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Research Article

Intersection graphs of quasinormal subgroups of general skew linear groups

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Abstract

The intersection graph of quasinormal subgroups of a group G, denoted by $\Gamma_{\mathrm{q}}(G)$, is a graph defined as follows: the vertex set consists of all nontrivial, proper quasinormal subgroups of G, and two distinct vertices H and K are adjacent if $H \cap K$ is nontrivial. In this paper, we show that when G is an arbitrary nonsimple group, the diameter of $\Gamma_{\mathrm{q}}(G)$ is in $\{0, 1, 2, \infty\}$. Besides, all general skew linear groups $\mathrm{GL}_n(D)$ over a division ring D can be classified depending on the diameter of $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$.

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1. Introduction

Constructing a graph over an algebraic structure is an interesting topic that has attracted several mathematicians for approximately 150 years. In this direction, we would like to associate abstract structures, for instance, groups, rings, or modules as fairly concrete objects, for instance, graphs. Through characteristics of the graphs, properties of the original structures can be indirectly surveyed. In [6], the likely first literature of its idea, A. Cayley proposed a graphical representation of a group whose vertices are the group's elements. After another 100 years, in [5], J. Bosak first proposed the intersection graphs of semigroups with vertices as proper subsemigroups. Inspired by Bosak's work, Csákány and Pollák researched the intersection graph of subgroups with vertices as proper subgroups in a finite group. Recently, the intersection graphs whose vertices are sub-structures of groups, rings, or modules are studied deeply in the works of Akbari *et al.* (see [1–3]). More dated results can be found in [4, 15].

In this paper, we study a class of subgraphs of the intersection graph of a group G, where G is not necessarily finite. The intersection graph of G, denoted by $\Gamma(G)$, is a graph whose vertices are all nontrivial proper subgroups of G, and two distinct vertices H and K are adjacent if and only if $H \cap K \neq \langle 1 \rangle$. The main object of this paper is $\Gamma_{\mathrm{q}}(G)$, the subgraph of $\Gamma(G)$ induced on the nontrivial proper quasinormal subgroups of G. Recall that a subgroup Q of G is *quasinormal* (also *permutable*) in G if $QH = HQ$ for

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every $H \leq G$. Many interesting properties of quasinormal subgroups have already been discovered, for example, [12, 18]. Recently, quasinormal subgroups of the general linear group $\mathrm{GL}_n(D)$ over a division ring is researched in [8, 9]. Continuing this topic, we study on $\Gamma_q(G)$ in the generality and in the case $G = \mathrm{GL}_n(D)$, the general linear group over a division ring D.

This paper is organized as follows. In Section 2, we recall some notions in group and graph theory, as well as some basic properties of quasinormal subgroups and of the general skew linear groups. In Section 3, we consider the intesection graph of quasinormal subgroups of an arbitrary nonsimple group. Firstly, we assert that a nonsimple group G is the direct product of two simple subgroups if and only if $\Gamma_q(G)$ is disconnected. Secondly, we prove that $\mathrm{Diam}(\Gamma_q(G)) \in \{0, 1, 2, \infty\}$ for an arbitrary nonsimple group G. Lastly, we show that $\mathrm{Girth}(\Gamma_q(G)) = 3$ if $\Gamma_q(G)$ includes a cycle. We consider the intersection graph of quasinormal subgroups of the general linear group $\mathrm{GL}_n(D)$ of degree n over D in Section 4 and 5. In Section 4, we list all conditions for the graph $\Gamma_q(\mathrm{GL}_n(D))$ to be complete. Finally, in Section 5, we classify all general linear groups $\mathrm{GL}_n(D)$ by the diameter of $\Gamma_q(\mathrm{GL}_n(D))$, where $n \geq 2$. Also in this final section, we classify all general linear groups $\mathrm{GL}_n(F)$ of degree $n \geq 1$ over a field F by the diameter of the graph $\Gamma_q(\mathrm{GL}_n(F))$.

2. Preliminaries

In this section, we recall some basic definitions, notations, and essential lemmas that are used throughout this paper. Let G be a group and H a subgroup. We denote by $[G, G]$ the commutator group, by $\mathrm{Z}(G)$ the center of G, and by $[G : H]$ the index of H in G; moreover, define $H^G = \left\langle \bigcup_{g \in G} H^g \right\rangle$ and $H_G = \bigcap_{g \in G} H^g$ in which $H^g = g^{-1}Hg$ for $g \in G$. The subgroup H is called *central* if $H \subseteq \mathrm{Z}(G)$; otherwise, H is *noncentral*. The group G is called *centerless* if $\mathrm{Z}(G) = \langle 1 \rangle$.

H is called *quasinormal* (also *permutable*) in G if $HK = KH$ for all subgroups K of G. We have some basic properties of quasinormal subgroups as followings.

Lemma 2.1. *Let G be a group and H a quasinormal subgroup of G. Then,*

(1) *For each subgroup K of G, $H \cap K$ is a quasinormal subgroup of K;*
(2) *For each normal subgroup N of G, HN/N is a quasinormal subgroup of G/N;*
(3) *For each element g of G, H^g is a quasinormal subgroup of G;*
(4) *For each quasinormal subgroup K of G, HK is a quasinormal subgroup of G.*

Proof. It immediately follows from the definition of quasinormality. □

Lemma 2.2 ([18, Corollary C2])**.** *A simple group has no nontrivial proper quasinormal subgroup.*

Lemma 2.3 ([18, Lemma 2.3])**.** *Let G be a group and $g_1, g_2, \ldots, g_n \in G$. If H is a quasinormal subgroup of G, then $[H^{g_1}H^{g_2} \cdots H^{g_n} : H]$ is finite.*

We say that H is *subnormal* in G if there is a finite series of subgroups

$$H = N_r \trianglelefteq N_{r-1} \trianglelefteq \ldots \trianglelefteq N_1 \trianglelefteq N_0 = G.$$

Besides, G is called to be *radical* over subgroup H if for every $g \in G$, there exists a positive integer n_g such that $g^{n_g} \in H$. Stonehewer says that every quasinormal subgroup of a finitely generated group is subnormal [18]. Here, we give some connections between quasinormality and subnormality by two following lemmas.

Lemma 2.4. *Let G be a group and H a quasinormal subgroup of G. Then, either H is subnormal in G or G is radical over H.*

Proof. This result can be found in [8] or [9]. □

Lemma 2.5. *A quasinormal subgroup of finite index is subnormal.*

Proof. Assume that G is a group and H is a quasinormal subgroup of G such that $[G : H]$ is finite. Then, the quotient group G/H_G is finite. By Lemma 2.1, the subgroup H/H_G is quasinormal in G/H_G. It implies that H/H_G is subnormal in G/H_G. Hence, H is subnormal in G. □

Now we turn to general linear groups over division rings. Let D be a division ring. We denoted by D^* the multiplicative group of D and by D' the commutator group of D^*. In this paper, if no hypothesis is added, then $\mathrm{GL}_n(D)$ is the general linear group of degree n over D and $\mathrm{SL}_n(D)$ is the special linear group, where n is a positive integer. Also, the identity element of $\mathrm{GL}_n(D)$ is denoted by I_n. The following lemma lists some interesting properties of $\mathrm{GL}_n(D)$ and $\mathrm{SL}_n(D)$.

Lemma 2.6. *Let D be a division ring and $n \geq 2$ an integer. Then*

(1) $\mathrm{GL}_n(D)/\mathrm{SL}_n(D) \cong D^*/D'$;
(2) $\mathrm{Z}(\mathrm{GL}_n(D)) = \{c\mathrm{I}_n | c \in F^*\}$;
(3) $\mathrm{Z}(\mathrm{SL}_n(D)) = \{c\mathrm{I}_n | c \in F^* \text{ and } c^n \in D'\}$.

If $\mathrm{GL}_n(D) \ncong \mathrm{GL}_2(\mathbb{F}_2)$ *and* $\mathrm{GL}_n(D) \ncong \mathrm{GL}_2(\mathbb{F}_3)$*, then*

(4) $\mathrm{SL}_n(D) = [\mathrm{GL}_n(D), \mathrm{GL}_n(D)] = [\mathrm{SL}_n(D), \mathrm{SL}_n(D)]$;
(5) $\mathrm{SL}_n(D)/\mathrm{Z}(\mathrm{SL}_n(D))$ *is a simple group;*
(6) N *is a noncentral subnormal subgroup of* $\mathrm{GL}_n(D)$ *if and only if* N *contains* $\mathrm{SL}_n(D)$.

Proof. The statement can be established from [10, §20 and §21] and [14, Theorem 5.13]. □

In [8] and [9], authors proved some properties of quasinormal subgroups of $\mathrm{GL}_n(D)$. We have the following results.

Lemma 2.7 ([8, Theorem 3.3])**.** *Every quasinormal subgroup of* $\mathrm{GL}_n(D)$ *is normal, where* $n \geq 2$.

Lemma 2.8 ([9, Theorem 1 and 3])**.** *Every locally solvable subnormal or locally solvable quasinormal subgroup of D^* is central.*

Finally, we recall some notions in graph theory. Let Γ be a graph with the vertex set V and the edge set E. For two distinct vertices u and v in V, we denote $u \backsim v$ if u and v are adjacent. The *null graph* is a graph that has no vertex. In a graph Γ, a *path* from vertex v_0 to vertex v_n is a sequence of vertices $v_0 \backsim v_1 \backsim \cdots \backsim v_n$, where $v_0, v_1, \ldots, v_{n-1}$ are distinct. If $v_0 = v_n$, then we call the sequence a *cycle*. The number n as above is call the *length* of this path. If two vertices u and v are connected, then the *distance* between u and v, denoted by $\mathrm{dist}(u, v)$, is the length of a shortest path from u to v; otherwise, $\mathrm{dist}(u, v) = \infty$. In particular, $\mathrm{dist}(u, u) = 0$. The *diameter* of a graph Γ, denote $\mathrm{Diam}(\Gamma)$, is the supremum distance between two vertices in Γ. If Γ is a graph with at least one cycle, then the *girth* of Γ, denoted by $\mathrm{Girth}(\Gamma)$, is the length of shortest cycle in Γ.

3. Intersection graph of quasinormal subgroups of an arbitrary group

Let G be an arbitrary nontrivial group and $\Gamma_{\mathrm{q}}(G)$ the intersection graph of quasinormal subgroups of G. Lemma 2.2 leads to the fact that if G is a simple group, then $\Gamma_{\mathrm{q}}(G)$ is the null graph. In this section, we present some characteristics of $\Gamma_{\mathrm{q}}(G)$ in case G is nonsimple. Namely, we prove that if G is nonsimple, then the diameter of $\Gamma_{\mathrm{q}}(G)$ is in $\{0, 1, 2, \infty\}$. Moreover, if $\Gamma_{\mathrm{q}}(G)$ contains a cycle, then $\Gamma_{\mathrm{q}}(G)$ is a connected graph and $\mathrm{Girth}(\Gamma_{\mathrm{q}}(G)) = 3$.

We begin with the following lemma.

Lemma 3.1. *Let G be a group and H a proper quasinormal subgroup of G. Then, $H\cdot H^g$ is also a proper quasinormal subgroup of G for every $g \in G$.*

Furthermore, for a fixed $g \in G$, if $H \neq H^g$ then either $H \backsim H^g$ or $H \backsim H\cdot H^g \backsim H^g$.

Proof. The part (4) of Lemma 2.1 follows that $H\cdot H^g$ is quasinormal in G for every $g \in G$. Assume to the contrary that $G = H\cdot H^g$ for some $g \in G$. According to Lemma 2.3, $[G : H]$ is finite. Then, in view of Lemma 2.5, H is subnormal in G. Because H is proper in G, so is H^G. On the other hand, because $H \cdot H^g \subseteq H^G$, we have $H^G = G$, which is a contradiction. Hence, $H\cdot H^g$ is a proper quasinormal subgroup of G for every $g \in G$.

Now we prove the remain assertion. Suppose that $H \neq H^g$. Because $H\cdot H^g$ includes H and H^g, we have $H \backsim H\cdot H^g$ or $H^g \backsim H\cdot H^g$. Hence, if $H^g = H\cdot H^g$ or $H = H\cdot H^g$, then $H \backsim H^g$; and if $H \neq H\cdot H^g$ and $H^g \neq H\cdot H^g$, then $H \backsim H\cdot H^g \backsim H^g$. □

Theorem 3.2. *Let G be a nonsimple group. The following statements are equivalent:*

(1) *G is the direct product of two simple subgroups;*
(2) *$\Gamma_q(G)$ is disconnected;*
(3) *$\Gamma_q(G)$ has more than one vertex and every vertex of $\Gamma_q(G)$ is isolated.*

Proof. The implication (3) ⇒ (2) is trivial.

The implication (2) ⇒ (1): Assume that $\Gamma_q(G)$ is disconnected. Then, there exists a pair of disconnected vertices H and K. By Lemma 2.1, HK is a quasinormal subgroup of G. Since H and K are disconnected, we have $G = HK$. Suppose that H is not normal in G, i.e. $H^g \nsubseteq H$ for some $g \in G$. Select $x \in H^g \setminus H$. Since $G = HK$, there exist $h \in H$ and $k \in K$ such that $x = hk$. This implies that $h^{-1}x \in K \cap (H \cdot H^g)$. On the other hand, Lemma 3.1 follows that $H \cdot H^g$ is a vertex of $\Gamma_q(G)$. Since H and K is not connected, $K \cap (H \cdot H^g) = \langle 1\rangle$. That means $x = h \in H$, a contradiction. Thus, H is normal in G. Similarly, K is also normal in G. Now we claim that H is simple. Deny the claim, suppose that there exists a subgroup $N \trianglelefteq H$ such that $N \neq \langle 1\rangle$ and $N \neq H$. Clearly, NK is a nontrivial proper normal subgroup of G. This implies that $H \backsim NK \backsim K$, a contradiction. Thus, H is simple. Similarly, K is also simple. Hence, G is the direct product of simple subgroups H and K.

The implication (1) ⇒ (3): Assume that $G = HK$, the direct product of some simple subgroups H and K. We claim that every vertex of $\Gamma_q(G)$ is a simple subgroup. For each vertex Q of $\Gamma_q(G)$, if $Q \cong H$, then Q is simple. Suppose that $Q \ncong H$. By Lemma 2.1, $Q \cap H$ is quasinormal in H. Since H is simple, $Q \cap H = \langle 1\rangle$ or $Q \cap H = H$ by Lemma 2.2. If $Q \cap H = H$, then $H \subseteq Q$. By part (2) of Lemma 2.1, Q/H is quasinormal in G/H. The group G/H is simple because $G/H \cong K$. This leads to the fact that $Q = H$ or $Q = G$, a contradiction. Thus, $Q \cap H = \langle 1\rangle$. By Lemma 2.1, QH/H is quasinormal in G/H. It follows that $QH = H$ or $QH = G$. If $QH = H$, then $Q = \langle 1\rangle$. Thus, $QH = G$. Then, $Q \cong G/H$, and so $Q \cong K$. Then, Q is simple. Hence, the claim is proved. It is obvious to that all vertices of $\Gamma_q(G)$ are isolated. We complete the proof. □

Theorem 3.3. *Let G be a nonsimple group. Then,* $\mathrm{Diam}(\Gamma_q(G)) \in \{0, 1, 2, \infty\}$.

Proof. Clearly, $\mathrm{Diam}(\Gamma_q(G)) = \infty$ if $\Gamma_q(G)$ is disconnected; $\mathrm{Diam}(\Gamma_q(G)) = 0$ if $\Gamma_q(G)$ has only one vertex. Assume that $\Gamma_q(G)$ has at least two vertices and is connected. We prove the claim: $\mathrm{dist}(H, K) \in \{1, 2\}$ for any pair of vertices H and K. Indeed, if $H \cap K \neq \langle 1\rangle$ or $HK \neq G$, then $\mathrm{dist}(H, K) \in \{1, 2\}$. Suppose that $H \cap K = \langle 1\rangle$ or $HK = G$. If H is not normal in G, then $H^g \nsubseteq H$ for some $g \in G$. Lemma 3.1 tells us $H\cdot H^g$ is a vertex. Select $x \in H^g \setminus H$. Since $G = HK$, we have $x = hk$ for some $h \in H$ and $k \in K$. This implies that $k \neq 1$ and $k \in K \cap H\cdot H^g$. Then, either $H \backsim H\cdot H^g = K$ or $H \backsim H\cdot H^g \backsim K$, that is, $\mathrm{dist}(H, K) \in \{1, 2\}$. Similarly, if K is not normal in G, then $\mathrm{dist}(H, K) \in \{1, 2\}$. Now we consider that both H and K are normal in G. So G is the direct product of H and K. Theorem 3.2 follows that H or K is nonsimple. Without loss of generality, it may be assumed that H is nonsimple, that is, there exists a normal subgroup N of H such that $N \neq \langle 1 \rangle$ and $N \neq H$. Then, $\text{dist}(H,K) = 2$ since $H \backsim NK \backsim K$. Thus, the claim is established. The claim is equivalent to that $\text{Diam}(\Gamma_q(G)) \in \{1,2\}$. The theorem is proved. □

Theorem 3.4. *Let G be a nonsimple group. If $\Gamma_q(G)$ contains a cycle, then $\Gamma_q(G)$ is connected and* $\text{Girth}(\Gamma_q(G)) = 3$.

Proof. Assume that $\Gamma_q(G)$ has a cycle $H_1 \backsim H_2 \backsim H_3 \backsim \ldots \backsim H_k \backsim H_1$ where $k \geq 3$. Then, H_1, H_2 and H_k are distinct and not isolated. Thank to Theorem 3.2, $\Gamma_q(G)$ is connected. To determine the girth of $\Gamma_q(G)$, we consider two following cases:

Case 1. *H_1 or H_2 is not normal in G:* We can suppose H_1 is not normal in G. Then, $H_1^g \not\subseteq H_1$ for some $g \in G$. By Lemma 3.1, $H_1 \backsim H_1 \cdot H_1^g$. If $H_1 \cdot H_1^g \neq H_2$, then $H_1 \backsim H_1 \cdot H_1^g \backsim H_2 \backsim H_1$, or else $H_1 \backsim H_2 \backsim H_k \backsim H_1$. Thus, $\text{Girth}(\Gamma_q(G)) = 3$.

Case 2. *H_1 and H_2 are normal in G:* Then $H_1 \cap H_2$ is a vertex of $\Gamma_q(G)$. If $H_1 \cap H_2 = H_1$, i.e. $H_1 \subseteq H_2$, then $H_1 \backsim H_2 \backsim H_k \backsim H_1$. Similarly, if $H_1 \cap H_2 = H_2$, then $H_1 \backsim H_2 \backsim H_3 \backsim H_1$. If $H_1 \cap H_2$ is different from both H_1 and H_2, then $H_1 \backsim H_1 \cap H_2 \backsim H_2 \backsim H_1$. Thus, $\text{Girth}(\Gamma_q(G)) = 3$.

Our proof is complete. □

4. Completeness of $\Gamma_q(\text{GL}_n(D))$

Let $\text{GL}_n(D)$ be the general linear group of degree $n \geq 1$ over a division ring D. The aim of this section is to give necessary and sufficient conditions on D and n for $\Gamma_q(\text{GL}_n(D))$ to be complete. More precisely, such conditions are obtained in Theorem 4.6 for the case $n = 1$ and in Theorem 4.8 and 4.9 for $n \geq 2$. However, out of necessity, we first present a series of lemmas to support these results.

Lemma 4.1. *Let F be a field.*

(1) *If F^* is isomorphic to an additive subgroup of the rational field $\mathbb{Q}$, then $F \cong \mathbb{F}_2$.*
(2) *If F^* is isomorphic to a subgroup of a quasicyclic p-group C_{p^∞}, then either $F \cong \mathbb{F}_9$ or $F \cong \mathbb{F}_{2^k+1}$, where $2^k + 1$ is a prime integer.*

Proof. (1) Let H be an additive subgroup of $\mathbb{Q}$ such that $F^* \overset{\varphi}{\cong} H$. This implies that F^* is torsion-free; hence, $\text{char}(F) = 2$. We assume that there exists an element $a \in F \backslash \mathbb{F}_2$ and derive a contradiction. Let $\varphi(a) = k/m$ and $\varphi(a+1) = \ell/n$. Since $\varphi(a^{\ell m}) = \varphi((a+1)^{kn}) = k\ell$, we obtain that $a^{\ell m} = (a+1)^{kn}$, which means that a is algebraic over $\mathbb{F}_2$. It follows that the field $\mathbb{F}_2(a)$ is finite, and so a is periodic. As F^* is torsion-free, it follows that $a = 1$, a contradiction. This implies that $F = \mathbb{F}_2$, completing the proof.

(2) It follows from [7, Theorem] that F^* is not isomorphic to C_{p^∞}. This means $F^* \cong \text{C}_{p^n}$. By [7, Proposition 2.6], we obtain that $F \cong \mathbb{F}_9$ or $F \cong \mathbb{F}_{2^k+1}$, where $\text{char}(F) = 2^k + 1$. □

Lemma 4.2. *Let G be an abelian group. Then, $\Gamma_q(G)$ is complete if and only if G is isomorphic to some subgroup of $\mathbb{Q}$ or of C_{p^∞}.*

Proof. Since G is abelian, $\Gamma_q(G)$ is also the intersection graph of subgroups of G. Hence, the statement is exactly [1, Theorem 2]. □

Three next lemmas draw some traits of quasinormal subgroups in the general linear groups over division rings.

Lemma 4.3. *Let $G = \text{GL}_n(D)$ be a noncommutative general skew linear group of degree $n \geq 1$ over division ring D. Then, the intersection of any two noncentral quasinormal subgroups of G is nontrivial.*

Proof. First, we consider that $n \geq 2$. If $G \cong \mathrm{GL}_2(\mathbb{F}_2)$ or $G \cong \mathrm{GL}_2(\mathbb{F}_3)$, then the conclusion obviously holds. When $G \ncong \mathrm{GL}_2(\mathbb{F}_2)$ and $G \ncong \mathrm{GL}_2(\mathbb{F}_3)$, Lemma 2.6(6) and Lemma 2.7 lead to that every noncentral quasinormal subgroup of G contains $\mathrm{SL}_n(D)$. This yields the conclusion.

Next, consider $n = 1$, that is $G = D^*$. Let H and K be noncentral quasinormal subgroup of D^* and $N = H \cap K$. According to Lemma 2.4, we can naturally consider three following cases.

Case 1. *H and K are subnormal in D^*:* By [17, 14.4.5], N is noncentral.

Case 2. *H is subnormal in D^*, and D^* is radical over K:* The second implies that H is radical over N. If $N = \langle 1 \rangle$, then H is periodic. By [13, Theorem 8], H is central, a contradiction. Thus N is nontrivial.

Case 3. *D^* is radical over both H and K:* Then, D^* is also radical over N. By [13, Theorem 8], $N \neq \langle 1 \rangle$ because D^* is noncommutative. □

Lemma 4.4. *Let D be a division ring with center F, and let G be a subgroup of D^* such that D^* is radical over G. Then, every locally soluble normal subgroup of G is contained in F.*

In particular, $\mathrm{Z}(G) = G \cap F$.

Proof. We can suppose that D is noncommutative. By [9, Lemma 8], G is nonabelian and is an absolutely irreducible subgroup in D^*, that is $D = F[G]$. Suppose that N is a locally soluble normal subgroup of G. By [20, 4.Corollary], N contains an abelian normal subgroup A of G such that N/A locally finite. Put $H = \mathrm{C}_N(A)$. The first claim is that H is normal in G. Indeed, for every $h \in H$, $g \in G$, and $a \in A$, $a^{g^{-1}hg} = (a^g)^{hg} = a$. This fact leads to the fact that $h^g \in H$. The next claim is that $H/\mathrm{Z}(H)$ is locally finite. Because N/A is locally finite and $H \subseteq N$, H/A is also locally finite. Clearly, $A \subseteq \mathrm{Z}(H)$. By [16, 1.4.5], $(H/A)/(\mathrm{Z}(H)/A) \cong H/\mathrm{Z}(H)$. Consequently, $H/\mathrm{Z}(H)$ is locally finite. According to [19, 3.8], $G/\mathrm{C}_G(H)$ is periodic, that is G is radical over $\mathrm{C}_G(H)$. Since $\mathrm{C}_G(H) \subseteq \mathrm{C}_D(H)$, D is radical over its division subring $\mathrm{C}_D(H)$. Then, $\mathrm{C}_D(H) = D$ by [11, Theorem B]. This implies that $H \subseteq F$, and so $A \subseteq F$. Then, $N = \mathrm{C}_N(A) = H \subseteq F$.

What is left to show that $\mathrm{Z}(G) = G \cap F$. Obviously, $G \cap F \subseteq \mathrm{Z}(G)$. Since $\mathrm{Z}(G)$ is an abelian normal subgroup of G, $\mathrm{Z}(G) \subseteq G \cap F$. Hence, $\mathrm{Z}(G) = G \cap F$. □

It is easy to see that Lemma 4.4 is an extension of [9, Theorem 2]. Besides, we can extend [9, Theorem 3] as follows.

Lemma 4.5. *Let D be a division ring with center F, and let Q be a quasinormal subgroup of D^*. Then, every locally soluble normal subgroup of Q is contained in F.*

In particular, $\mathrm{Z}(Q) = Q \cap F$.

Proof. Let N be a locally soluble normal subgroup of Q. If Q is subnormal in D^*, then N is also subnormal in D^*. According to [9, Theorem 1], $N \subseteq F$. If Q is not subnormal in D^*, then D^* is radical over Q. By Lemma 4.4, $N \subseteq F$. □

Now we have the following theorem to describe cases that the intersection graph of quasinormal subgroups of the multiphicative group of a division ring.

Theorem 4.6. *Let D be a division ring with center F. Then, $\Gamma_{\mathrm{q}}(D^*)$ is complete if and only if one of the following conditions holds*

1. *D is isomorphic to either $\mathbb{F}_9$ or $\mathbb{F}_{2^k+1}$, where 2^k+1 is a prime integer greater than 3;*
2. *D^* is nonsimple and $F \cong \mathbb{F}_2$;*
3. *D is noncommutative, every nontrivial quasinormal subgroup of D^* is not centerless, and F is isomorphic to one of fields $\mathbb{F}_9$, $\mathbb{F}_{2^k+1}$ or $\mathbb{F}_{2^\ell}$, where 2^k+1 and $2^\ell-1$ are prime integers.*

Proof. Assume that $\Gamma_{\mathrm{q}}(D^*)$ is complete. If D is commutative, then Lemma 4.2 implies that D^* is isomorphic to some subgroup of the additive group of rationals $\mathbb{Q}$ or of a quasicyclic p-group C_{p^∞}. Since D^* must be nontrivial and nonsimple, in view of Lemma 4.1, D is isomorphic to either $\mathbb{F}_9$ or $\mathbb{F}_{2^k+1}$, where 2^k+1 is a prime integer greater than 3. Thus, (1) is proved. Suppose that D is noncommutative. If $F^* = \langle 1\rangle$, then $F \cong \mathbb{F}_2$. By Lemma 4.3, D^* is nonsimple, and so (2) is proved. If $F^* \neq \langle 1\rangle$, we claim that $F \cong \mathbb{F}_9$, $F \cong \mathbb{F}_{2^\ell}$, or $F \cong \mathbb{F}_{2^k+1}$, where 2^k+1 and $2^\ell-1$ are prime integers. If F^* is simple, i.e. F^* is a cyclic group of prime index, then either $F \cong \mathbb{F}_3$ or $F \cong \mathbb{F}_{2^\ell}$, where $2^\ell - 1$ is a prime integer. If F^* is nonsimple, then $\Gamma_{\mathrm{q}}(F^*)$, an induced subgraph of $\Gamma_{\mathrm{q}}(D^*)$, is complete. Thank to (1), $F \cong \mathbb{F}_9$ or $F \cong \mathbb{F}_{2^k+1}$, where 2^k+1 is a prime integer greater than 3. Thus, the claim is proved. Since F^* is a vertex of $\Gamma_{\mathrm{q}}(D^*)$, every nontrivial quasinormal subgroup of D^* intersects to F^* is nontrivial. By Lemma 4.5, every nontrivial quasinormal subgroup of D^* is not centerless. Thus, (3) is asserted.

Conversely, assume that D satisfies one of conditions (1), (2), or (3). If (1) holds, then $D^* \cong \mathrm{C}_{2^3}$ or $D^* \cong \mathrm{C}_{2^k}$ for $k \geq 2$. By Lemma 4.2, $\Gamma_{\mathrm{q}}(D^*)$ is complete. If (2) holds, then $\Gamma_{\mathrm{q}}(D^*)$ is not a null graph and every vertex of $\Gamma_{\mathrm{q}}(D^*)$ is noncentral quasinormal subgroups. By Lemma 4.3, $\Gamma_{\mathrm{q}}(D^*)$ is complete. Suppose that (3) holds. We have that the order of F^* is either 2^k or $2^\ell - 1$, where $k \geq 1$ and $2^\ell - 1$ is a prime integer. Thus, the subgraph induced on central quasinormal subgroups of $\Gamma_{\mathrm{q}}(D^*)$ is complete. Additionally, every nontrivial quasinormal subgroup of D^* is not centerless. Hence, $\Gamma_{\mathrm{q}}(D^*)$ is complete. The proof is complete. □

Now we turn to the general linear groups $\mathrm{GL}_n(D)$ of degree $n \geq 2$. Firstly, because $\mathrm{GL}_2(\mathbb{F}_2)$ and $\mathrm{GL}_2(\mathbb{F}_3)$ are quite special, they are considered independently in the following lemma.

Lemma 4.7. (1) $\mathrm{GL}_2(\mathbb{F}_2)$ *has only one nontrivial proper normal subgroup that is* $[\mathrm{GL}_2(\mathbb{F}_2), \mathrm{GL}_2(\mathbb{F}_2)]$. *Moreover, that is the only vertex of* $\Gamma_{\mathrm{q}}(\mathrm{GL}_2(\mathbb{F}_2))$.

(2) $\mathrm{GL}_2(\mathbb{F}_3)$ *has only three nontrivial proper normal subgroups that are*

$$\langle -\mathrm{I}_2\rangle,\ \left\langle \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle,\ [\mathrm{GL}_2(\mathbb{F}_3), \mathrm{GL}_2(\mathbb{F}_3)].$$

Moreover, $\Gamma_{\mathrm{q}}(\mathrm{GL}_2(\mathbb{F}_3))$ *is a complete graph with the vertices as above.*

Proof. (1) It is a well-known fact that $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$, the symmetric group of degree 3. Because S_3 has only three normal subgroups that are $\langle 1\rangle$, $A_3 = [S_3, S_3]$ and itself, $[\mathrm{GL}_2(\mathbb{F}_2), \mathrm{GL}_2(\mathbb{F}_2)]$ is the unique normal subgroup of $\mathrm{GL}_2(\mathbb{F}_2)$ except $\langle \mathrm{I}_2\rangle$ and the whole group. According to Lemma 2.7, $\Gamma_{\mathrm{q}}(\mathrm{GL}_2(\mathbb{F}_2))$ has only one vertex that is $[\mathrm{GL}_2(\mathbb{F}_2), \mathrm{GL}_2(\mathbb{F}_2)]$.

(2) It is easy to prove that $\langle -\mathrm{I}_2\rangle$, $[\mathrm{GL}_2(\mathbb{F}_3), \mathrm{GL}_2(\mathbb{F}_3)]$, and

$$N = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$$

are normal in $\mathrm{GL}_2(\mathbb{F}_3)$. By utilizing `GAP`, a programming language or discrete algebra computation, we can determine that $\mathrm{GL}_2(\mathbb{F}_3)$ has only five normal subgroups by running following commands:

```
gap> LoadPackage("sonata");
gap> NormalSubgroups(GL(2,3));
```

Consequently, $\mathrm{GL}_2(\mathbb{F}_3)$ has only three normal subgroups apart from $\langle \mathrm{I}_2\rangle$ and $\mathrm{GL}_2(\mathbb{F}_3)$. By Lemma 2.7, $\langle -\mathrm{I}_2\rangle$, N and $[\mathrm{GL}_2(\mathbb{F}_3), \mathrm{GL}_2(\mathbb{F}_3)]$ are the vertices of $\Gamma_{\mathrm{q}}(\mathrm{GL}_2(\mathbb{F}_3))$. Since $\langle -\mathrm{I}_2\rangle \leq N \leq [\mathrm{GL}_2(\mathbb{F}_3), \mathrm{GL}_2(\mathbb{F}_3)]$, the graph $\Gamma_{\mathrm{q}}(\mathrm{GL}_2(\mathbb{F}_3))$ is complete.

□

Theorem 4.8. *Let D be a division ring with center F and $n \geq 2$ an integer. Assume that $\mathrm{Z}(D') \neq \langle 1 \rangle$. Then, $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete if and only if one of following conditions holds*

(1) $F \cong \mathbb{F}_9$;
(2) $F \cong \mathbb{F}_{2^k+1}$, *where* $2^k + 1$ *is a prime integer;*
(3) $F \cong \mathbb{F}_{2^\ell}$, *where* $2^\ell - 1$ *is a prime integer.*

Proof. Assume that $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete. Since $\mathrm{Z}(D') \neq \langle 1 \rangle$, there exists $d \in D' \cap F$ such that $d \neq 1$. Then, $F \ncong \mathbb{F}_2$, and so $F^* \neq \langle 1 \rangle$. According to part (2) of Lemma 2.6, $\mathrm{Z}(\mathrm{GL}_n(D)) \cong F^*$, and so either F^* is a simple group or $\Gamma_{\mathrm{q}}(F^*) \cong \Gamma_{\mathrm{q}}(\mathrm{Z}(\mathrm{GL}_n(D)))$. We now consider these cases:

Case 1. *F^* is a simple group:* Then $F \cong \mathbb{F}_3$ or $F \cong \mathbb{F}_{2^\ell}$, where $2^\ell - 1$ is a prime integer. If $F \cong \mathbb{F}_3$, then we assert (2) in case that $k = 1$. If $F \cong \mathbb{F}_{2^\ell}$, then (3) is proved.

Case 2. *$\Gamma_{\mathrm{q}}(F^*) \cong \Gamma_{\mathrm{q}}(\mathrm{Z}(\mathrm{GL}_n(D)))$:* Then $\Gamma_{\mathrm{q}}(F^*)$ is complete. By Lemma 4.3, either $F \cong \mathbb{F}_9$ or $F \cong \mathbb{F}_{2^k+1}$, where $2^k + 1$ is a prime integer greater than 3. Thus, (1) and (2) are proved.

Conversely, suppose that F satisfy one of conditions (1), (2), or (3). Since $\mathrm{Z}(\mathrm{GL}_n(D)) \cong F^*$, either $\mathrm{Z}(\mathrm{GL}_n(D))$ is simple or $\mathrm{Z}(\mathrm{GL}_n(D)) \cong \mathrm{C}_{2^k}$, where $k > 1$. By Lemma 4.2, the subgraph induced on central quasinormal subgroups of $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete. On the other hand, thank to part (3) and (6) of Lemma 2.6, the assumption $\mathrm{Z}(D') \neq \langle 1 \rangle$ leads to the fact that every noncentral quasinormal subgroup of $\mathrm{GL}_n(D)$ contains $\mathrm{SL}_n(D)$ and is not centerless. Hence, $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete. □

Theorem 4.9. *Let D be a division ring with center F and $n \geq 2$ an integer. Assume that $\mathrm{Z}(D') = \langle 1 \rangle$. Then, $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete if and only if one of following conditions holds*

(1) $n = 2$ *and* $D \cong \mathbb{F}_2$;
(2) $n \geq 2$, $D^* \neq D'$, *and* $F \cong \mathbb{F}_2$;
(3) n *is even and* $F \cong \mathbb{F}_9$;
(4) n *is even and* $F \cong \mathbb{F}_{2^k+1}$, *where* $2^k + 1$ *is a prime integer;*
(5) n *is a multiple of* $2^\ell - 1$ *and* $F \cong \mathbb{F}_{2^\ell}$, *where* $2^\ell - 1$ *is a prime integer.*

Proof. Assume that $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete. Let us consider possible cases:

Case 1. *F is isomorphic to $\mathbb{F}_2$:* First, suppose that $D \cong \mathbb{F}_2$. If $n > 2$, then $\mathrm{GL}_n(\mathbb{F}_2)$ is a simple group, and so $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is the null graph, a contradiction. Thus, $n = 2$, and so we get (1). Next, suppose that $D \ncong \mathbb{F}_2$. Then, D^* is nonabelian and $\mathrm{Z}(\mathrm{GL}_n(D)) = \langle \mathrm{I}_n \rangle$. It follows that all nontrivial quasinormal subgroups of $\mathrm{GL}_n(D)$ are noncentral. By part (1) and (6) of Lemma 2.6, if $D^* = D'$, then $\mathrm{GL}_n(D)$ is a simple group, and so $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is the null graph, a contradiction. Thus, $D^* \neq D'$, and so (2) is asserted.

Case 2. *F is not isomorphic to $\mathbb{F}_2$ and F^* is a simple group:* Then F^* is a cyclic group of prime order p. Since $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete, $\mathrm{SL}_n(D) \cap \mathrm{Z}(\mathrm{GL}_n(D)) \neq \langle \mathrm{I}_n \rangle$. Thus, $\mathrm{Z}(\mathrm{SL}_n(D)) \neq \langle \mathrm{I}_n \rangle$. Because $\mathrm{Z}(D') = \langle 1 \rangle$, it follows from part (3) of Lemma 2.6, n must be a multiple of p. If $p = 2$, then $F \cong \mathbb{F}_3$ and n is an even integer, and so (4) holds in case that $k = 1$. If p is an odd prime integer, then $F \cong \mathbb{F}_{2^\ell}$ and n is a multiple of $p = 2^\ell - 1$, and so (5) is proved.

Case 3. *F is not isomorphic to $\mathbb{F}_2$ and F^* is a nonsimple group:* Then $\Gamma_{\mathrm{q}}(F^*)$ is not a null graph. Since $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete, it follows from part (2) of Lemma 2.6, $\Gamma_{\mathrm{q}}(F^*)$ is also complete. In light of Theorem 4.6, either $F \cong \mathbb{F}_9$ or $F \cong \mathbb{F}_{2^k+1}$, where $2^k + 1$ is a prime integer greater than 3. Notice that, the central quasinormal subgroup $\langle -\mathrm{I}_n \rangle$ of $\mathrm{GL}_n(D)$ is not trivial. Since $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete, $\langle -\mathrm{I}_n \rangle \subseteq \mathrm{SL}_n(D)$. The assumption $\mathrm{Z}(D') = \langle 1 \rangle$ leads to the fact that n is even. Hence, (3) and (4) are proved.

We now turn to the converse of the statement.

First, if $\mathrm{GL}_n(D)$ satisfies (1), then $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete by Lemma 4.7.

Second, suppose that $\mathrm{GL}_n(D)$ satisfies (2). Because $\mathrm{GL}_n(D)/\mathrm{SL}_n(D) \cong D^*/D'$ and $D^* \neq D'$, the group $\mathrm{GL}_n(D)$ is nonsimple. On the other hand, the vertices of $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ are noncentral subgroups of $\mathrm{GL}_n(D)$ because $\mathrm{Z}(\mathrm{GL}_n(D)) = \langle \mathrm{I}_n \rangle$. From Lemma 4.3, $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete.

Next, suppose that $\mathrm{GL}_n(D)$ is under (3) or (4). If $n = 2$ and $k = 1$, then $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete by Lemma 4.7. Now, we assume that if $n = 2$, then $F \ncong \mathbb{F}_3$. Since n is even, $\mathrm{Z}(\mathrm{GL}_n(D))$ is adjacent to $\mathrm{SL}_n(D)$ because both contain $-\mathrm{I}_n$. Consequently, all nontrivial quasinormal subgroups of $\mathrm{GL}_n(D)$ contain $-\mathrm{I}_n$. Hence, $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete.

Finally, if $\mathrm{GL}_n(D)$ is under (5), then $\mathrm{Z}(\mathrm{GL}_n(D))$ is a simple group of prime order $2^\ell - 1$. Then, $\mathrm{Z}(\mathrm{GL}_n(D))$ is the unique central subgroup as a vertex of $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$. On the other hand, n is a multiple of $2^\ell - 1$ and every element $a \in F^*$ such that $a \neq 1$ has order $2^\ell - 1$. By parts (2) and (3) of Lemma 2.6, $\mathrm{Z}(\mathrm{GL}_n(D)) = F^*\mathrm{I}_n = \mathrm{Z}(\mathrm{SL}_n(D))$. Thus, all nontrivial quasinormal subgroups of $\mathrm{GL}_n(D)$ contain $F^*\mathrm{I}_n$, and so $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete. □

5. Diameter of $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$, $n \geq 2$

In this section, we seperate all general linear groups $\mathrm{GL}_n(D)$, where $n \geq 2$, according to diameter of the graph $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$. In particular, if D is a field, then we can fully describe the diameter of this graph.

The following lemma lets us know whether $\mathrm{GL}_n(D)$ is a simple group.

Lemma 5.1. *Let D be a division ring with center F and $n \geq 2$ an integer. Assume that $\mathrm{GL}_n(D) \ncong \mathrm{GL}_2(\mathbb{F}_2)$. Then, $\mathrm{GL}_n(D)$ is a simple group if and only if $F \cong \mathbb{F}_2$ and $D^* = D'$.*

Proof. Assume that $\mathrm{GL}_n(D)$ is a simple group. Since $\mathrm{GL}_n(D)$ is nonabelian, we have $\mathrm{Z}(\mathrm{GL}_n(D)) = \langle \mathrm{I}_n \rangle$ and $\mathrm{GL}_n(D) = \mathrm{SL}_n(D)$. The parts (1) and (2) of Lemma 2.6 allow us to conclude that $F \cong \mathbb{F}_2$ and $D^* = D'$.

Conversely, assume that $F \cong \mathbb{F}_2$ and $D^* = D'$. Then, $\mathrm{Z}(\mathrm{GL}_n(D)) = \langle \mathrm{I}_n \rangle$ and $\mathrm{GL}_n(D) = \mathrm{SL}_n(D)$. Clearly, if $D \neq F$, then $\mathrm{GL}_n(D)$ is simple, where $n \geq 2$. If $D = F$, then part (5) of Lemma 2.6 yields that $\mathrm{SL}_n(D)$ is simple, where $n \geq 3$. Hence, $\mathrm{GL}_n(D)$ is a simple group. □

In the above lemma, whether there exists a non-commutative division ring D such that $D^* = D'$, or more generally, D^* is radical over D', is still an open problem up to now (see [14]).

Now, the following lemma helps us to recognize if $\mathrm{Diam}(\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))) = 0$. Notice that Lemma 2.7 follows that $\mathrm{Diam}(\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))) = 0$ means $\mathrm{GL}_n(D)$ has a unique trivial proper normal subgroup, where $n \geq 2$. Thanks to this lemma, together with Theorem 4.8 and 4.9, we can infer the case $\mathrm{Diam}(\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))) = 1$.

Lemma 5.2. *Let D be a division ring with center F and $n \geq 2$ an integer. Then, $\mathrm{GL}_n(D)$ has exactly one nontrivial proper normal subgroup if and only if one of the following conditions is satisfied*

(1) *$n = 2$ and $D \cong \mathbb{F}_2$;*
(2) *$F \cong \mathbb{F}_2$ and $[D^* : D']$ is a prime integer;*
(3) *$F \cong \mathbb{F}_{p+1}$ and $D^* = D'$ where either $p = 2$ or $p = 2^\ell - 1$ is a prime integer.*

Proof. Suppose that $\mathrm{GL}_n(D)$ has only one normal subgroup N such that $\langle 1 \rangle \neq N \subsetneq \mathrm{GL}_n(D)$. There are two cases to be considered:

Case 1. *D is commutative:* Lemma 5.1 says that $\mathrm{GL}_k(\mathbb{F}_2)$, where $k > 2$, is a simple group. Thus, $\mathrm{GL}_n(D) \ncong \mathrm{GL}_k(\mathbb{F}_2)$, where $k > 2$. Additionally, if $\mathrm{GL}_n(D) \ncong \mathrm{GL}_2(\mathbb{F}_2)$ and $\mathrm{GL}_n(D) \ncong \mathrm{GL}_2(\mathbb{F}_3)$, then $\mathrm{GL}_n(D)$ always contains $\mathrm{Z}(\mathrm{GL}_n(D))$ and $\mathrm{SL}_n(D)$, two nontrivial proper normal subgroups. This is a contradiction. So, $\mathrm{GL}_n(D) \cong \mathrm{GL}_2(\mathbb{F}_2)$ or $\mathrm{GL}_n(D) \cong \mathrm{GL}_2(\mathbb{F}_3)$. Thanks to Lemma 4.7, $\mathrm{GL}_n(D) \cong \mathrm{GL}_2(\mathbb{F}_2)$ and we have the assertion (1).

Case 2. *D is noncommutative:* We consider two subcases:

Subcase 2.1. *N is noncentral:* Then, $N = \mathrm{SL}_n(D)$ and $\mathrm{Z}(\mathrm{GL}_n(D)) = \langle \mathrm{I}_n \rangle$. Since $\mathrm{Z}(\mathrm{GL}_n(D)) \cong F^*$, one has $F \cong \mathbb{F}_2$. Moreover, $\mathrm{GL}_n(D)/\mathrm{SL}_n(D)$ is a simple abelian group. Since $\mathrm{GL}_n(D)/\mathrm{SL}_n(D) \cong D^*/D'$, the index $[D^* : D']$ is a prime. Thus, we have the assertion (2).

Subcase 2.2. *N is central:* This yields $\mathrm{GL}_n(D) = \mathrm{SL}_n(D)$ and $N = \mathrm{Z}(\mathrm{GL}_n(D))$ such that N is a simple abelian group. Because $\mathrm{Z}(\mathrm{GL}_n(D)) \cong F^*$, one has $F \cong \mathbb{F}_{p+1}$ where $p = 2$ or $p = 2^\ell - 1$ a prime integer. Moreover, the fact that $\mathrm{GL}_n(D) = \mathrm{SL}_n(D)$ leads to $D^* = D'$. Hence, this subcase leads to the assertion (3).

Now we turn to the converse direction. We prove if one of conditions (1), (2) or (3) holds, then $\mathrm{GL}_n(D)$ has only one subgroup N such that $\langle 1 \rangle \neq N \subsetneq G$. Indeed, if the condition (1) holds, then Lemma 4.7 indicates that $N = [\mathrm{GL}_2(\mathbb{F}_2), \mathrm{GL}_2(\mathbb{F}_2)]$. If $\mathrm{GL}_n(D)$ is under the condition (2), then $\mathrm{Z}(\mathrm{GL}_n(D)) = \langle \mathrm{I}_n \rangle$ and the index $[\mathrm{GL}_n(D) : \mathrm{SL}_n(D)]$ is a prime integer. Thus, with this condition, we have $N = \mathrm{SL}_n(D)$. Finally, if $\mathrm{GL}_n(D)$ satisfies the condition (3), then $\mathrm{Z}(\mathrm{GL}_n(D))$ is an abelian group of index p and $\mathrm{GL}_n(D) = \mathrm{SL}_n(D)$. This follows that $N = \mathrm{Z}(\mathrm{GL}_n(D))$. Hence, we complete the proof. ☐

The following lemma shows us whether $\mathrm{GL}_n(D)$ is the direct product of two simple subgroups.

Lemma 5.3. *Let D be a division ring with center F and $n \geq 2$ an integer. Then, $\mathrm{GL}_n(D)$ is the direct product of two simple subgroups if and only if the following conditions are satisfied*

(1) *F is isomorphic to $\mathbb{F}_{p+1}$, where p is a prime such that $p = 2$ or $p = 2^\ell - 1$;*
(2) *n is not a multiple of p;*
(3) *$\mathrm{Z}(D') = \langle 1 \rangle$; and*
(4) *$[D^* : D'] = p$.*

Proof. Assume that $\mathrm{GL}_n(D) = HK$, the direct product of two simple normal subgroups H and K. Since $\mathrm{GL}_n(D)$ is nonabelian, H or K must be noncentral. Suppose that both H and K are noncentral. By Lemma 4.7 and Lemma 5.1, we have $\mathrm{GL}_n(D) \ncong \mathrm{GL}_k(\mathbb{F}_2)$, where $k \geq 2$, and $\mathrm{GL}_n(D) \ncong \mathrm{GL}_2(\mathbb{F}_3)$. Therefore, H and K contain $\mathrm{SL}_n(D)$. This is a contradiction because $H \cap K = \langle \mathrm{I}_n \rangle$. Thus, without losing the generality, we can conclude that H is central and K is noncentral. Since H and K are simple, H is a cyclic group of prime order and $K = \mathrm{SL}_n(D)$. Additionally, it follows from Theorem 3.2 that $H = \mathrm{Z}(\mathrm{GL}_n(D)) \cong F^*$. Thus, F is isomorphic to $\mathbb{F}_{p+1}$, where either $p = 2$ or $p = 2^\ell - 1$ a prime integer, and so the assertion (1) is established. Because $H \cap K = \langle \mathrm{I}_n \rangle$, the assertion (3) of Lemma 2.6 leads to the fact that n is not a multiple of p and $\mathrm{Z}(D') = \langle 1 \rangle$. Thus, we proved assertions (2) and (3). Finally, notice that

$$D^*/D' \cong \mathrm{GL}_n(D)/\mathrm{SL}_n(D) = HK/K \cong H \cong F^*.$$

Consequently, $[D^* : D'] = p$, and so we proved the assertion (4).

We turn to the converse of the statement. The hypothesis (1) leads to the fact that $\mathrm{Z}(\mathrm{GL}_n(D))$ is a simple group. Then, $\mathrm{Z}(\mathrm{GL}_n(D))$ is the only nontrivial central normal subgroup of $\mathrm{GL}_n(D)$. Meanwhile, $[\mathrm{GL}_n(D) : \mathrm{SL}_n(D)] = p$ due to the condition (4). This means $\mathrm{SL}_n(D)$ is a maximal normal subgroup of $\mathrm{GL}_n(D)$. According to the assertion (3) of Lemma 2.6, conditions (2) and (3) yield that $\mathrm{Z}(\mathrm{GL}_n(D)) \cap \mathrm{SL}_n(D) = \mathrm{Z}(\mathrm{SL}_n(D)) = \langle \mathrm{I}_n \rangle$. It follows from part (5) of Lemma 2.6 that $\mathrm{SL}_n(D)$ is simple. Moreover, because of the maximality of $\mathrm{SL}_n(D)$ in $\mathrm{GL}_n(D)$, we have $\mathrm{GL}_n(D) = \mathrm{Z}(\mathrm{GL}_n(D)) \cdot \mathrm{SL}_n(D)$. Hence, $\mathrm{GL}_n(D)$ is the direct product of $\mathrm{Z}(\mathrm{GL}_n(D))$ and $\mathrm{SL}_n(D)$, and so our proof is complete. □

Theorem 5.4. *Let D be a division ring with center F and $n \geq 2$ an integer. Then, the intersection graph of quasinormal subgroups $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is classified by the diameter as Table 1.*

Table 1. Classification of $\mathrm{GL}_n(D)$ by $\mathrm{Diam}(\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D)))$, where $n \geq 2$

No.	***Assumptions***			***Diameter of***
	n	F	D	$\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$
(1)	$n \geq 2$	$F \cong \mathbb{F}_2$	$D^* = D' \neq \langle 1 \rangle$	$\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ *is null*
(2)	$n \geq 3$	$D = F \cong \mathbb{F}_2$		
(3)	$n \geq 2$, *and* $p \nmid n$	$F \cong \mathbb{F}_{p+1}$	$\mathrm{Z}(D') = \langle 1 \rangle$, *and* $[D^* : D'] = p$	∞
(4)	$n = 2$	$F \cong \mathbb{F}_2$	$D \cong \mathbb{F}_2$	0
(5)	$n \geq 2$	$F \cong \mathbb{F}_2$	$[D^* : D']$ *is a prime integer*	
(6)	$n \geq 2$	$F \cong \mathbb{F}_{p+1}$	$D^* = D'$	
(7)	$n \geq 2$	$F \cong \mathbb{F}_9$, $\mathbb{F}_{2^k+1}$, $\mathbb{F}_{p+1}$	$\mathrm{Z}(D') \neq \langle 1 \rangle$	1
(8)	$n \geq 2$	$F \cong \mathbb{F}_2$	$D^* \neq D'$ *and* $[D^* : D']$ *is not a prime integer*	
(9)	n *is even*	$F \cong \mathbb{F}_9$, $\mathbb{F}_{2^k+1}$	$\mathrm{Z}(D') = \langle 1 \rangle$	
(10)	$n \geq 2$, *and* $p \mid n$	$F \cong \mathbb{F}_{p+1}$	$\mathrm{Z}(D') = \langle 1 \rangle$	
(11)	*All assumptions for $n \geq 2$, F and D were not listed above*			2

where k is some positive integer such that $2^k + 1$ is prime and p is either 2 or some prime integer of the form $2^\ell - 1$.

Proof. The cases (1) and (2) is Lemma 5.1. The case (3) is Lemma 5.3. If $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))$ is complete, then $\mathrm{Diam}(\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D)))$ is either 0 or 1. Lemma 5.2 fully describes the case $\mathrm{Diam}(\Gamma_{\mathrm{q}}(\mathrm{GL}_n(D))) = 0$, and so we have the assertions (4), (5), and (6). Hence, by Theorem 4.8 and 4.9, we have the assertions (7), (8), (9), and (10). □

From the above theorem, we can describe specifically the case where $D = F$ is a field. Furthermore, the following lemma helps us to fully enumerate $\mathrm{GL}_n(F)$ for all $n \geq 1$.

Lemma 5.5. *Let F be a field. Then,*

(1) $\Gamma_{\mathrm{q}}(F^*)$ *is null if and only if F is isomorphic to either $\mathbb{F}_2$ or $\mathbb{F}_{q+1}$, where q is a prime integer;*

(2) $\mathrm{Diam}(\Gamma_{\mathrm{q}}(F^*)) = \infty$ *if and only if one of following conditions holds*

  (a) $|F| = c$, *where c is a odd prime integer such that $(c-1)/2$ is also a odd prime integer;*

  (b) $|F| = 3^k$, *where k is a positive integer such that $(3^k - 1)/2$ is a prime integer; or*

  (c) $|F| = 2^k$, *where k is a positive integer such that $2^k - 1$ is a product of two prime integers.*

(3) $\mathrm{Diam}(\Gamma_{\mathrm{q}}(F^*)) = 0$ *if and only if* $F \cong \mathbb{F}_5$*;*

(4) $\mathrm{Diam}(\Gamma_{\mathrm{q}}(F^*)) = 1$ *if and only if* $F \cong \mathbb{F}_9$ *or* $F \cong \mathbb{F}_{2^k+1}$*, where* $2^k + 1$ *is a prime integer and* $k \geq 3$*;*

Proof. First, we prove (1). If $\Gamma_{\mathrm{q}}(F^*)$ is null, then F is a finite field such that F^* is trivial or simple, i.e. $|F^*|$ is 1 or q, where q is a prime integer. The converse is trivial.

Next, we consider the assertion (2). Assume that $\Gamma_{\mathrm{q}}(F^*)$ is disconnected. By Theorem 3.2, F^* is the direct product of simple subgroups H and K of F^*. Since H and K are abelian, there exist some prime integers p and q such that $|H| = p$, $|K| = q$, and so $|F^*| = pq$. It is easy to see that $p \neq q$ because F^* is cyclic. We can suppose that $|F| = c^k$ where c is a prime integer and k is a positive integer. If $k = 1$, then $c - 1 = pq$. This implies that c and $(c-1)/2$ are odd prime integers, and so (a) holds. If $k > 1$, then $pq = c^k - 1 = (c-1)(c^{k-1} + \cdots + 1)$. If $c - 1 = 1$ then $c = 2$, that is $|F^*| = 2^k - 1 = pq$, and (c) is asserted. If $c - 1 \neq 1$, then, without losing the generality, we can say that $p = c - 1$. Since p and c are prime, we have $c = 3$ and $p = 2$. Hence, $q = (3^k - 1)/2$ is a prime integer, and so (b) holds.

The converse of part (2) is proved quite easily.

Finally, we prove assertions (3) and (4). Clearly, $\Gamma_{\mathrm{q}}(F^*)$ is a complete graph if and only if $\mathrm{Diam}(\Gamma_{\mathrm{q}}(F^*)) = 0$ or $\mathrm{Diam}(\Gamma_{\mathrm{q}}(F^*)) = 1$. Theorem 4.6 leads to the fact that if $\Gamma_{\mathrm{q}}(F^*)$ is complete, then F^* is a cyclic group of order 2^k, where $k \geq 2$. From that, we have $\mathrm{Diam}(\Gamma_{\mathrm{q}}(F^*)) = 0$ if $k = 2$ and $\mathrm{Diam}(\Gamma_{\mathrm{q}}(F^*)) = 1$ if $k > 2$. Hence, the proof of assertions (3) and (4) is complete. □

By the combination of Theorem 5.4 and Lemma 5.5, we have the following corollary.

Corollary 5.6. *Let F be a field and n a positive integer. Then, the intersection graph of quasinormal subgroups $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(F))$ is classified by the diameter as Table 2.*

Table 2. Classification of $\mathrm{GL}_n(F)$ **by** $\mathrm{Diam}(\Gamma_{\mathrm{q}}(\mathrm{GL}_n(F)))$

No.	***Assumptions***		***Diameter of*** $\Gamma_{\mathrm{q}}(\mathrm{GL}_n(F))$
	n	F	
(1)	$n = 1$	$F \cong \mathbb{F}_2$, $\mathbb{F}_{q+1}$, *where q is a prime integer*	$\Gamma_{\mathrm{q}}(\mathrm{GL}_n(F))$ *is null*
(2)	$n > 2$	$F \cong \mathbb{F}_2$	
(3)	$n = 1$	$F \cong \mathbb{F}_q$, *where q and $(q-1)/2$ are odd prime integers*	∞
(4)	$n = 1$	$F \cong \mathbb{F}_{3^m}$, *where $(3^m - 1)/2$ is prime*	
(5)	$n = 1$	$F \cong \mathbb{F}_{2^m}$, *where $2^m - 1$ is a product of two prime integers*	
(6)	$n \geq 2$ *and* $p \nmid n$	$F \cong \mathbb{F}_{p+1}$	
(7)	$n = 1$	$F \cong \mathbb{F}_5$	0
(8)	$n = 2$	$F \cong \mathbb{F}_2$	
(9)	$n = 1$	$F \cong \mathbb{F}_9$, $\mathbb{F}_{2^k+1}$, $(k \geq 3)$	1
(10)	n *is even*	$F \cong \mathbb{F}_9$, $\mathbb{F}_{2^k+1}$	
(11)	$p \mid n$	$F \cong \mathbb{F}_{p+1}$	
(12)	*All assumptions of n and F were not listed above*		2

where k is some positive integer such that $2^k + 1$ is prime and p is either 2 or some prime integer of the form $2^\ell - 1$.

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