Hacettepe Journal of Mathematics and Statistics
Volume 39 (3) (2010), 295 – 303

A STUDY ON RARE $\mathrm{m_X}$ SETS AND RARELY $\mathrm{m_X}$ CONTINUOUS FUNCTIONS

Sharmistha Bhattacharya (Halder)*

Received 15 : 02 : 2008 : Accepted 28 : 10 : 2009

Abstract

The aim of this paper is to introduce the concept of rare m_X sets, dense m_X sets and rarely m_X continuous functions. Some properties of these sets are studied and it is shown that only the null set is both a rare m_X set and an open m_X set.

Keywords: m_X set, Rare set, Rarely continuous function.

2000 AMS Classification: 54 A 05.

1. Introduction

The family of all semi-open (resp. preopen, α - open, β - open, regular open) sets in (X, τ) is denoted by SO(X) (resp. PO(X), $\alpha(X)$, $\beta(X)$, RO(X)). Obviously, SO(X) (PO(X), $\alpha(X)$, $\beta(X)$ and RO(X)) are minimal structures on X. The concept of m_X - open sets and m_X - closed sets in ordinary topological space has been introduced by Popa and Noiri [10]. It is to be noted that none of these sets forms even a supra topological space. So, a new concept of open m_X set and closed m_X set are defined which form a supra topological space and a dual supra topological space. Then, a new concept of rare m_X [resp., dense m_X] set is introduced, and it is shown that only the null [resp., the universal] set is a rare m_X [resp dense m_X] set and an open m_X [resp closed m_X] set jointly.

It can be shown that these sets form a pseudo [resp., dual pseudo] supra topological space. The concept of closed rare m_X set and open dense m_X set are introduced. Also the concept of rarely m_X-m_Y continuous function is introduced, and it can be shown that if (X, m_X), (Y, m_Y) are topological spaces then rarely continuous functions and rarely m_X-m_Y continuous functions coincide.

In the next section, some of the important preliminaries required are cited. These are required to proceed further through this paper.

*Department of Mathematics, Tripura University, Suryamaninagar, Tripura, India.
E-mail: halder_731@rediffmail.com

2. Preliminaries

In this section some of the important required preliminaries are given.

2.1. Definition. [10] A subfamily m_X of $P(X)$ is called a *minimal structure* on X if $\emptyset \in \mathrm{m}_X$ and $X \in \mathrm{m}_X$. Each member of m_X is said to be an m_X *open set*, and the complement of an m_X open set is said to be an m_X *closed set*. We denote by (X, m_X) a nonempty set X with a minimal structure m_X on X.

2.2. Definition. [10] Let X be a nonempty set and m_X a minimal structure on X. For a subset A of X, $\mathrm{m}_X\text{-}\mathrm{Cl}\,A$ and $\mathrm{m}_X\text{-}\mathrm{Int}\,A$ are defined as follows:

$$\mathrm{m}_X\text{-}\mathrm{Cl}\,A = \bigcap\{F : A \subseteq F,\ X \setminus F \in \mathrm{m}_X\},$$
$$\mathrm{m}_X\text{-}\mathrm{Int}\,A = \bigcup\{G : G \subseteq A,\ G \in \mathrm{m}_X\}.$$

2.3. Theorem. [10] *Let (X, m_X) be a minimal structure. For any subsets A, B of X the following hold:*

(a) $\mathrm{m}_X\text{-}\mathrm{Int}\,(X \setminus A) = X \setminus \mathrm{m}_X\text{-}\mathrm{Cl}\,(A)$.
(b) $A \in \mathrm{m}_X \implies \mathrm{m}_X\text{-}\mathrm{Int}\,A = A$.
(c) $X \setminus A \in \mathrm{m}_X \implies \mathrm{m}_X\text{-}\mathrm{Cl}\,A = A$.
(d) *If* $A \subseteq B$, $\mathrm{m}_X\text{-}\mathrm{Int}\,A \subseteq \mathrm{m}_X\text{-}\mathrm{Int}\,B$ *and* $\mathrm{m}_X\text{-}\mathrm{Cl}\,A \subseteq \mathrm{m}_X\text{-}\mathrm{Cl}\,B$.
(e) $\mathrm{m}_X\text{-}\mathrm{Int}\,(\mathrm{m}_X\text{-}\mathrm{Int}\,A) = \mathrm{m}_X\text{-}\mathrm{Int}\,A$ *and* $\mathrm{m}_X\text{-}\mathrm{Cl}\,(\mathrm{m}_X\text{-}\mathrm{Cl}\,A) = \mathrm{m}_X\text{-}\mathrm{Cl}\,A$.

2.4. Definition. [6] A *rare set* is a set S such that $\mathrm{Int}S = \emptyset$, and a *dense set* is a set S such that $\mathrm{Cl}\,S = X$.

2.5. Definition. [9] A function $f : X \longrightarrow Y$ is called a *rarely continuous function* if for each $x \in X$ and each $G \in O(Y, f(x))$, there exist a rare set R_G with $G \cap \mathrm{Cl}(R_G) = \emptyset$ and $U \in O(X, x)$ such that $f(U) \subseteq G \cup R_G$.

2.6. Definition. [3] A topological space (X, T) is said to be *sub maximal* if every dense subset of X is open in X.

2.7. Definition. [7] A minimal structure m_X on X is said to *have property (B)* if every union of m_X - open sets is a m_X - open set

2.8. Definition. [11] Let X be a nonempty set, which has a minimal structure m_X, and let A be a subset of X. The m_X - frontier of A, denoted by $\mathrm{m}_X\text{-}\mathrm{Fr}(A)$, is defined by $\mathrm{m}_X\text{-}\mathrm{Fr}(A) = \mathrm{m}_X\text{-}\mathrm{Cl}\,(A) \cap \mathrm{m}_X\text{-}\mathrm{Cl}\,(X - A)$

2.9. Definition. [2] Let $f : (X, \mathrm{m}_X) \longrightarrow (Y, \mathrm{m}_Y)$ be a mapping from a minimal structure (X, m_X) to a minimal structure (Y, m_Y). Then, f is said to be an m_X - *continuous mapping* if the inverse image of an m_X open set is an m_X open set.

Throughout the paper, an ordinary topological space is denoted as T and the complement of a set A is denoted as A^C.

3. On open m_X sets

In this section, the concepts of open m_X set and closed m_X set are introduced and the corresponding spaces formed by this sets is identified.

3.1. Definition. Let m_x be a minimal structure on X. A subset A of X is said to be an *open* m_X [resp., *closed* m_X] *set* if $\mathrm{m}_X\text{-}\mathrm{Int}\,A = A$ [resp., $\mathrm{m}_X\text{-}\mathrm{Cl}\,A = A$].

3.2. Example. Let $X = \{a, b, c\}$, $\mathrm{m}_X = \{\emptyset, X, \{a\}, \{b\}, \{c\}\}$. Let $A = \{b, c\}$. Then $\mathrm{m}_X\text{-}\mathrm{Int}\,A = \bigcup\{G : G \subseteq A, G \in \mathrm{m}_X\} = \{b\} \cup \{c\} \cup \emptyset = \{b, c\} = A$. Therefore A is an open m_X set. Similarly, A^C is a closed m_X set. But A is not an m_X open set

3.3. Theorem. *A subset A of X is an open m_X set iff $X \setminus A$ is a closed m_X set.*

Proof. Let A be an open m_X set. Then from the definition, $\mathrm{m}_X\text{-}\operatorname{Int} A = A$ iff $X \setminus [\mathrm{m}_X\text{-}\operatorname{Int} A] = X \setminus A$ iff $\mathrm{m}_X\text{-}\operatorname{Cl}(X \setminus A) = X \setminus A$ iff $X \setminus A$ is a closed m_X set. □

3.4. Remark. From Theorem 2.3 (e) it is clear that $\mathrm{m}_X\text{-}\operatorname{Int} A$ [resp., $\mathrm{m}_X\text{-}\operatorname{Cl} A$] is an open m_X [resp., a closed m_X] set.

3.5. Remark. If a subset A of X is an m_X open [resp m_X closed] set, then it is an open [resp., a closed] m_X set, but not conversely which follows from the Example 3.2.

3.6. Theorem. *In a minimal structure (X, m_X) having property B, m_X open sets and open m_X sets coincide.*

Proof. Since arbitrary union of open m_X sets is an m_X open set, so, $\mathrm{m}_X\text{-}\operatorname{Int} A = A$ is an m_X open set. That is, an open m_X set is an m_X open set. □

3.7. Remark. Let (X, m_X) be a minimal structure formed from the topological space (X, T). Then if a subset A of (X, m_X) is a closed set then it is a closed m_X set since $\mathrm{Cl}A \subseteq \mathrm{m}_X\text{-}\operatorname{Cl} A$. Similarly, if a subset A of (X, m_X) is an open set then it is an open m_X set.

3.8. Theorem. *The collection of all open m_X sets forms a supra topological space.*

Proof. The sets $X, \emptyset$ are obviously open m_X sets. Let $A = \bigcup \{A_\alpha : \alpha \in \Delta\}$ be a union of open m_X sets, i.e. $\mathrm{m}_X\text{-}\operatorname{Int}(A_\alpha) = A_\alpha$ for each α. Now, $A_\alpha \subseteq A$ so $A_\alpha = \mathrm{m}_X\text{-}\operatorname{Int} A_\alpha \subseteq \mathrm{m}_X\text{-}\operatorname{Int} A$ for each α. Hence

$$A = \bigcup \{A_\alpha : \alpha \in \Delta\} \subseteq \mathrm{m}_X\text{-}\operatorname{Int} A,$$

which gives $A \subseteq \mathrm{m}_X\text{-}\operatorname{Int} A$. But we know that $\mathrm{m}_X\text{-}\operatorname{Int} A \subseteq A$, so $A = \mathrm{m}_X\text{-}\operatorname{Int} A$, that is an arbitrary union of open m_X sets is an open m_X set. So, the collection of all open m_X sets forms a supra topological space. □

3.9. Remark. Arbitrary (even finite) intersections (unions) of open (closed) m_X sets need not be open (closed) m_X sets. This follows from the following example. Let $X = \{a, b, c\}$, $\mathrm{m}_X = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b\}\}$. Let $A = \{a\}$, $B = \{c\}$. Then $\mathrm{m}_X\text{-}\operatorname{Cl} A = \{a\}$ and $\mathrm{m}_X\text{-}\operatorname{Cl} B = \{c\}$. But $A \cup B = \{a, c\}$, $\mathrm{m}_X\text{-}\operatorname{Cl}(A \cup B) = X \neq (\mathrm{m}_X\text{-}\operatorname{Cl} A) \cup (\mathrm{m}_X\text{-}\operatorname{Cl} B)$. Similarly, $\mathrm{m}_X\text{-}\operatorname{Int} A^C \cap \mathrm{m}_X\text{-}\operatorname{Int} B^C \neq \mathrm{m}_X\text{-}\operatorname{Int}(A^C \cap B^C)$.

3.10. Remark. From the above theorem and remark it is clear that the collection of all open m_X sets forms a supra topological space, and the collection of all closed m_X sets forms a dual supra topological space. These may be denoted by (X, T_{m_X}) and $(X, T^C_{m_X})$, respectively.

4. On rare m_X sets

In this section, the concept of rare m_X set and dense m_X set are defined, and the corresponding spaces are obtained. Also it is shown that the non-null [resp non-universal] rare m_X [resp dense m_X] sets are not open [resp closed] m_X sets. The concept of open dense m_X set and closed rare m_X set is also introduced and the corresponding topological spaces obtained in this section of the paper.

4.1. Definition. A subset A of X is said to be a *rare* [resp., *dense*] m_X *set* if $\mathrm{m}_X\text{-}\operatorname{Int} A = \emptyset$ [resp., $\mathrm{m}_X\text{-}\operatorname{Cl} A = X$].

4.2. Example. Let $X = \{a, b, c, d\}$ and $\mathrm{m}_X = \{X, \emptyset, \{a, b\}\}$. Let $A = \{b, c\}$. Then $\mathrm{m}_X\text{-}\operatorname{Int} A = \emptyset$. Again, $A^C = \{a, d\}$, $\mathrm{m}_X\text{-}\operatorname{Cl} A^C = X$. Therefore, A is a rare m_X-set and A^C a dense m_X set.

4.3. Remark. If $\mathrm{m}_X = T$ then rare m_X set and rare set are the same concept. Also, dense m_X set and the dense sets are the same concept.

4.4. Theorem. *A subset A of X is a rare m_X set iff A^C is a dense m_X set.*

Proof. Obvious. □

4.5. Theorem. *A subset A of X is both an open [resp., closed] m_X set and a rare [resp., dense] m_X set iff it is the null [resp., universal] set.*

Proof. Let A be a set which is an open m_X set (i.e. $\mathrm{m}_X\text{-}\mathrm{Int}\,A = A$) and a rare m_X set (i.e. $\mathrm{m}_X\text{-}\mathrm{Int}\,A = \emptyset$). Then i.e. $\mathrm{m}_X\text{-}\mathrm{Int}\,A = A = \emptyset$, i.e. A is the null set. The proof of the second result is similar. □

4.6. Remark. Since m_X open sets are open m_X sets, a non-null m_X open set would be a rare m_X set, which gives a contradiction. Hence no non-null m_X open set is a rare m_X set. Similarly the only m_X closed set which is also a dense m_X set, is X.

4.7. Remark. A subset A of X can be both a rare m_X set and a dense m_X set. This follows from the following example. Let $X = \{a, b, c\}$ and $\mathrm{m}_X = \{\emptyset, X, \{a, b\}\}$. Consider the set $A = \{a, c\}$. Then $\mathrm{m}_X\text{-}\mathrm{Cl}\,A = X$ and $\mathrm{m}_X\text{-}\mathrm{Int}\,A = \emptyset$. So, A is both a rare m_X set and a dense m_X set.

4.8. Theorem. *A subset A of (X, m_X) is both a rare m_X set and a dense m_X set iff there there exists neither an m_X open set contained in A nor an m_X closed set containing A, except for $\emptyset$ and X.*

Proof. Obvious. □

4.9. Remark. If a subset A of (X, m_X) is both a rare m_X set and a dense m_X set then A is neither an m_X open set nor an m_X closed set. The converse need not be true, as follows from the following example. Let $X = \{a, b, c\}$, $\mathrm{m}_X = \{\emptyset, X, \{a\}\}$. Let $A = \{b\}$. This is neither an m_X closed set nor an m_X open set, but $\mathrm{m}_X\text{-}\mathrm{Cl}\,A = \{b, c\} \neq X$ i.e. A is not a dense m_X set.

4.10. Theorem.

(1) *X is a dense m_X set, but not a rare m_X set.*
(2) *$\emptyset$ is a rare m_X set, but not a dense m_X set.*
(3) *Arbitrary intersections [resp., unions] of rare [resp., dense] m_X sets are rare [resp., dense] m_X sets.*

Proof. (1) and (2) are obvious from the definition.

To prove (3), let $A = \bigcap\{A_\alpha : \alpha \in \Delta\}$ be an intersection of rare m_X sets, i.e. $\mathrm{m}_X\text{-}\mathrm{Int}\,A_\alpha = \emptyset$ for each $\alpha \in \Delta$. Then $\bigcap\{\mathrm{m}_X\text{-}\mathrm{Int}\,A_\alpha : \alpha \in \Delta\} = \emptyset$. We know that

$$\emptyset = \bigcap\{\mathrm{m}_X\text{-}\mathrm{Int}\,A_\alpha : \alpha \in \Delta\} \supseteq \mathrm{m}_X\text{-}\mathrm{Int}\bigcap\{A_\alpha : \alpha \in \Delta\},$$

i.e. $\mathrm{m}_X\text{-}\mathrm{Int}\bigcap(A_\alpha : \alpha \in \Delta)\} = \mathrm{m}_X\text{-}\mathrm{Int}\,A = \emptyset$. Hence arbitrary intersections of rare m_X sets are rare m_X sets. Similarly it can be proved that arbitrary unions of dense m_X sets are dense m_X sets. □

4.11. Remark. Finite unions of rare m_X sets need not be rare m_X sets. This follows from the following example. Let $X = \{a, b, c\}$, $\mathrm{m}_X = \{X, \emptyset, \{a\}, \{b, c\}$. Let $A = \{b\}$ and and B $= \{c\}$. Then $\mathrm{m}_X\text{-}\mathrm{Int}\,A = \emptyset$, $\mathrm{m}_X\text{-}\mathrm{Int}\,B = \emptyset$. Here $A \cup B = \{b, c\}$, $\mathrm{m}_X\text{-}\mathrm{Int}\,(A \cup B) = \{b, c\} \neq \emptyset$.

Similarly, $\mathrm{m}_X\text{-}\mathrm{Cl}\,(A^C) = X$ and $\mathrm{m}_X\text{-}\mathrm{Cl}\,(B^C) = X$, but $\mathrm{m}_X\text{-}\mathrm{Cl}\,(A^C \cap B^C) = \{a\} \neq X$.

4.12. Remark. The collection of all rare m_X set forms a pseudo supra topological space, and the collection of all dense m_X sets forms a dual pseudo supra topological space.

4.13. Theorem. *A subset A of X is a dense [resp., rare] m_X set iff for every open [resp., closed] m_X set G satisfying $A \subseteq G$ [resp., $A \supseteq G$] we have $m_X\text{-}\operatorname{Cl} A \supseteq G$ [resp., $m_X\text{-}\operatorname{Int} A \subseteq G$].*

Proof. Let us first assume that A is a dense m_X set and take an open m_X set G with $A \subseteq G$. Then $m_X\text{-}\operatorname{Cl} A = X \supseteq G$.

Conversely, let the given condition hold and take $G = X$. Then G is an open m_X set and $A \subseteq G$, so $m_X\text{-}\operatorname{Cl} A \supseteq G = X$, i.e. $m_X\text{-}\operatorname{Cl} A = X$ so A is a dense m_X set. The other part can be proved similarly. □

4.14. Remark. A subset A of X is a rare m_X set if there exist no non-null open m_X set contained in A.

4.15. Theorem. *The union [resp., intersection] of a dense [resp., rare] m_X set and a closed [resp., open] m_X set is a dense [resp., rare] m_X set.*

Proof. Let A be a dense m_X set and F a closed m_X set. If U is a open m_X set with $A \cup F \subseteq U$ then $A \subseteq U$ and so $m_X\text{-}\operatorname{Cl} A \supseteq U$ by Theorem 4.13. Now,

$$m_X\text{-}\operatorname{Cl}(A \cup F) \supseteq m_X\text{-}\operatorname{Cl} A \cup F \supseteq U \cup F \supseteq U,$$

i.e. the union of a dense m_X set and a closed m_X set is a dense m_X set by Theorem 4.13. The result for rare m_X sets can be similarly proved. □

4.16. Theorem. $m_X\text{-}\operatorname{Cl} A$ *[resp.,* $m_X\text{-}\operatorname{Int} A$*] is a dense [resp rare] m_X set whenever A is a dense [resp rare] m_X set.*

Proof. Clear from Theorem 2.3 (e). □

4.17. Remark. Let (X, m_X) be a minimal structure formed from the topological space (X, T). Then if a subset A of (X, m_X) is a dense set it is a dense m_X set since, $\operatorname{Cl} A \subseteq m_X\text{-}\operatorname{Cl} A$. Similarly, if a subset A of (X, m_X) is a rare set then it is a rare m_X set.

4.18. Definition. A subset A of X is said to be a *closed rare* [resp., *open dense*] m_X *set* if the set A is both a closed m_X set and a rare m_X set [resp., an open m_X set and a dense m_X set].

4.19. Example. Let $X = \{a, b, c, d\}$ and $m_X = \{X, \emptyset, \{a\}, \{b\}\}$. Let $A = \{c, d\}$, $m_X\text{-}\operatorname{Int}(A) = \emptyset$ and $m_X\text{-}\operatorname{Cl} A = A$ then A is a closed rare m_X set.

4.20. Remark. Let A be a dense m_X set. Then there exists an open dense m_X set G containing A.

4.21. Theorem. *A subset A of X is a closed [resp., open] rare [resp., dense] m_X set iff A is a closed m_X set which does not contain any non-null open m_X set.*

Proof. Obvious. □

4.22. Theorem. *If a subset A of X is a dense m_X set then* $m_X\text{-}\operatorname{Fr}(A) = X \backslash m_X\text{-}\operatorname{Int}(A)$.

Proof. Let A be a dense m_X set i.e. $m_X\text{-}\operatorname{Cl} A = X$. Then,

$$\begin{aligned} m_X\text{-}\operatorname{Fr}(A) &= m_X\text{-}\operatorname{Cl}(A) \cap m_X\text{-}\operatorname{Cl}(X \setminus A) \\ &= X \cap m_X\text{-}\operatorname{Cl}(X \setminus A) \\ &= m_X\text{-}\operatorname{Cl}(X \setminus A) \\ &= X \setminus m_X\text{-}\operatorname{Int}(A). \end{aligned}$$

□

4.23. Theorem. *If a subset A of X is a rare* m_X *set, then* $\mathrm{m}_X\text{-}\mathrm{Fr}(A) = \mathrm{m}_X\text{-}\mathrm{Cl}\,(A)$.

Proof. Similar to the above. □

4.24. Theorem. *A subset A of X is a both dense* m_X *set and a rare* m_X *set iff* $\mathrm{m}_X\text{-}\mathrm{Fr}(A) = X$.

Proof. Necessity follows from the above two theorems. Conversely, Let $\mathrm{m}_X\text{-}\mathrm{Fr}(A) = X$. Then $\mathrm{m}_X\text{-}\mathrm{Cl}\,(A) \cap \mathrm{m}_X\text{-}\mathrm{Cl}\,(X \setminus A) = X$, i.e. $\mathrm{m}_X\text{-}\mathrm{Cl}\,(A) = X$ and $\mathrm{m}_X\text{-}\mathrm{Cl}\,(X \setminus A) = X$. The first equality shows A is a dense m_X set, and from the second $\mathrm{m}_X\text{-}\mathrm{Cl}\,(X \setminus A) = X \setminus \mathrm{m}_X\text{-}\mathrm{Int}\,A = X$, i.e. $\mathrm{m}_X\text{-}\mathrm{Int}\,A = \emptyset$, so A is a rare m_X set. □

4.25. Theorem.

(1) *$\emptyset$ is a closed rare* m_X *set.*
(2) *Arbitrary intersections of closed rare* m_X *sets are closed rare* m_X *sets.*

Proof. (1) and (2) are true from Theorem 4.10 and Theorem 3.8. □

4.26. Remark. X is not closed rare m_X set, since X is a closed m_X set but not a rare m_X set. Also, arbitrary unions of closed rare m_X sets need not be closed rare m_X sets.

4.27. Remark. The collection of all open dense m_X sets, together with the set $\emptyset$, forms a supra topological space. This is called the *pseudo open dense* m_X *supra topological space*, and is denoted by $(X, \mathrm{OD}(\mathrm{m}_X))$. Likewise, the collection of all closed rare m_X sets forms a dual pseudo closed rare supra topological space which is denoted by $(X, \mathrm{CR}(\mathrm{m}_X))$.

4.28. Theorem. *Let $g : (X, \mathrm{m}_X) \to (Y, m_Y)$ be* m_X*-*m_Y *continuous and injective. Then g preserves rare* m_X *sets.*

Proof. Suppose that R_G is a rare m_X set but that $g(R_G)$ is not a rare m_Y set. Then $\mathrm{m}_Y\text{-}\mathrm{Int}\,g(R_G) \neq \emptyset$, so there exist a non-null open m_Y set V contained in $g(R_G)$, i.e. $V \subseteq g(R_G)$. Since g is injective, $g^{-1}(V) \subseteq R_G$. Since g is m_X-m_Y continuous, $g^{-1}(V)$ is an open m_X set. But this contradicts the fact that R_G is rare. So, g preserves rare m_X sets. □

5. On rarely m_X-m_Y continuous functions

Popa introduced the concept of rarely continuous function in [9]. M. Caldas and S. Jafari [4] introduced the concept of rarely δ - continuous function. In this section a more general form of rarely m_X-m_Y continuous function is introduced and some theorems are stated and proved.

5.1. Definition. A function $f : (X, \mathrm{m}_X) \to (Y, m_Y)$ is called a *rarely* m_X-m_Y *continuous function* if for each open m_Y set G containing $f(x)$ there exists a rare m_Y set R_G with $G \cap \mathrm{m}_Y\text{-}\mathrm{Cl}\,(R_G) = \emptyset$ and an open m_X set U containing x such that $f(U) \subseteq G \cup R_G$.

5.2. Remark. If $(X, \mathrm{m}_X) = (X, T)$, $(Y, m_Y) = (Y, V)$ then rarely m_X-m_Y continuous functions and rarely continuous functions are the same concept.

5.3. Theorem. *The following statements are equivalent for a function $f : X \to Y$.*

(1) *The function f is rarely* m_X*-*m_Y *continuous at $x \in X$.*
(2) *For each open* m_Y *set G containing $f(x)$ there exists a open* m_X *set U containing x such that* $\mathrm{m}_Y\text{-}\mathrm{Int}\,f(U) \cap \mathrm{m}_Y\text{-}\mathrm{Int}\,(Y \setminus G) = \emptyset$.
(3) *For each open* m_Y *set G containing $f(x)$ there exists a open* m_X *set U containing x such that* $\mathrm{m}_Y\text{-}\mathrm{Int}\,f(U) \subseteq \mathrm{m}_Y\text{-}\mathrm{Cl}\,(G)$.

Proof. (1)$\Longrightarrow$(2) Let G be an open m_Y set containing $f(x)$. Since G is an open m_Y set,

$$G = m_Y\text{-}\operatorname{Int} G \subseteq m_Y\text{-}\operatorname{Int}(m_Y\text{-}\operatorname{Cl} G).$$

Also, $m_Y\text{-}\operatorname{Int}(m_Y\text{-}\operatorname{Cl} G)$ is an open m_Y set containing $f(x)$. Since f is a rarely m_X continuous function there exists a rare m_Y set R_G with $m_Y\text{-}\operatorname{Int}(m_Y\text{-}\operatorname{Cl} G) \cap m_Y\text{-}\operatorname{Cl} R_G = \emptyset$, and an m_X open set U containing x so that $f(U) \subseteq m_Y\text{-}\operatorname{Int}(m_Y\text{-}\operatorname{Cl} G) \cup R_G$. So, we have

$$\begin{aligned} & m_Y\text{-}\operatorname{Int} f(U) \cap m_Y\text{-}\operatorname{Int}(Y \setminus G) \\ &\qquad \subseteq m_Y\text{-}\operatorname{Int}[m_Y\text{-}\operatorname{Int}(m_Y\text{-}\operatorname{Cl} G) \cup R_G] \cap [Y \setminus m_Y\text{-}\operatorname{Cl}(G)] \\ &\qquad \subseteq [m_Y\text{-}\operatorname{Int}(m_Y\text{-}\operatorname{Cl} G) \cup m_Y\text{-}\operatorname{Int} R_G] \cap [Y \setminus m_Y\text{-}\operatorname{Cl}(G)] \\ &\qquad \subseteq m_Y\text{-}\operatorname{Cl} G \cap (Y \setminus m_Y\text{-}\operatorname{Cl} G) = \emptyset. \end{aligned}$$

(2)$\Longrightarrow$(3) From (3), $m_Y\text{-}\operatorname{Int} f(U) \cap m_Y\text{-}\operatorname{Int}(Y \setminus G) = \emptyset$, so

$$m_Y\text{-}\operatorname{Int} f(U) \subseteq Y \setminus m_Y\text{-}\operatorname{Int}(Y \setminus G) = m_Y\text{-}\operatorname{Cl} G.$$

(3)$\Longrightarrow$(1) Let there exist an open m_Y set G containing $f(x)$ with the properties given in (3). Then there exists a open m_X set U containing x such that $m_Y\text{-}\operatorname{Int} f(U) \subseteq m_Y\text{-}\operatorname{Cl} G$. We have

$$\begin{aligned} f(U) &= [f(U) \setminus m_Y\text{-}\operatorname{Int} f(U)] \cup m_Y\text{-}\operatorname{Int} f(U) \\ &\subseteq [f(U) \setminus m_Y\text{-}\operatorname{Int} f(U)] \cup m_Y\text{-}\operatorname{Cl} G \\ &= [f(U) \setminus m_Y\text{-}\operatorname{Int} f(U)] \cup G \cup (m_Y\text{-}\operatorname{Cl} G \setminus G) \\ &= [(f(U) \setminus m_Y\text{-}\operatorname{Int} f(U)) \cap (Y \setminus G)] \cup G \cup (m_Y\text{-}\operatorname{Cl} G \setminus G). \end{aligned}$$

Let $R_1 = [f(U) \setminus m_Y\text{-}\operatorname{Int} f(U)] \cap (Y \setminus G)$ and $R_2 = m_Y\text{-}\operatorname{Cl} G \setminus G$. Then, $R_G = R_1 \cup R_2$ is a rare set such that $m_Y\text{-}\operatorname{Cl} R_G \cap G = \emptyset$ and $f(U) \subseteq G \cup R_G$. Therefore from Definition 5.3, f is a rarely m_X-m_Y continuous function. □

5.4. Theorem. *Let $f : (X, m_X) \to (Y, m_Y)$ be a rarely m_X-m_Y continuous function. Then the following statements hold:*

(1) *For each open m_Y set G containing $f(x)$, there exists a rare m_Y set R_G with $G \cap m_Y\text{-}\operatorname{Cl}(R_G) = \emptyset$ such that $x \in m_X\text{-}\operatorname{Int}(f^{-1}(G \cup R_G))$.*
(2) *For each open m_Y set G containing $f(x)$, there exists a rare m_Y set R_G with $m_Y\text{-}\operatorname{Cl} G \cap R_G = \emptyset$ such that $x \in m_X\text{-}\operatorname{Int}(f^{-1}(m_Y\text{-}\operatorname{Cl} G \cup R_G)$.*

Proof. (1) Suppose that G is an open m_Y set containing $f(x)$. Then there exists a rare m_Y set R_G with $G \cap m_Y\text{-}\operatorname{Cl}(R_G) = \emptyset$, and an open m_X set U containing x such that $f(U) \subseteq G \cup R_G$. It follows that $x \in U \subseteq f^{-1}(G \cup R_G)$, which implies that $x \in m_X\text{-}\operatorname{Int}(f^{-1}(G \cup R_G))$.

(2) Suppose that G is an open m_Y set containing $f(x)$. Then there exist a rare m_Y set R_G with $G \cap m_Y\text{-}\operatorname{Cl}(R_G) = \emptyset$ such that $x \in m_X\text{-}\operatorname{Int}(f^{-1}(G \cup R_G))$. Since $G \cap m_Y\text{-}\operatorname{Cl}(R_G) = \emptyset$, $R_G \subseteq (Y \setminus m_Y\text{-}\operatorname{Cl} G) \cup (m_Y\text{-}\operatorname{Cl} G \setminus G)$. Now, we have $R_G \subseteq (R_G \cup (Y \setminus m_Y\text{-}\operatorname{Cl} G)) \cup (m_Y\text{-}\operatorname{Cl} G \setminus G)$. Let $R_1 = (R_G \cup (Y \setminus m_Y\text{-}\operatorname{Cl} G))$. It follows that R_1 is a rare m_Y set with $m_Y\text{-}\operatorname{Cl} G \cap R_1 = \emptyset$. Therefore, $x \in m_X\text{-}\operatorname{Int}(f^{-1}(G \cup R_G)) \subseteq m_X\text{-}\operatorname{Int}(f^{-1}(m_Y\text{-}\operatorname{Cl} G \cup R_1)$. □

5.5. Theorem. *If f is a rarely m_X-m_Y continuous function then there exists a clopen set G such that for each open m_X set U containing x, $f(U)$ is a dense m_Y set.*

Proof. If G is a clopen subset of Y then $G \cup R_G$ is a dense m_Y set. Then, $m_Y\text{-}\operatorname{Cl}(f(U)) \subseteq m_Y\text{-}\operatorname{Cl}(G \cup R_G) = Y$. Hence $m_Y\text{-}\operatorname{Cl} f(U) = Y$ i.e. $f(U)$ is a dense m_Y set. □

5.6. Remark. Let $f : X \to Y$ be a rarely m_X-m_Y continuous function. Then there exists an open m_Y set G containing $f(x)$ and a rare m_Y set R_G such that $G \cap \mathrm{m}_Y\text{-}\operatorname{Cl}(R_G)$ is also a rare m_Y set.

5.7. Theorem. *Let (X, m_X) be an open dense m_X set. Then a function $f : (X, \mathrm{m}_X) \to (Y, \mathrm{m}_Y)$ is a rarely m_X-m_Y continuous function iff for each open m_Y set G containing $f(x)$ there exists a rare m_Y set R_G disjoint from the set G, and an open m_X set U containing x such that $f(U) \subseteq G \cup R_G$.*

Proof. Obvious. □

5.8. Theorem. *Let $f : (X, \mathrm{m}_X) \to (Y, \mathrm{m}_Y)$ be a rarely m_X-m_Y continuous function. Then the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for every x in X, is a rarely m_X-$\mathrm{m}_X \times \mathrm{m}_Y$ continuous function.*

Proof. Suppose that $x \in X$ and that W is any open $\mathrm{m}_X \times \mathrm{m}_Y$ set containing $g(x)$. It follows that there exists an open m_X set U and an open m_Y set V such that $(x, f(x)) \in U \times V \subseteq W$. Since f is rarely m_X-$\mathrm{m}_X \times \mathrm{m}_Y$ continuous, from Theorem 5.3 there exists an open m_X set G containing x such that $\mathrm{m}_X \times \mathrm{m}_Y\text{-}\operatorname{Int} f(G) \subseteq \mathrm{m}_Y\text{-}\operatorname{Cl} V$. Let $E = U \cap G$. It follows that E is an open m_X set containing x for which

$$\mathrm{m}_X \times \mathrm{m}_Y\text{-}\operatorname{Int}[g(E)] \subseteq \mathrm{m}_X \times \mathrm{m}_Y\text{-}\operatorname{Int}(U \times f(G)) \subseteq U \times \mathrm{m}_Y\text{-}\operatorname{Cl} V \subseteq \mathrm{m}_X \times \mathrm{m}_Y\text{-}\operatorname{Cl} W.$$

Therefore, g is a rarely m_X-$\mathrm{m}_X \times \mathrm{m}_Y$ continuous function. □

5.9. Theorem. *Let $f : (X, \mathrm{m}_X) \to (Y, \mathrm{m}_Y)$ be a rarely m_X-m_Y continuous function and $g : (Y, \mathrm{m}_Y) \to (Z, \mathrm{m}_Z)$ a one to one m_Y-m_Z continuous function. Then $g \circ f : (X, \mathrm{m}_X) \to (Z, \mathrm{m}_Z)$ is rarely m_X-m_Z continuous.*

Proof. Suppose that $x \in X$ and $g \circ f(x) \in V$, where V is an open m_Z set in Z. By hypothesis, g is m_Y-m_Z continuous, therefore there exists an m_Y open set $G \subseteq Y$ containing $f(x)$ such that $g(G) \subseteq V$. Since f is rarely m_X-m_Y continuous, there exists a rare m_Y set R_G with $G \cap \mathrm{m}_Y\text{-}\operatorname{Cl}(R_G) = \emptyset$, and an m_X open set U containing x such that $f(U) \subseteq G \cup R_G$. It follows from Theorem 4.28 that $g(R_G)$ is a rare m_Z set in Z. Since R_G is a subset of $Y \setminus G$, and g is injective, we have $\mathrm{m}_Y\text{-}\operatorname{Cl}(g(R_G)) \cap V = \emptyset$. This implies that $g \circ f(U) \subseteq V \cup g(R_G)$, hence the result. □

6. Conclusion

For a topological space (X, T), $\mathrm{RO}(X)$, $\mathrm{GO}(X)$, etc. are all m_X structures, and so if we put $\mathrm{m}_X = \mathrm{RO}(X)$, etc. then the pair (X, m_X) is an m_X space. For families of this type we can apply the results in this paper.

References

[1] Abdel Monsef, M. E and Ramadan, A. E. *On fuzzy supra topological spaces*, Indian J. Pure and Appl. Math. **18**, 322–329, 1987.

[2] Bhattacharya (Halder), S. and Chanda, S. *Some properties of m_X–Λ (resp. m_X-V) sets and its applications*, Acta Ciencia Indica, **XXXIV** (2), 513–??, 2008

[3] Bourbaki, N. *General Topology, Part I* (Addison Wesley, Reading, Mass., 1996).

[4] Caldas, M. and Jafari, S. *A strong form of rare continuity*, Soochow Journal of Mathematics **32** (1), 161–169, 2006.

[5] Jafari, S. *On some properties of rarely continuous functions*, Univ. Bacau. Stud. Cerc. St. Ser. Mat. **7**, 65–73, 1997.

[6] Long, P. E. and Herrington, L. L. *Properties of rarely continuous functions*, Glasnik Mat. **17** (37), 229–236, 1982.

[7] Maki, H. *On generalizing semi open and preopen sets* (Report for Meeting on Topological Spaces, Theory and its Applications, Yatsushiro College of Technology, 13 - 18, August 1996).

[8] Mocano, M. *Generalization of open functions*, Univ Bacau. Stud. Cerc. Sl. Ser. Mat **13**, 67–78, 2003.

[9] Popa, V. *Sur certain decomposition de la continuite dans les espaces topologiques*, Glasnik Mat. Setr III, **14** (34), 359–362, 1979.

[10] Popa, V. and Noiri, T. *On M - continuous functions*, Anal. Univ. "Dunarea de Jos" Galati Ser. Mat. Fiz. Mec. Teor. Fasc. II **18** (23), 31–41, 2000.

[11] Popa, V. and Noiri, T. *Slightly m - continuous functions*, Radovi Matematicki **10**, 219–232, 2001.