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THE IMAGES UNDER THE MODULAR GROUP AND EXTENDED MODULAR GROUP

Özden Koruoğlu*

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Abstract

In this paper we obtain the image of each element of $\mathbb{C}$ under the modular and extended modular group by continued fractions. The modular group takes the upper half of the complex plane into itself. Using these images, unlike known methods, we can show that the extended modular group takes the upper half of the complex plane into $\mathbb{H}$.

Keywords: Modular group, Extended modular group, Continued fraction, Upper half plane.

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1. Introduction

The modular group Γ is defined as

$$\mathrm{PSL}(2,\mathbb{Z}) = \left\{ \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}.$$

The transformations

$$T(z) = -\frac{1}{z} \text{ and } S(z) = -\frac{1}{z+1}$$

generate Γ, and a presentation is then given by

$$\Gamma = \left\langle T, S \mid T^2 = S^3 = I \right\rangle. \tag{1.1}$$

This group is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$, the free product of two finite cyclic groups. The modular group and its normal subgroups are of interest in many areas of Mathematics, for example number theory, automorphic function theory, group theory and graph theory (see [8, 9, 10, 11, 12]).

A transformation of the form

$$TS : z \longmapsto z+1,\ TS^2 : z \longmapsto \frac{z}{z+1}$$

*Balıkesir University, Necatibey Faculty of Education, 10100 Balıkesir, Turkey. E-mail: ozdenk@balikesir.edu.tr

is called a **block**, and any element $W \in \Gamma$ can be expressed in block form as

$$S^i(TS^2)^{m_1}(TS)^{n_1}\cdots(TS^2)^{m_k}(TS)^{n_k}T^j \tag{1.2}$$

for $i = 0, 1, 2,\ j = 0, 1$ and $m_1, n_1, \ldots, m_k, n_k$ are natural numbers (see [2]). This facilitates the study of words in Γ, Fine in [2] uses this to study the trace classes in Γ, while Koruoğlu, Şahin and İkikardeş extended this to the extended modular group (see [6]).

A *finite continued fraction* is an expression of the form

$$r_0 + \cfrac{1}{r_1 + \frac{1}{r_2 + \frac{1}{r_3 + \cdots + \frac{1}{r_n}}}}$$

where $r_1, r_2, \ldots, r_n$ are positive real numbers, called the partial quotients, and r_0 is any real number. The notation $[r_0; r_1, r_2, \ldots, r_n]$ is used for the finite continued fraction above. Continued fractions have been used in many ways to study both the subgroup structure and action on the complex plane of the modular group and of the more general Hecke groups (see [7] and [13]).

In this paper we develop a method to determine the image of an element $z \in \mathbb{C}$ under the action of a given $W \in \Gamma$ in terms the the block structure of W. This action is defined in terms of a finite continued fraction.

We develop a similar method for the extended modular group $\overline{\Gamma} = PGL(2, \mathbb{Z})$, formed by extending Γ by the reflection $R(z) = 1/\overline{z}$. This group has also been extensively studied (see [1, 3, 5, 14, 15]).

2. Images under the modular group

In this section we develop formulas in terms of continued fractions for the image of an element $z \in \mathbb{C}$ under the action of a given $W \in \Gamma$. The finite continued fraction is determined by the block structure of W.

2.1. Theorem. *For a positive integer n, the following holds:*

$$(TS)^n : z \longmapsto z + n \text{ and } (TS^2)^n : z \longmapsto \frac{z}{nz+1} = \frac{1}{n + \frac{1}{z}}. \tag{2.1}$$

Proof. For any positive integer n this may be proved by utilizing presentations of the matrices. □

In the following theorem we obtain the image of an element of $\mathbb{C}$ under the group elements of the modular group under the assumption that $i = 0$ and $j = 0$ in the general expression (1.2). Then we give the image of an element of the modular group for the remaining conditions in the next four corollaries.

2.2. Theorem. *For a given word in block form in Γ, suppose the form of the word is*

$$(TS^2)^{m_1}(TS)^{n_1}\cdots(TS^2)^{m_k}(TS)^{n_k}.$$

Then, the images of $\zeta \in \mathbb{C}$ are,

Conditions	*Images*
$m_1 \neq 0,\ n_k \neq 0$	$\dfrac{1}{[m_1; n_1, m_2, n_2, \ldots, m_k, n_k + \zeta]}$
$m_1 \neq 0,\ n_k = 0,\ m_k \neq 0$	$\dfrac{1}{[m_1; n_1, m_2, n_2, \ldots, m_k, \zeta]}$
$m_1 = 0,\ n_1 \neq 0,\ n_k \neq 0$	$[n_1; m_2, n_2, \ldots, m_k, n_k + \zeta]$
$m_1 = 0,\ n_1 \neq 0,\ n_k = 0,\ m_k \neq 0$	$[n_1; m_2, n_2, \ldots, m_k, \zeta]$

Proof. If $m_1 \neq 0$ and $n_k \neq 0$, then the form of the word is,

$$(TS^2)^{m_1}(TS)^{n_1}\cdots(TS^2)^{m_k}(TS)^{n_k}$$

From (1.3), initially,

$$(TS)^{n_k}(\zeta) = n_k + \zeta.$$

Then, the second step is

$$(TS^2)^{m_k}(n_k+\zeta) = \frac{1}{m_k + \frac{1}{n_k+\zeta}} = \frac{1}{[m_k;n_k+\zeta]}.$$

Benefiting from the properties in (1.3), continuing these steps the result given below is easily found.

$$(TS^2)^{m_1}(TS)^{n_1}\cdots(TS^2)^{m_k}(TS)^{n_k}(\zeta) = \frac{1}{[m_1;n_1,m_2,n_2,\ldots,m_k,n_k+\zeta]}.$$

For the 2nd and 3rd conditions, the result is easily proved. Now, let us prove the result for the conditions $m_1 = 0,\ n_1 \neq 0,\ n_k = 0,\ m_k \neq 0$. In this case, the form of the word is,

$$(TS)^{n_1}(TS^2)^{m_2}\cdots(TS)^{n_{k-1}}(TS^2)^{m_k}.$$

Firstly, we get

$$(TS^2)^{m_k}(\zeta) = \frac{1}{m_k + \frac{1}{\zeta}} = \frac{1}{[m_k;\zeta]}.$$

Then, the second step is simply,

$$(TS)^{n_{k-1}}\left(\frac{1}{m_k+\frac{1}{\zeta}}\right) = n_{k-1} + \frac{1}{m_k + \frac{1}{\zeta}} = [n_{k-1};m_k,\zeta]$$

Therefore, finally we get

$$(TS)^{n_1}(TS^2)^{m_2}\cdots(TS)^{n_{k-1}}(TS^2)^{m_k}(\zeta) = [n_1;m_2,n_2,\ldots,m_k,\zeta].$$

□

We give the image of $\zeta \in \mathbb{C}$ for the conditions $i = 1, 2$ and $j = 1$ in the following the four corollaries.

2.3. Corollary. *If* $m_1 \neq 0,\ n_k \neq 0$, *then the images are*

Conditions	*Images*
$i=0,\ j=1$	$\frac{1}{[m_1;n_1,m_2,n_2,\ldots,m_k,n_k-\frac{1}{\zeta}]}$
$i=1,\ j=0$	$-\frac{1}{\frac{1}{[m_1;n_1,m_2,n_2,\ldots,m_k,n_k+\zeta]}+1}$
$i=1,\ j=1$	$-\frac{1}{\frac{1}{[m_1;n_1,m_2,n_2,\ldots,m_k,n_k-\frac{1}{\zeta}]}+1}$
$i=2,\ j=0$	$-1-[m_1;n_1,m_2,n_2,\ldots,m_k,n_k+\zeta]$
$i=2,\ j=1$	$-1-[m_1;n_1,m_2,n_2,\ldots,m_k,n_k-\frac{1}{\zeta}]$

2.4. Corollary. *If* $m_1 \neq 0,\ n_k = 0,\ m_k \neq 0$, *then the images are*

Conditions	*Images*
$i=0,\ j=1$	$\dfrac{1}{[m_1; n_1, m_2, n_2, \ldots, m_{k,} -\frac{1}{\zeta}]}$
$i=1,\ j=0$	$-\dfrac{1}{\dfrac{1}{[m_1; n_1, m_2, n_2, \ldots, m_{k,}\zeta]}+1}$
$i=1,\ j=1$	$-\dfrac{1}{\dfrac{1}{[m_1; n_1, m_2, n_2, \ldots, m_{k,} -\frac{1}{\zeta}]}+1}$
$i=2,\ j=0$	$-1-[m_1; n_1, m_2, n_2, \ldots, m_{k,}\zeta]$
$i=2,\ j=1$	$-1-[m_1; n_1, m_2, n_2, \ldots, m_{k,} -\frac{1}{\zeta}]$

2.5. Corollary. *If* $m_1 = 0,\ n_1 \neq 0,\ n_k \neq 0$, *then the images are*

Conditions	*Images*
$i=0,\ j=1$	$[n_1; m_2, n_2, \ldots, m_{k,}n_k - \frac{1}{\zeta}]$
$i=1,\ j=0$	$-\dfrac{1}{[n_1; m_2, n_2, \ldots, m_{k,}n_k + \zeta]+1}$
$i=1,\ j=1$	$-\dfrac{1}{[n_1; m_2, n_2, \ldots, m_{k,}n_k - \frac{1}{\zeta}]+1}$
$i=2,\ j=0$	$-1-[n_1; m_2, n_2, \ldots, m_{k,}n_k + \zeta]$
$i=2,\ j=1$	$-1-[n_1; m_2, n_2, \ldots, m_{k,}n_k - \frac{1}{\zeta}]$

2.6. Corollary. *If* $m_1 = 0,\ n_1 \neq 0,\ n_k = 0,\ m_k \neq 0$, *then the images are*

Conditions	*Images*
$i=0,\ j=1$	$[n_1; m_2, n_2, \ldots, m_{k,} -\frac{1}{\zeta}]$
$i=1,\ j=0$	$-\dfrac{1}{[n_1; m_2, n_2, \ldots, m_{k,}\zeta]+1}$
$i=1,\ j=1$	$-\dfrac{1}{[n_1; m_2, n_2, \ldots, m_{k,} -\frac{1}{\zeta}]+1}$
$i=2,\ j=0$	$-1-[n_1; m_2, n_2, \ldots, m_{k,}\zeta]$
$i=2,\ j=1$	$-1-[n_1; m_2, n_2, \ldots, m_{k,} -\frac{1}{\zeta}]$

2.7. Example. Consider the word $W = STSTS^2TS^2TSTSTS$ in the modular group. In block form this word is

$$W = S(TS)(TS^2)^2(TS)^3$$

Hence, from Corollary 2.5, the image of $i \in C$ under this word is $\frac{-129+i}{314}$.

3. The images under the extended modular group

If T, S and R are given as before, then the extended modular group has the presentation

$$\overline{\Gamma} = \langle T, S, R \mid T^2 = S^3 = R^2 = (TR)^2 = (SR)^2 = I \rangle = D_2 *_{Z_2} D_3. \tag{3.1}$$

Here we extend our results to actions by an element $W \in \overline{\Gamma}$.

From (1.4), we have

$$T^2 = S^3 = R^2 = (TR)^2 = (SR)^2 = I.$$

Therefore, as is easily seen, there are two forms of a word in the extended modular group. These are,

$$W(T, S, R) = W(T, S) = S^i (TS^2)^{m_1} (TS)^{n_1} \cdots (TS^2)^{m_k} (TS)^{n_k} T^j$$

and

$$W(T, S, R) = RS^i (TS^2)^{m_1} (TS)^{n_1} \cdots (TS^2)^{m_k} (TS)^{n_k} T^j,$$

for $i = 0, 1, 2$ and $j = 0, 1$. The exponents of the blocks are natural numbers.

3.1. Corollary. *If the sum of the exponent of R in $W(T, S, R)$, is even (i.e. we have $W(T, S)$), the images of $W(T, S)$ can be computed as in Theorem 2.2 and Corollaries 2.3, 2.4, 2.5 and 2.6.* □

3.2. Corollary. *If the sum of the exponents of R in $W(T, S, R)$ is odd, the form of that word is*

$$W(T, S, R) = RS^i (TS^2)^{m_1} (TS)^{n_1} \cdots (TS^2)^{m_k} (TS)^{n_k} T^j.$$

Now, let us extract the word

$$S^i (TS^2)^{m_1} (TS)^{n_1} \cdots (TS^2)^{m_k} (TS)^{n_k} T^j$$

from the whole word. Let the image of ζ under this extract be ξ. In this case, the image of the word $W(T, S, R)$ is

$$R(\xi) = \frac{1}{\overline{\xi}}.$$ □

Now, let us give an application of the results we found so far. We know $\mathrm{PSL}(2, \mathbb{Z})$ takes the upper half $\mathbb{H}$ of the complex plane into itself [4].

3.3. Corollary. *The extended modular group, takes the upper half $\mathbb{H}$ of the complex plane into itself.*

Proof. If the word in the extended modular group is $W(T, S)$, the result is clear because the modular group takes the upper half $\mathbb{H}$ of the complex plane into itself. Now, let the word in the extended modular group be $W(T, S, R)$. From Corollary 3.2, if $\xi = x + iy$, $(y > 0)$ in R, then it is found that

$$R(\xi) = \frac{1}{\overline{\xi}} = \frac{x + iy}{|\xi|^2}$$

and this give us the expected result. Moreover, the extended modular group clearly takes the real axis into itself. □

3.4. Example. Consider the word

$$W = RTS^2RTS^2R$$

in the extended modular group. In block form this word is

$$W = R(TS^2)(TS),$$

owing to the relations in the presentation of the extended modular group i.e. $TR = RT$ and $RS = S^2R$. From Theorem 2.2 and Corollary 3.2, the image of $i \in \mathbb{C}$ under this element is

$$R(TS^2)(TS)(i) = \frac{3+i}{2}.$$

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