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FUZZY STABILITY OF A FUNCTIONAL EQUATION RELATED TO INNER PRODUCT SPACES

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Abstract

The fuzzy stability problems for the Cauchy quadratic functional equation and the Jensen quadratic functional equation in fuzzy Banach spaces have been investigated by Moslehian *et al.* Th. M. Rassias introduced the following equality

$$\sum_{i,j=1}^{m} \|x_i - x_j\|^2 = 2m \sum_{i=1}^{m} \|x_i\|^2, \quad \sum_{i=1}^{m} x_i = 0,$$

for a fixed integer $m \geq 3$. By the above equality, we define the following functional equation

$$\sum_{i,j=1}^{m} f(x_i - x_j) = 2m \sum_{i=1}^{m} f(x_i), \quad \sum_{i=1}^{m} x_i = 0. \tag{0.1}$$

In this paper, we prove the generalized Hyers-Ulam stability of the functional equation (0.1) in fuzzy Banach spaces.

Keywords: Fuzzy Banach space, Functional equation related to inner product space, Generalized Hyers-Ulam stability.

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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [41] concerning the stability of group homomorphisms. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [30] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [30] has had a lot of influence in the development of the *generalized Hyers-Ulam stability* of functional equations.

A generalization of the Th.M. Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. During the last two decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [4, 7, 15], [21]–[27], [32]–[39]).

A square norm on an inner product space satisfies the parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. The first author to treat the stability of the quadratic equation was F. Skof [40] by proving that if f is a mapping from a normed space X into a Banach space Y satisfying $\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \varepsilon$ for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : X \to Y$ such that $\|f(x)-g(x)\| \le \frac{\varepsilon}{2}$.

Cholewa [6] and Czerwik [8, 9] got important results on the generalized Hyers-Ulam stability problem for the quadratic functional equation.

A square norm on an inner product space satisfies

$$\sum_{i,j=1}^{3} \|x_i - x_j\|^2 = 6\sum_{i=1}^{3} \|x_i\|^2$$

for all $x_1, x_2, x_3 \in \mathbb{R}$ with $x_1 + x_2 + x_3 = 0$ (see [31]).

From the above equality we can define the functional equation

$$h(x-y) + h(2x+y) + h(x+2y) = 3h(x) + 3h(y) + 3h(x+y),$$

which can be also called a *quadratic functional equation*. In fact, $h(x) = ax^2$ in $\mathbb{R}$ satisfies the above quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*.

In [28], Park investigated the functional equation (0.1) and proved the generalized Hyers-Ulam stability of the functional equation (0.1) in real Banach spaces. In [29], Park and Jang proved the generalized Hyers-Ulam stability of the functional equation (0.1) in fuzzy Banach spaces by using the fixed point method.

Katsaras [16] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [10, 18, 42]. In particular, Bag and Samanta [2], following Cheng and Mordeson [5], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [17]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 19, 20] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the functional equation (0.1) in the fuzzy normed vector space setting.

1.1. Definition. [2, 19, 20] Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x, t) = 0$ for $t \leq 0$;
(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N_5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [19, 20].

1.2. Definition. [2, 19, 20] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be *convergent*, or to *converge*, if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.

1.3. Definition. [2, 19, 20] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [3]).

This paper is organized as follows: In Section 2, we prove the generalized Hyers-Ulam stability of the functional equation (0.1) in fuzzy Banach spaces for the even case. In Section 3, we prove the generalized Hyers-Ulam stability of the functional equation (0.1) in fuzzy Banach spaces for the odd case.

Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

2. Generalized Hyers-Ulam stability of the functional equation (0.1): the even case

In this section, we prove the generalized Hyers-Ulam stability of the functional equation (0.1) in fuzzy Banach spaces for the even case.

2.1. Lemma. [28] *Let V and W be real vector spaces. If a mapping $f : V \to W$ satisfies*

$$\sum_{i,j=1}^{m} f(x_i - x_j) = 2m \sum_{i=1}^{m} f(x_i) \tag{2.1}$$

for all $x_1, \ldots, x_m \in V$ with $\sum_{i=1}^{m} x_i = 0$, then the mapping $f : V \to W$ is realized as the sum of an additive mapping and a quadratic mapping. □

For a given mapping $f : X \to Y$, we define

$$Df(x_1, \ldots, x_m) := \sum_{i,j=1}^{m} f(x_i - x_j) - 2m \sum_{i=1}^{m} f(x_i)$$

for all $x_1, \ldots, x_m \in X$ with $\sum_{i=1}^m x_i = 0$.

2.2. Theorem. *Let $\varphi : X^m \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x_1, \ldots, x_m) := \sum_{j=1}^{\infty} 4^{-j} \varphi\left(2^j x_1, \ldots, 2^j x_m\right) < \infty \tag{2.2}$$

for all $x_1, \ldots, x_m \in X$. Let $f : X \to Y$ be an even mapping with $f(0) = 0$ such that

$$\lim_{t\to\infty} N\left(Df(x_1, \ldots, x_m), t\varphi(x_1, \ldots, x_m)\right) = 1 \tag{2.3}$$

uniformly on X^m. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^{-n} f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$N\left(Df(x_1, \ldots, x_m), \delta\widetilde{\varphi}(x_1, \ldots, x_m)\right) \geq \alpha \tag{2.4}$$

for all $x_1, \ldots, x_m \in X$, then

$$N\left(f(x) - Q(x), \delta\widetilde{\varphi}(x, -x, \underbrace{0, \ldots, 0}_{m-2 \text{ times}})\right) \geq \alpha \tag{2.5}$$

for all $x \in X$.

Furthermore, the quadratic mapping $Q : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N(f(x) - Q(x), t\widetilde{\varphi}(x, -x, \underbrace{0, \ldots, 0}_{m-2 \text{ times}})) = 1 \tag{2.6}$$

uniformly on X.

Proof. For a given $\varepsilon > 0$, by (2.3), we can find some $t_0 > 0$ such that

$$N\left(Df(x_1, \ldots, x_m), t\varphi(x_1, \ldots, x_m)\right) \geq 1 - \varepsilon \tag{2.7}$$

for all $t \geq t_0$. Letting $x_1 = x$, $x_2 = -x$ and $x_3 = \ldots = x_m = 0$ in (2.7), we get

$$N\left(2f(2x) - 8f(x), t\varphi(x, -x, \underbrace{0, \ldots, 0}_{m-2 \text{ times}})\right) \geq 1 - \epsilon \tag{2.8}$$

for all $x \in X$. By induction on n, we will show that

$$N\left(f(2^n x) - 4^n f(x), t\sum_{k=1}^{n} 4^{n-k} \varphi\left(2^{k-1}x, -2^{k-1}x, \underbrace{0, \ldots, 0}_{m-2 \text{ times}}\right)\right) \geq 1 - \varepsilon \tag{2.9}$$

for all $t \geq t_0$, all $x \in X$ and all $n \in \mathbb{N}$.

It follows from (2.8) that

$$N\left(f(2x) - 4f(x), t\varphi\left(x, -x, \underbrace{0, \ldots, 0}_{m-2 \text{ times}}\right)\right) \geq 1 - \epsilon$$

for all $x \in X$. Thus we get (2.9) for $n = 1$.

Assume that (2.9) holds for $n \in \mathbb{N}$. Then

$$N\left(4^{n+1}f(x) - f\left(2^{n+1}x\right), t\sum_{k=1}^{n+1}4^{n-k+1}\varphi\left(2^{k-1}x, -2^{k-1}x, \underbrace{0,\ldots,0}_{m-2\text{ times}}\right)\right)$$

$$\geq \min\left\{N\left(4^{n+1}f(x) - 4f\left(2^{n}x\right), t_0\sum_{k=1}^{n}4^{n-k}\varphi\left(2^{k-1}x, -2^{k-1}x, \underbrace{0,\ldots,0}_{m-2\text{ times}}\right)\right),\right.$$

$$\left. N\left(4f\left(2^{n}x\right) - f\left(2^{n+1}x\right), t_0\varphi\left(2^{n}x, -2^{n}x, \underbrace{0,\ldots,0}_{m-2\text{ times}}\right)\right)\right\}$$

$$\geq \min\{1-\varepsilon, 1-\varepsilon\} = 1-\varepsilon.$$

This completes the induction argument. Letting $t = t_0$ and replacing n and x by p and $2^n x$ in (2.9), respectively, we get

$$N\left(\frac{f\left(2^{n}x\right)}{4^{n}} - \frac{f\left(2^{n+p}x\right)}{4^{n+p}}, \frac{t_0}{4^{n+p}}\sum_{k=1}^{p}4^{p-k}\varphi\left(2^{n+k-1}x, -2^{n+k-1}x, \underbrace{0,\ldots,0}_{m-2\text{ times}}\right)\right) \geq 1-\varepsilon \tag{2.10}$$

for all integers $n \geq 0, p > 0$.

It follows from (2.2) and the equality

$$\sum_{k=1}^{p}4^{-n-k}\varphi\left(2^{n+k-1}x, -2^{n+k-1}x, \underbrace{0,\ldots,0}_{m-2\text{ times}}\right) = \sum_{k=n+1}^{n+p}4^{-k}\varphi\left(2^{k-1}x, -2^{k-1}x, \underbrace{0,\ldots,0}_{m-2\text{ times}}\right)$$

that for a given $\delta > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$t_0\sum_{k=n+1}^{n+p}4^{-k}\varphi\left(2^{k-1}x, -2^{k-1}x, \underbrace{0,\ldots,0}_{m-2\text{ times}}\right) < \delta$$

for all $n \geq n_0$ and $p > 0$. Now we deduce from (2.9) that

$$N\left(4^{-n}f\left(2^{n}x\right) - 4^{-(n+p)}f\left(2^{n+p}x\right), \delta\right)$$

$$\geq N\left(4^{-n}f\left(2^{n}x\right) - 4^{-(n+p)}f\left(2^{n+p}x\right), \frac{t_0}{4^{n+p}}\sum_{k=1}^{p}4^{p-k}\varphi\left(2^{n+k-1}x, -2^{n+k-1}x, \underbrace{0,\ldots,0}_{m-2\text{ times}}\right)\right)$$

$$\geq 1-\varepsilon$$

for each $n \geq n_0$ and all $p > 0$. Thus the sequence $\left\{4^{-n}f\left(2^{n}x\right)\right\}$ is Cauchy in Y. Since Y is a fuzzy Banach space, the sequence $\left\{4^{-n}f\left(2^{n}x\right)\right\}$ converges to some $Q(x) \in Y$. So we can define a mapping $Q : X \to Y$ by $Q(x) := N\text{-}\lim_{n\to\infty} 4^{-n}f\left(2^{n}x\right)$, namely, for each $t > 0$ and $x \in X$, $\lim_{n\to\infty} N\left(4^{-n}f\left(2^{n}x\right) - Q(x), t\right) = 1$.

It is obvious that $Q : X \to Y$ is even, since $f : X \to Y$ is even.

Let $x_1, \dots, x_m \in X$. Fix $t > 0$ and $0 < \varepsilon < 1$. Since

$$\lim_{n\to\infty} 4^{-n}\varphi\left(2^n x, -2^n x, \underbrace{0, \dots, 0}_{m-2 \text{ times}}\right) = 0,$$

there is an $n_1 > n_0$ such that $t_0\varphi\left(2^n x, -2^n x, \underbrace{0, \dots, 0}_{m-2 \text{ times}}\right) < \frac{4^n t}{(m^2+m+2)}$ for all $n \geq n_1$. Hence for each $k \geq n_1$, we have

$$\begin{aligned} N(DQ(x_1 \dots, x_m), t) &= N\left(\sum_{i,j=1}^{m} Q(x_i - x_j) - 2m\sum_{i=1}^{m} Q(x_i), t\right) \\ &\geq \min_{1\leq i,j\leq m}\left\{N\left(Q(x_i - x_j) - 4^{-k} f\left(2^k x_i - 2^k x_j\right), \frac{t}{m^2+m+2}\right),\right. \\ &\qquad N\left(2mQ(x_i) - 2m4^{-k} f\left(2^k x_i\right), \frac{t}{m^2+m+2}\right), \\ &\qquad\qquad \left. N\left(Df\left(2^k x_1, \dots, 2^k x_m\right), \frac{2t}{(m^2+m+2)}\right)\right\}. \end{aligned}$$

The first m^2+m terms on the right-hand side of the above inequality tend to 1 as $k \to \infty$, and the last term is greater than

$$N\left(Df\left(2^k x_1, \dots, 2^k x_m\right), t_0\varphi\left(2^k x_1, \dots, 2^k x_m\right)\right),$$

which is greater than or equal to $1 - \varepsilon$. Thus

$$N(DQ(x_1, \dots, x_m), t) \geq 1 - \varepsilon$$

for all $t > 0$. Since $N(DQ(x_1, \dots, x_m), t) = 1$ for all $t > 0$, by (N_2), $DQ(x_1, \dots, x_m) = 0$ for all $x \in X$. By [28, Lemma 2.1], the mapping $Q : X \to Y$ is quadratic.

Now let for some positive δ and α, (2.4) hold. Let

$$\varphi_n(x_1, \dots x_m) := \sum_{k=1}^{n} 4^{-k}\varphi\left(2^k x_1, \dots 2^k x_m\right)$$

for all $x_1, \dots, x_m \in X$. Let $x \in X$. By the same reasoning as in the beginning of the proof, one can deduce from (2.4) that

$$N\left(4^n f(x) - f(2^n x), \delta\sum_{k=1}^{n} 4^{n-k}\varphi\left(2^k x, -2^k x, \underbrace{0, \dots, 0}_{m-2 \text{ times}}\right)\right) \geq \alpha \tag{2.11}$$

for all positive integers n. Let $t > 0$. We have

$$
\begin{aligned}
&N\left(f(x)-Q(x), \delta\varphi_n\left(x,-x,\underbrace{0,\ldots,0}_{m-2\text{ times}}\right)+t\right)\\
&\geq \min\left\{N\left(f(x)-4^{-n}f\left(2^n x\right), \delta\varphi_n\left(x,-x,\underbrace{0,\ldots,0}_{m-2\text{ times}}\right)\right),\right.\\
&\qquad\qquad\left. N\left(4^{-n}f\left(2^n x\right)-Q(x),t\right)\right\}
\end{aligned}
\tag{2.12}
$$

Combining (2.11) and (2.12) and the fact that $\lim_{n\to\infty} N\left(4^{-n}f\left(2^n x\right)-Q(x),t\right)=1$, we observe that

$$
N\left(f(x)-Q(x), \delta\varphi_n\left(x,-x,\underbrace{0,\ldots,0}_{m-2\text{ times}}\right)+t\right) \geq \alpha
$$

for large enough $n \in \mathbb{N}$. Since the function $N(f(x)-Q(x),\cdot)$ is continuous, we see that

$$
N\left(f(x)-Q(x), \delta\widetilde{\varphi}\left(x,-x,\underbrace{0,\ldots,0}_{m-2\text{ times}}\right)+t\right) \geq \alpha.
$$

Letting $t \to 0$, we conclude that

$$
N\left(f(x)-Q(x), \delta\widetilde{\varphi}\left(x,-x,\underbrace{0,\ldots,0}_{m-2\text{ times}}\right)\right) \geq \alpha.
$$

To end the proof, it remains to prove the uniqueness assertion. Let T be another quadratic mapping satisfying (2.1) and (2.6). Fix $c > 0$. Given $\varepsilon > 0$, by (2.6) for Q and T, we can find some $t_0 > 0$ such that

$$
\begin{aligned}
&N\left(f(x)-Q(x), t\widetilde{\varphi}\left(x,-x,\underbrace{0,\ldots,0}_{m-2\text{ times}}\right)\right) \geq 1-\varepsilon,\\
&N\left(f(x)-T(x), t\widetilde{\varphi}\left(x,-x,\underbrace{0,\ldots,0}_{m-2\text{ times}}\right)\right) \geq 1-\varepsilon
\end{aligned}
$$

for all $x \in X$ and all $t \geq t_0$. Fix some $x \in X$ and find some integer n_0 such that

$$
t_0 \sum_{k=n}^{\infty} 4^{-k}\varphi\left(2^k x,-2^k x,\underbrace{0,\ldots,0}_{m-2\text{ times}}\right) < \frac{c}{2}
$$

for all $n \geq n_0$. Since

$$\begin{aligned}
&\sum_{k=n}^{\infty} 4^{-k}\varphi\left(2^k x, -2^k x, \underbrace{0,\ldots,0}_{m-2 \text{ times}}\right) \\
&\qquad = 4^{-n}\sum_{k=n}^{\infty} 4^{(n-k)}\varphi\left(2^{k-n}2^n x, -2^{k-n}2^n x, \underbrace{0,\ldots,0}_{m-2 \text{ times}}\right) \\
&\qquad = 4^{-n}\sum_{l=0}^{\infty} 4^{-l}\varphi\left(2^l 2^n x, -2^l 2^n x, \underbrace{0,\ldots,0}_{m-2 \text{ times}}\right) \\
&\qquad = 4^{-n}\widetilde{\varphi}\left(2^n x, -2^n x, \underbrace{0,\ldots,0}_{m-2 \text{ times}}\right),
\end{aligned}$$

we have

$$\begin{aligned}
&N(Q(x)-T(x),c) \\
&\quad \geq \min\left\{N\left(4^{-n}f\left(2^n x\right)-Q(x),\frac{c}{2}\right), N\left(T(x)-4^{-n}f\left(2^n x\right),\frac{c}{2}\right)\right\} \\
&\quad = \min\left\{N\left(f\left(2^n x\right)-Q\left(2^n x\right),4^n\frac{c}{2}\right), N\left(T\left(2^n x\right)-f\left(2^n x\right),4^n\frac{c}{(}2)\right)\right\} \\
&\quad \geq \min\left\{N\left(f\left(2^n x\right)-Q\left(2^n x\right),4^n t_0\sum_{k=n}^{\infty}4^{-k}\varphi\left(2^k x,-2^k x,\underbrace{0,\ldots,0}_{m-2 \text{ times}}\right)\right),\right. \\
&\qquad\qquad \left. N\left(T\left(2^n x\right)-f\left(2^n x\right),4^n t_0\sum_{k=n}^{\infty}4^{-k}\varphi\left(2^k x,-2^k x,\underbrace{0,\ldots,0}_{m-2 \text{ times}}\right)\right)\right\} \\
&\quad = \min\left\{N\left(f\left(2^n x\right)-Q\left(2^n x\right),t_0\widetilde{\varphi}\left(2^n x,-2^n x,\underbrace{0,\ldots,0}_{m-2 \text{ times}}\right)\right),\right. \\
&\qquad\qquad \left. N\left(T\left(2^n x\right)-f\left(2^n x\right),t_0\widetilde{\varphi}\left(2^n x,-2^n x,\underbrace{0,\ldots,0}_{m-2 \text{ times}}\right)\right)\right\} \\
&\quad \geq 1-\varepsilon.
\end{aligned}$$

It follows that $N(Q(x)-T(x),c)=1$ for all $c>0$. Thus $Q(x)=T(x)$ for all $x \in X$. $\square$

2.3. Corollary. *Let $\theta \geq 0$ and let p be a real number with $p>2$. Let $f: X \to Y$ be an even mapping such that*

$$\lim_{t\to\infty} N\left(Df(x_1,\ldots x_m), t\theta\sum_{i=1}^{m}\|x_i\|^p\right)=1 \tag{2.13}$$

uniformly on X^m. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^{-n}f(2^n x)$ exists for each $x \in X$ and defines a quadratic mapping $Q: X \to Y$ such that if for some $\delta>0, \alpha>0$

$$N\left(Df\left(x_1,\ldots,x_m\right),\delta\theta\sum_{i=1}^{m}\|x_i\|^p\right) \geq \alpha$$

for all $x_1 \dots, x_m \in X$, then

$$N\left(f(x)-Q(x), \frac{2^p}{2^p-4}\delta\theta\|x\|^p\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the quadratic mapping $Q : X \to Y$ is the unique mapping such that

$$\lim_{t\to\infty} N\left(f(x)-Q(x), \frac{2^p}{2^p-4}t\theta\|x\|^p\right) = 1$$

uniformly on X.

Proof. Define $\varphi(x_1, \dots, x_m) := \theta\sum_{i=1}^m(\|x_i\|^p)$ and apply Theorem 2.2 to get the result, as desired. □

Similarly, we can obtain the following. We will omit the proof.

2.4. Theorem. *Let $\varphi : X^m \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x_1, \dots, x_m) := \sum_{n=0}^{\infty} 4^n\varphi\left(\frac{x_1}{2^n}, \dots, \frac{x_m}{2^n}\right) < \infty$$

for all $x_1, \dots, x_m \in X$. Let $f : X \to Y$ be an even mapping satisfying (2.3) and $f(0) = 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$N\left(Df\left(x_1, \dots, x_m\right), \delta\varphi(x_1, \dots, x_m)\right) \geq \alpha$$

for all $x_1, \dots, x_m \in X$, then

$$N\left(f(x)-Q(x), \delta\widetilde{\varphi}\left(x, -x, \underbrace{0, \dots, 0}_{m-2\ times}\right)\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the quadratic mapping $Q : X \to Y$ is the unique mapping such that

$$\lim_{t\to\infty} N\left(f(x)-Q(x), t\widetilde{\varphi}\left(x, -x, \underbrace{0, \dots, 0}_{m-2\ times}\right)\right) = 1$$

uniformly on X. □

2.5. Corollary. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let $f : X \to Y$ be an even mapping satisfying (2.13). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f(\frac{2^n}{x})$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that if for some $\delta > 0, \alpha > 0$*

$$N\left(Df\left(x_1, \dots, x_m\right), \delta\theta\sum_{i=1}^m(\|x_i\|^p)\right) \geq \alpha$$

for all $x_1, \dots, x_m \in X$, then

$$N\left(f(x)-Q(x), \frac{2^p}{4-2^p}\delta\theta\|x\|^p\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the quadratic mapping $Q : X \to Y$ is the unique mapping such that

$$\lim_{t\to\infty} N\left(f(x)-Q(x), \frac{2^p}{4-2^p}t\theta\|x\|^p\right) = 1$$

uniformly on X.

Proof. Define $\varphi(x_1, \dots, x_m) := \theta \sum_{i=1}^{m} (\|x_i\|^p)$ and apply Theorem 2.4 to get the result, as desired. $\square$

3. Generalized Hyers-Ulam stability of the functional equation (0.1): the odd case

In this section, we prove the generalized Hyers-Ulam stability of the functional equation (0.1) in fuzzy Banach spaces for the odd case.

3.1. Theorem. *Let $\varphi : X^m \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x_1, \dots, x_m) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x_1}{2^j}, \dots, \frac{x_m}{2^j}\right) < \infty$$

for all $x_1, \dots, x_m \in X$. Let $f : X \to Y$ be an odd mapping such that

$$\lim_{t\to\infty} N\left(Df(x_1, \dots, x_m), t\widetilde{\varphi}(x_1, \dots, x_m)\right) = 1 \tag{3.1}$$

uniformly on X^m. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$N\left(Df(x_1, \dots, x_m), \delta\widetilde{\varphi}(x_1, \dots, x_m)\right) \geq \alpha$$

for all $x_1, \dots, x_m \in X$, then

$$N\left(f(x) - A(x), \delta\widetilde{\varphi}(x, x, -2x, \underbrace{0, \dots, 0}_{m-3 \ times})\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $A : X \to Y$ is a unique mapping such that

$$\lim_{t\to\infty} N(f(x) - A(x), t\widetilde{\varphi}(x, x, -2x, \underbrace{0, \dots, 0}_{m-3 \ times})) = 1$$

uniformly on X.

Proof. For a given $\varepsilon > 0$, by (3.1), we can find some $t_0 > 0$ such that

$$N\left(Df(x_1, \dots, x_m), t\varphi(x_1 \dots, x_m)\right) \geq 1 - \varepsilon \tag{3.2}$$

for all $t \geq t_0$. Letting $x_1 = x$, $x_2 = x$, $x_3 = -2x$ and $x_3 = \ldots = x_m = 0$ in (3.2), we get

$$N\left(f(2x) - 2f(x), t\varphi(x, x, -2x, \underbrace{0, \dots, 0}_{m-3 \text{ times}})\right) \geq 1 - \epsilon$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. $\square$

3.2. Corollary. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping such that*

$$\lim_{t\to\infty} N\left(Df(x_1, \dots x_m), t\theta \sum_{i=1}^{m} \|x_i\|^p\right) = 1 \tag{3.3}$$

uniformly on X^m. Then $A(x) := N\text{-}\lim_{n\to\infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that if for some $\delta > 0, \alpha > 0$

$$N\left(Df\left(x_1, \dots, x_m\right), \delta\theta \sum_{i=1}^{m} \|x_i\|^p\right) \geq \alpha$$

for all $x_1 \ldots, x_m \in X$*, then*

$$N\left(f(x) - A(x), \frac{2^p}{2^p - 2}\delta\theta\|x\|^p\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $A : X \to Y$ *is a unique mapping such that*

$$\lim_{t\to\infty} N\left(f(x) - A(x), \frac{2^p}{2^p - 2}t\theta\|x\|^p\right) = 1$$

uniformly on X.

Proof. Define $\varphi(x_1, \ldots, x_m) := \theta \sum_{i=1}^{m}(\|x_i\|^p)$ and apply Theorem 3.1 to get the result, as desired. □

Similarly, we can obtain the following. We will omit the proof.

3.3. Theorem. *Let* $\varphi : X^m \to [0, \infty)$ *be a function such that*

$$\widetilde{\varphi}(x_1, \ldots, x_m) := \sum_{n=0}^{\infty} 2^{-n}\varphi\left(2^n x_1, \ldots, 2^n x_m\right) < \infty$$

for all $x_1, \ldots, x_m \in X$*. Let* $f : X \to Y$ *be an odd mapping satisfying* (3.1)*. Then* $A(x) := N\text{-}\lim_{n\to\infty} 2^{-n} f(2^n x)$ *exists for each* $x \in X$ *and defines an additive mapping* $A : X \to Y$ *such that if for some* $\delta > 0, \alpha > 0$

$$N\left(Df\left(x_1, \ldots, x_m\right), \delta\varphi(x_1, \ldots, x_m)\right) \geq \alpha$$

for all $x_1, \ldots, x_m \in X$*, then*

$$N\left(f(x) - A(x), \delta\widetilde{\varphi}\left(x, x, -2x, \underbrace{0, \ldots, 0}_{m-3 \ times}\right)\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $A : X \to Y$ *is a unique mapping such that*

$$\lim_{t\to\infty} N\left(f(x) - A(x), t\widetilde{\varphi}\left(x, x, -2x, \underbrace{0, \ldots, 0}_{m-3 \ times}\right)\right) = 1$$

uniformly on X.

3.4. Corollary. *Let* $\theta \geq 0$ *and let* p *be a real number with* $0 < p < 1$*. Let* $f : X \to Y$ *be an odd mapping satisfying* (3.3)*. Then* $A(x) := N\text{-}\lim_{n\to\infty} 2^{-n} f(2^n x)$ *exists for each* $x \in X$ *and defines an additive mapping* $A : X \to Y$ *such that if for some* $\delta > 0, \alpha > 0$

$$N\left(Df\left(x_1, \ldots, x_m\right), \delta\theta\sum_{i=1}^{m}(\|x_i\|^p)\right) \geq \alpha$$

for all $x_1, \ldots, x_m \in X$*, then*

$$N\left(f(x) - A(x), \frac{2^p}{2 - 2^p}\delta\theta\|x\|^p\right) \geq \alpha$$

for all $x \in X$.

Furthermore, the additive mapping $A : X \to Y$ *is a unique mapping such that*

$$\lim_{t\to\infty} N\left(f(x) - A(x), \frac{2^p}{2 - 2^p}t\theta\|x\|^p\right) = 1$$

uniformly on X.

Proof. Define $\varphi(x_1, \ldots, x_m) := \theta \sum_{i=1}^{m}(\|x_i\|^p)$ and apply Theorem 3.3 to get the result, as desired. □

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