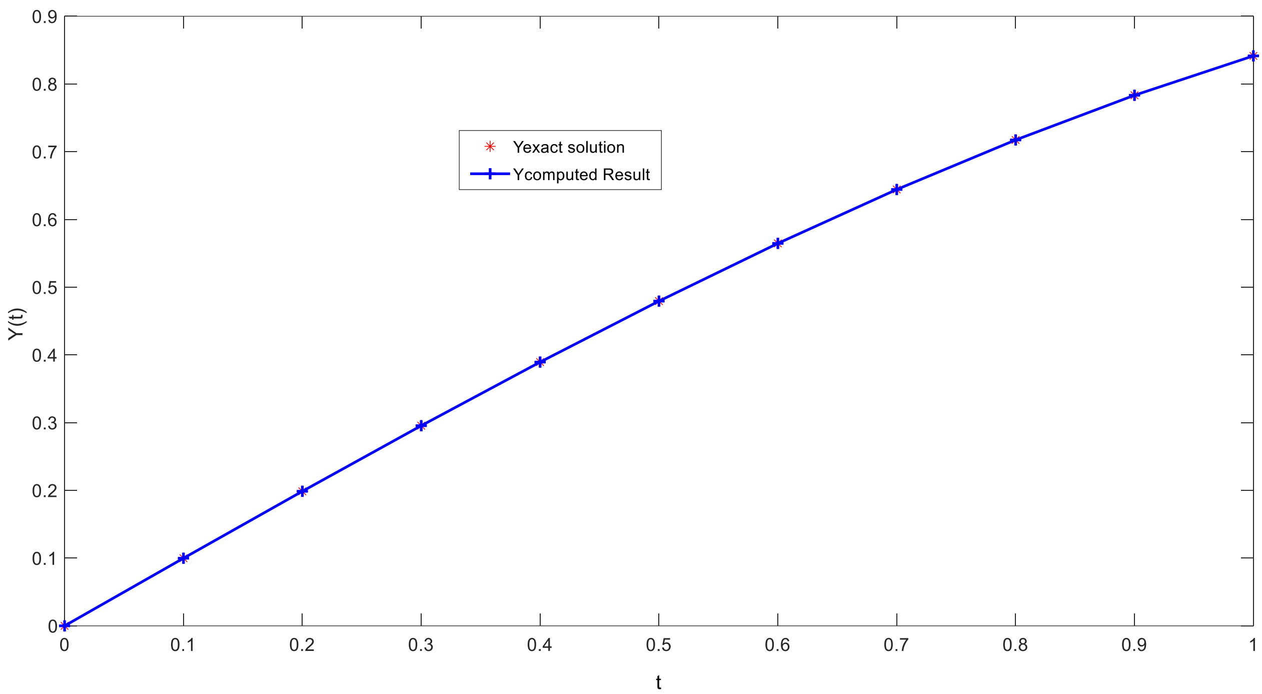

Figure 5.2: Presenting the results of Example 5.2 using both analytical and numerical approaches.

t	y_{Exact}	$y_{computed}$	ERROR in New Method	ERROR in [13]
0.1	0.099833416646828152307	0.099833416646828152301	6.0E-21	1.822016E-14
0.2	0.19866933079506121546	0.19866933079506121544	2.0E-20	2.271482E-14
0.3	0.29552020666133957511	0.29552020666133957508	3.0E-20	4.241108E-14
0.4	0.38941834230865049167	0.38941834230865049164	3.0E-20	1.364169E-14
0.5	0.47942553860420300027	0.47942553860420300024	3.0E-20	6.502551E-14
0.6	0.56464247339503535720	0.56464247339503535714	6.0E-20	9.103963E-14
0.7	0.64421768723769105367	0.64421768723769105357	1.0E-20	1.951339E-14
0.8	0.71735609089952276163	0.71735609089952276154	9.0E-20	7.155093E-14
0.9	0.78332690962748338846	0.78332690962748338836	1.0E-20	5.921081E-14
1.0	0.84147098480789650665	0.84147098480789650656	9.0E-20	8.457038E-14

Table 5.2: Results and Absolute Errors of Example 5.2.

5.1. Discussion of the Results

In this article, we investigated the effectiveness of a new hybrid block method by testing it on two numerical problems: one involving stiff ODEs and the other involving oscillatory ODEs. The stiff problem was previously solved using a seven-step block LMM by [12], while the oscillatory problem was previously solved using a similar derivation of the order seven block method by [13]. Tables 5.1 and 5.2 display the comparative results of problem 5.1 in equation (1.24) and problem 5.2 in equation (1.25), respectively. The new hybrid block method was evaluated against the exact solutions of the two numerical problems, and the results are shown in Figures 5.1-5.2. Our findings demonstrate that the recently developed hybrid block integrator is highly computationally efficient and offers superior performance in precision and stability compared to current methods.

6. Conclusion

This paper presents a novel hybrid block integrator that uses a continuous collocation and interpolation approach to solve stiff and oscillatory first-order ODEs. The hybrid LMM used in this study employs a unique basis function that combines Hermite polynomials and exponential functions, which differs from the approaches used by other researchers. Additionally, the derived LMM is distinct from previous methods. The hybrid block method is both convergent and consistent, with zero stability and an A-stable region of absolute stability. As such, it is well-suited for solving both stiff and oscillatory ODEs.

In terms of accuracy, the novel hybrid block method has outperformed previous methods of similar derivations. The use of combined basis functions in the generation of LMMs is worthy of universal acceptance. The technique indicates that continuous LMMs can be derived from any combination of polynomials and exponential functions utilizing an interpolation and collocation approach.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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