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CONTRA SEMI-I-CONTINUOUS FUNCTIONS

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Abstract

In this paper, we apply the notion of *semi-I-open* sets in ideal topological spaces to present and study a new class of functions called *contra semi-I-continuous* functions. Relationships between this new class and other classes of functions are established.

Keywords: Ideal topological space, Semi-I-open sets, Contra semi-I-continuous functions, Contra I-irresolute functions, Perfectly contra I-irresolute functions.

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1. Introduction

In 1996, Dontchev [1] introduced a new class of functions called contra-continuous functions. He defined a function $f : X \to Y$ to be *contra-continuous* if the preimage of every open set of Y is closed in X. A new weaker form of this class of functions, called *contra-semi-continuous* functions, is introduced and investigated by Dontchev and Noiri [2]. In this direction, we will introduce the concept of *contra semi-I-continuous* functions via the notion of semi-I-open sets. We also obtain some properties of such functions.

The subject of ideals in topological spaces has been studied by Kuratowski [6] and Vaidyanathaswamy [9]. An ideal on a topological space (X, τ) is defined as a non-empty collection I of subsets of X satisfying the following two conditions:

(1) If $A \in I$ and $B \subseteq A$, then $B \in I$;
(2) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

An ideal topological space is a topological space (X, τ) with an ideal I on X, and is denoted by (X, τ, I). For a subset $A \subseteq X$, the set

$$A^*(I) = \{x \in X : U \cap A \notin I\}$$

for every $U \in \tau$ with $x \in U$, is called the *local function* of A with respect to I and τ [5]. We simply write A^* instead of $A^*(I)$ in case there is no chance of confusion. It is well known that $\mathrm{Cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

Throughout this paper, for a subset A of a topological space (X, τ), $\mathrm{Cl}\,(A)$ and $\mathrm{Int}\,(A)$ denote the closure and the interior of A, respectively.

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A subset A of an ideal topological space (X, τ, I) is said to be *semi-I-open* [3] (resp., *semi-open* [7], *I-open* [5]) if $A \subseteq \mathrm{Cl}^*(\mathrm{Int}(A))$ (resp., $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$, $A \subseteq \mathrm{Int}(A^*)$). The *complement* of a semi-I-open (resp., semi-open, I-open) set is called *semi-I-closed* [3] (resp., *semi-closed* [7], I-closed [5]). If the set A is semi-I-open and semi-I-closed, then it is called *semi-I-clopen*.

2. Contra semi-I-continuous functions

2.1. Definition. A function $f : (X, \tau, I) \to (Y, \rho)$ is called *contra semi-I-continuous* if $f^{-1}(V)$ is semi-I-closed in X for each open set V in Y.

2.2. Theorem. *For a function $f : (X, \tau, I) \to (Y, \rho)$, the following are equivalent:*

a) *f is contra semi-I-continuous.*
b) *For every closed subset F of Y, $f^{-1}(F)$ is semi-I-open in X.*
c) *For each $x \in X$ and each closed subset F of Y with $f(x) \in F$, there exists a semi-I-open subset U of X with $x \in U$ such that $f(U) \subseteq F$.*

Proof. The implications (a) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (c) are obvious.

(c) $\Longrightarrow$ (b) Let F be any closed subset of Y. If $x \in f^{-1}(F)$ then $f(x) \in F$, and there exists a semi-I-open subset U_x of X with $x \in U_x$ such that $f(U_x) \subseteq F$. Therefore, we obtain $f^{-1}(F) = \cup\{U_x : x \in f^{-1}(F)\}$. Now, by [4, Theorem 3.4] we have that $f^{-1}(F)$ is semi-I-open. □

Recall that a function $f : (X, \tau, I) \to (Y, \rho)$ is called *contra semi-continuous* [2] if the preimage of every open subset of Y is semi-closed in X.

Since every semi-I-open set is semi-open, then every contra semi-I-continuous function is contra semi-continuous, but the converse need not be true as shown by the following example:

2.3. Example. Let $X = \{1, 2, 3, 4\}$, $\tau = \{\emptyset, X, \{2\}, \{3\}, \{2, 3\}, \{1, 2\}, \{1, 2, 3\}, \{2, 3, 4\}\}$ and $I = \{\emptyset, \{3\}\}$. Let $f : (X, \tau, I) \to (X, \tau)$ be defined by $f(1) = 2$, $f(2) = 1$, $f(3) = 4$ and $f(4) = 3$. Observe that f is contra semi-continuous. But f is not contra semi-I-continuous, since $\{1, 2\}$ is open and $f^{-1}(\{1, 2\}) = \{1, 2\}$ is not semi-I-closed.

A function $f : (X, \tau, I) \to (Y, \rho)$ is called *contra I-continuous* if the preimage of every open subset of Y is an I-closed subset of X.

The following two examples show that the concepts of contra-I-continuity and contra semi-I-continuity are independent of each other.

2.4. Example. Let $X = \{1, 2\}$, $\tau = \{\emptyset, X, \{1\}\}$ and $I = \{\emptyset, \{1\}\}$. Let $f : (X, \tau, I) \to (X, \tau)$ be defined by $f(1) = 2$ and $f(2) = 1$. Observe that f is contra semi-I-continuous. But f is not contra-I-continuous since $\{1\}$ is open and $f^{-1}(\{1\}) = \{2\}$ is not I-closed.

2.5. Example. Let $X = Y = \{1, 2\}$, $\tau = \{\emptyset, X\}$, $I = \{\emptyset, \{1\}\}$ and $\rho = \{\emptyset, Y, \{1\}\}$. Let $f : (X, \tau, I) \to (Y, \rho)$ be the identity function. Observe that f is contra I-continuous. But f is not contra semi-I-continuous, since $\{1\}$ is open in Y and $f^{-1}(\{1\}) = \{1\}$ is not semi-I-closed in X.

Recall that a function $f : (X, \tau, I) \to (Y, \rho)$ is called *semi-I-continuous* [3] if the preimage of every open subset of Y is semi-I-open in X.

The following two examples show that the concepts of semi-I-continuity and contra semi-I-continuity are independent of each other.

2.6. Example. Let $X = \{1,2\}$, $\tau = \{\emptyset, X, \{1\}\}$ and $I = \{\emptyset, \{2\}\}$. Let $f : (X,\tau,I) \to (X,\tau)$ be defined by $f(1) = 2$ and $f(2) = 1$. Observe that f is contra semi-I-continuous. But f is not semi-I-continuous, since $\{1\}$ is open and $f^{-1}(\{1\}) = \{2\}$ is not semi-I-open.

2.7. Example. Let $X = \mathbb{R}$, τ the usual topology and $I = \{\emptyset, \{1\}\}$. The identity function $f : (R,\tau,I) \to (R,\tau)$ is semi-I-continuous. But f is not contra semi-I-continuous, since the inverse image of $(0,1)$ is not semi-I-closed.

2.8. Theorem. *If a function $f : (X,\tau,I) \to (Y,\rho)$ is contra semi-I-continuous and Y is regular, then f is semi-I-continuous.*

Proof. Let $x \in X$ and let V be an open subset of Y with $f(x) \in V$. Since Y is regular, there exists an open set W in Y such that $f(x) \in W \subseteq \mathrm{Cl}\,(W) \subseteq V$. Since f is contra semi-I-continuous, by Theorem 2.2. there exists a semi-I-open set U in X with $x \in U$ such that $f(U) \subseteq \mathrm{Cl}\,(W)$. Then $f(U) \subseteq \mathrm{Cl}\,(W) \subseteq V$. Hence, by [4, Theorem 4.1], f is semi-I-continuous. □

Note that if $f : (X,\tau,I) \to (Y,\rho)$ is semi-I-continuous and Y is regular, then f need not be contra semi-I-continuous, as shown in Example 2.7.

2.9. Definition. A topological space (X,τ,I) is said to be *semi-I-connected* if X is not the union of two disjoint non-empty semi-I-open subsets of X.

2.10. Theorem. *If $f : (X,\tau,I) \to (Y,\rho)$ is a contra semi-I-continuous function from a semi-I-connected space X onto any space Y, then Y is not a discrete space.*

Proof. Suppose that Y is discrete. Let A be a proper non-empty clopen set in Y. Then $f^{-1}(A)$ is a proper non-empty semi-I-clopen subset of X, which contradicts the fact that X is semi-I-connected. □

2.11. Theorem. *A contra semi-I-continuous image of a semi-I-connected space is connected.*

Proof. Let $f : (X,\tau,I) \to (Y,\rho)$ be a contra semi-I-continuous function from a semi-I-connected space X onto a space Y. Assume that Y is disconnected. Then $Y = A \cup B$, where A and B are non-empty clopen sets in Y with $A \cap B = \emptyset$. Since f is contra semi-I-continuous, we have that $f^{-1}(A)$ and $f^{-1}(B)$ are semi-I-open non-empty sets in X with $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$ and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$. This means that X is not semi-I-connected, which is a contradiction. Then Y is connected. □

2.12. Theorem. *Let (X,τ,I) be a semi-I-connected space and (Y,ρ) a T_1-space. If $f : (X,\tau,I) \to (Y,\rho)$ is a contra semi-I-continuous function, then f is constant.*

Proof. Let $\Delta = \{f^{-1}(\{y\}) : y \in Y\}$. Since (Y,ρ) is T_1, Δ is a disjoint semi-I-open partition of X. If $|\Delta| \geq 2$, then X is the union of two non-empty semi-I-open sets. Since (X,τ,I) is semi-I-connected, $|\Delta| = 1$. Therefore, f is constant. □

2.13. Definition. A space (X,τ,I) is said to be *semi-I-T_2* if for each pair of distinct points x and y in X, there exists two semi-I-open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

2.14. Theorem. *Let $f : (X,\tau,I) \to (Y,\rho)$ be a contra semi-I-continuous injection. If Y is a Urysohn space, then X is semi-I-T_2.*

Proof. Let x and y be a pair of distinct points in X. Then $f(x) \neq f(y)$. Since Y is a Urysohn space, there exist open sets U and V of Y such that $f(x) \in U$, $f(y) \in V$ and $\mathrm{Cl}\,(U) \cap \mathrm{Cl}\,(V) = \emptyset$. Since f is contra semi-I-continuous at x and y, there exist semi-I-open sets A and B in X such that $x \in A$, $y \in B$ and $f(A) \subseteq \mathrm{Cl}\,(U)$, $f(B) \subseteq \mathrm{Cl}\,(V)$. Then, $f(A) \cap f(B) = \emptyset$, so $A \cap B = \emptyset$. Hence, X is semi-I-T_2. □

2.15. Definition. A space (X, τ, I) is said to be semi-I-normal if each pair of non-empty disjoint closed sets can be separated by disjoint semi-I-open sets.

2.16. Definition. [8] A space (X, τ) is said to be *ultra normal* if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

2.17. Theorem. *If $f : (X, \tau, I) \to (Y, \rho)$ is a contra semi-I-continuous, closed injection and Y is ultra normal, then X is semi-I-normal.*

Proof. Let C_1 and C_2 be disjoint closed subsets of X. Since f is closed and injective, $f(C_1)$ and $f(C_2)$ are disjoint closed subsets of Y. But Y is ultra normal, so $f(C_1)$ and $f(C_2)$ are separated by disjoint clopen sets V_1 and V_2, respectively. Since f is contra semi-I-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are semi-I-open, with $C_1 \subseteq f^{-1}(V_1)$, $C_2 \subseteq f^{-1}(V_2)$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Hence, X is semi-I-normal. □

2.18. Definition. [1] A space (X, τ) is said to be *strongly S-closed* if every closed cover of X has a finite subcover.

2.19. Definition. A space (X, τ, I) is said to be semi-I-compact if every semi-I-open cover of X has a finite subcover.

2.20. Definition. A subset A of a space (X, τ, I) is said to be *semi-I-compact relative to X* if every cover of A by semi-I-open subsets of X has a finite subcover.

2.21. Theorem. *If $f : (X, \tau, I) \to (Y, \rho)$ is contra semi-I-continuous and K is semi-I-compact relative to X, then $f(K)$ is strongly S-closed in Y.*

Proof. Let $\{C_\alpha : \alpha \in \Delta\}$ be any cover of $f(K)$ by closed subsets of the subspace $f(K)$. For each $\alpha \in \Delta$, there exists a closed set F_α of Y such that $C_\alpha = F_\alpha \cap f(K)$. For each $x \in K$, there exists $\alpha_x \in \Delta$ such that $f(x) \in F_{\alpha_x}$. Now by Theorem 2.2, there exists a semi-I-open set U_x of X with $x \in U_x$ such that $f(U_x) \subseteq F_{\alpha_x}$.

Since the family $\{U_x : x \in K\}$ is a cover of K by sets semi-I-open in X, there exists a finite subset K_0 of K such that $K \subseteq \bigcup\{U_x : x \in K_0\}$. Therefore we obtain $f(K) \subseteq \bigcup\{f(U_x) : x \in K_0\}$, which is a subset of $\bigcup\{F_{\alpha_x} : x \in K_0\}$. Thus $f(K) \subseteq \bigcup\{C_{\alpha_x} : x \in K_0\}$, and hence $f(K)$ is strongly S-closed. □

2.22. Corollary. *If $f : (X, \tau, I) \to (Y, \rho)$ is a contra semi-I-continuous surjection and X is semi-I-compact, then Y is strongly S-closed.*

2.23. Definition. A function $f : (X, \tau, I) \to (Y, \rho, J)$ is called *contra I-irresolute* if $f^{-1}(V)$ is semi-I-closed in X for each semi-J-open set V of Y.

Recall that a function $f : (X, \tau, I) \to (Y, \rho, J)$ is said to be *I-irresolute* [4] if $f^{-1}(V)$ is semi-I-open in X for each semi-J-open set V of Y. In fact, contra I-irresoluteness and I-irresoluteness are independent, as shown by the following two examples.

2.24. Example. The function f in Example 2.6 is contra I-irresolute but not I-irresolute.

2.25. Example. Let (X, τ, I) be the ideal topological space in Example 2.6. Then the identity function on the space (X, τ, I) is an example of an I-irresolute function which is not contra I-irresolute.

Also, every contra I-irresolute function is contra semi-I-continuous, but the converse is not true as shown by the following example.

2.26. Example. Let $X = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{1\}\}$ and $I = \{\emptyset, \{2\}\}$. Define a function $f : (X, \tau, I) \to (X, \tau, I)$ by $f(1) = 2$, $f(2) = 1$ and $f(3) = 3$. Then f is contra semi-I-continuous but f is not contra I-irresolute.

The following results can be easily verified and the proofs are omitted.

2.27. Theorem. *A function $f : (X, \tau, I) \to (Y, \rho, J)$ is contra I-irresolute if and only if the inverse image of each semi-J-closed set in Y is semi-I-open in X.* □

2.28. Theorem. *Let $f : (X, \tau, I) \to (Y, \rho, J)$ and $g : (Y, \rho, J) \to (Z, \sigma, K)$. Then*

1) *gof is contra I-irresolute if g is I-irresolute and f is contra I-irresolute.*
2) *gof is contra I-irresolute if g is contra I-irresolute and f is I-irresolute.* □

2.29. Theorem. *Let $f : (X, \tau, I) \to (Y, \rho, J)$ and $g : (Y, \rho, J) \to (Z, \sigma)$. Then*

1) *gof is contra semi-I-continuous if g is continuous and f is contra semi-I-continuous.*
2) *gof is contra semi-I-continuous if g is semi-I-continuous and f is contra I-irresolute.* □

Recall that a function $f : (X, \tau) \to (Y, \rho, J)$ is called *semi-I-open* [4] if for each $U \in \tau$, $f(U)$ is semi-I-open in Y.

2.30. Theorem. *Let $f : (X, \tau, I) \to (Y, \rho, J)$ be surjective, I-irresolute and semi-I-open, and let $g : (Y, \rho, J) \to (Z, \sigma)$ be any function. Then gof is contra semi-I-continuous if and only if g is contra semi-I-continuous.*

Proof. $\Longrightarrow$ Let gof be contra semi-I-continuous and C a closed subset of Z. Then $(gof)^{-1}(C)$ is a semi-I-open subset of X. That is $f^{-1}(g^{-1}(C))$ is semi-I-open. Since f is semi-I-open, $f(f^{-1}(g^{-1}(C)))$ is a semi-I-open subset of Y. So $g^{-1}(C)$ is semi-I-open in Y. Therefore, g is contra semi-I-continuous.

$\Longleftarrow$ Straightforward. □

2.31. Definition. A function $f : (X, \tau, I) \to (Y, \rho, J)$ is called *perfectly contra I-irresolute* if the inverse image of every semi-J-open set in Y is semi-I-clopen in X.

2.32. Example. Let $X = Y = \{1, 2, 3, 4\}$, $\tau = \{\emptyset, X, \{2\}, \{4\}, \{2, 4\}\}$, $I = \{\emptyset, \{4\}\}$, $\rho = \{\emptyset, Y, \{2\}\}$ and $J = \{\emptyset, \{2\}\}$. Let $f : (X, \tau, I) \to (Y, \rho, J)$ be defined by $f(1) = 3$, $f(2) = 4$, $f(3) = 1$ and $f(4) = 2$. Then f is perfectly contra I-irresolute.

Every perfectly contra I-irresolute function is contra I-irresolute and I-irresolute. The following two examples show that a contra I-irresolute function may not be perfectly contra I-irresolute, and an I-irresolute function may not be perfectly contra I-irresolute.

2.33. Example. The function in Example 2.6 is contra I-irresolute, but not perfectly contra I-irresolute.

2.34. Example. Let $X = \{1, 2, 3\}$, $\tau = \{\emptyset, X, \{1\}\}$ and $I = \{\emptyset, \{2\}\}$. Let $f : (X, \tau, I) \to (X, \tau, I)$ be the identity function. Then f is I-irresolute, but not perfectly contra I-irresolute.

2.35. Theorem. *For a function $f : (X, \tau, I) \to (Y, \rho, J)$ the following conditions are equivalent:*

1) *f is perfectly contra I-irresolute.*
2) *f is contra I-irresolute and I-irresolute.* □

References

[1] Dontchev, J. *Contra-continuous functions and strongly S-closed spaces*, Internat. J. Math. Math. Sci. **19**, 303–310, 1996.

[2] Dontchev, J. and Noiri, T. *Contra-semicontinuous functions*, Mathematica Pannonica 10(2), 159-168, 1999.

[3] Hatir, E. and Noiri, T. *On decomposition of continuity via idealization*, Acta Math. Hungar. **96**, 341–349, 2002.

[4] Hatir, E. and Noiri, T. *On semi-I-open sets and semi-I-continuous functions*, Acta Math. Hungar. **107** (4), 345–353, 2005.

[5] Janković, D. and Hamlett, T. R. *Compatible extensions of ideals*, Boll. Un. Mat. Ital. **7** (6-B), 453–465, 1992.

[6] Kuratowski, K. *Topology, Vol.I* (Academic Press, New York, 1966).

[7] Levine, N. *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly **70**, 36–41, 1963.

[8] Staum, R. *The algebra of bounded continuous functions into a non-archimedean field*, Pacific J. Math. 50, 169–185, 1974.

[9] Vaidyanathaswamy, R. *The localization theory in set topology*, Proc. Indian Acad. Sci. **20**, 51–61, 1954.