

SYMPLECTIC GEOGRAPHY PROBLEM IN DIMENSION SIX

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Abstract

In this note, the geography problem in dimension four is reviewed and then its extension to dimension six for the symplectic case is explained. Finally some examples in dimension six are provided.

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1. Introduction

An almost complex structure J on an even dimensional smooth manifold X^{2n} is a map from the tangent space TM to itself such that the restriction to each fiber has the property that J^2 is the negative of the identity map. If a manifold X has an almost complex structure J then the couple (X, J) is called an almost complex manifold. We are going to drop J and write X for such an almost complex manifold (X, J) . Complex manifolds are examples of almost complex manifolds and not all almost complex manifolds are complex. An almost complex manifold X has special characteristic classes, namely the Chern classes $c_i(X)$, which are elements in the even cohomology group $H^{2i}(X, \mathbb{Z})$. The zeroth Chern class of X is $1 \in H^0(X, \mathbb{Z})$. The total Chern class of X is $c(X) = 1 + c_1(X) + \dots + c_n(X)$. If X and Y are almost complex manifolds then $c(X \times Y)$ is $c(X)c(Y)$. Here the product of individual Chern classes is the cup product of the cohomology classes.

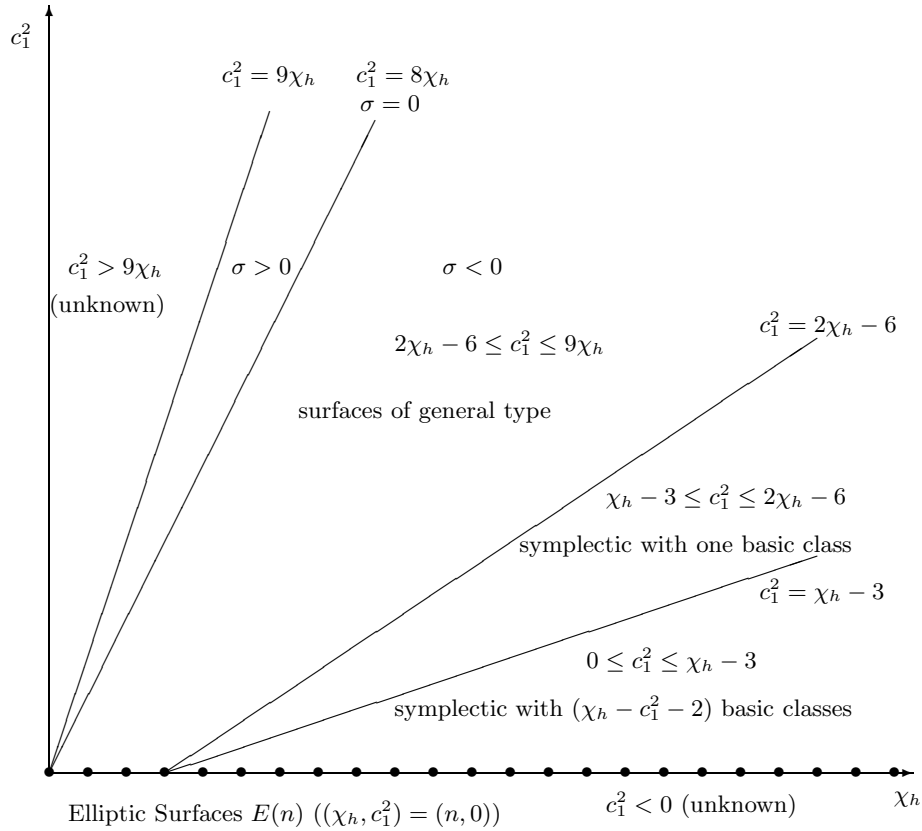
If n is equal to the sum of integers r_1, \dots, r_j , then the corresponding Chern number of X is the evaluation $c_{r_1} \cup \dots \cup c_{r_j}$ on the fundamental class $[X] \in H_{2n}(X, \mathbb{Z})$. The Chern numbers are integers. The top Chern number c_n is equal to the Euler characteristic of the manifold. In dimension four there are only two Chern numbers, which are c_1^2 and c_2 . The geography problem in dimension four has two parts. First, it is the search of smooth (or symplectic) 4-manifolds which admit a given couple (a, b) of integers as the Chern numbers. The second aim is to find the number of manifolds which admit the couple (a, b) as the Chern numbers. The latter is also known as the Botany problem [4].

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For an almost complex 4-manifold X we have the equalities $c_1^2(X) = 3\sigma(X) + 2e(X)$ and $c_2(X) = e(X)$, where $\sigma(X)$ is the signature of X and $e(X)$ is the Euler characteristic of X . An almost complex 4-manifold has another invariant which is called the holomorphic Euler characteristic $\chi_h(X)$. It turns out that $\chi_h(X)$ is equal to $(\sigma(X) + e(X))/4$, which is completely topological. So the invariant $\chi_h(X)$ is extended to the smooth category by this equality.

Figure 1 outlines the geography of smooth (or symplectic) 4-manifolds which is filled by manifolds which are constructed using some state of the art techniques. More information can be found in [4].

Figure 1



The Chern numbers of an almost complex 6-manifold M are c_3 , c_1^3 and c_1c_2 . Note that $c_3(M)$ is equal to the Euler characteristic $e(M)$ of M . The complex geography problem in dimension six (complex dimension three) is explored by Hunt in [8]. There are some studies on almost complex geography like [10] and [9]. We are going to focus on symplectic geography, i. e. distribution of the Chern numbers of an compatible almost complex structure on a symplectic 6-manifold. The next theorem by Halic shows that the restrictions on the Chern numbers of symplectic 6-manifolds are mild.

1.1. Theorem. (Halic [7]) *The Chern numbers of a symplectic 6-manifold M satisfy the conditions:*

$$c_3(M) \equiv c_1^3(M) \equiv 0 \pmod{2} \text{ and } c_1c_2(M) \equiv 0 \pmod{24}$$

Conversely, for a triple $(2r, 2s, 24t)$, where r, s and t are integers, there is a simply connected symplectic 6-manifold M with property $(c_3(M), c_1^2(M), c_1c_2(M))$ which is equal to $(2r, 2s, 24t)$. \square

Halic constructed the manifolds for the second part of the theorem by using an operation which is called the generalized symplectic fiber sum. This operation was first introduced by Gompf [5]. In the six dimensional case, let M_1 and M_2 be symplectic 6-manifolds. If X_1 and X_2 are codimension two symplectic submanifolds of M_1 and M_2 with trivial normal bundles, respectively, then the result of the fiber summing M_1 and M_2 through X_1 and X_2 is a symplectic 6-manifold M . The operation is done by removing a neighborhood of X_i in M_i , which is symplectomorphic to $X_i \times D^2$ [11]. Then $M_1 - \text{nb}(X_1)$ and $M_2 - \text{nb}(X_2)$ with a symplectomorphism of $X_1 \times (D^2 - 0)$ and $X_2 \times (D^2 - 0)$ turning $D^2 - 0$ inside out. Details can be found in [5].

In this note we are going to apply this construction as did Halic and get some manifolds on the geography picture. Our manifolds do not add anything new to filling the geography picture since it is already filled, but their smooth types are potentially different from those constructed by Halic. Some of these manifolds are symplectic Calabi-Yau manifolds (See Remark 3.3). Recently, similar manifolds are constructed by Akhmedov in [1].

2. Construction and Chern Numbers

Let X_1 and X_2 be closed, simply connected, smooth 4-manifolds with Lefschetz fibration structures on them. The base manifolds of the Lefschetz fibrations are S^2 . Further information on Lefschetz fibrations may be found in [6]. Let us assume that the generic fiber of the Lefschetz fibration on the respective X_i is an embedded surface Σ_i which has genus g_i . Assume that each singular fiber has a unique double point (fishtail) singularity. Let n_i be the number of singular fibers on each Lefschetz fibration. The Euler characteristic $e(X_i)$ of such a 4-manifold is $2(2 - 2g_i) + n_i$. In particular $e(X_i)$ is equal to $c_2(X_i)$. X_i being simply connected means that n_i is greater than $2g_i$.

Consider the product 6-manifolds $X_1 \times \Sigma_2$ and $X_2 \times \Sigma_1$ with the product symplectic structures on them. Let us apply the symplectic fiber sum operation [5, 7] to these 6-manifolds over $\Sigma_1 \times \Sigma_2$. As a result we get a closed, symplectic 6-manifold which we are going to denote by M .

As a symplectic manifold, M has a contractible family of almost complex structures which are compatible with the symplectic structure on it. With the formula for the Chern numbers for the symplectic fiber sum ([7]), calculations for the Chern numbers c_1^3 , c_1c_2 and c_3 of a generic almost complex structure on the product manifold M are given as follows:

$$\begin{aligned} c_3(M) &= c_3(X_1 \times \Sigma_2) + c_3(X_2 \times \Sigma_1) - 2c_2(\Sigma_1 \times \Sigma_2) \\ &= c_2(X_1)c_1(\Sigma_2) + c_2(X_2)c_1(\Sigma_1) - 2c_1(\Sigma_1)c_1(\Sigma_2) \\ &= c_2(X_1)(2 - 2g_2) + c_2(X_2)(2 - 2g_1) - 2(2 - 2g_1)(2 - 2g_2) \\ &= (2(2 - 2g_1) + n_1)(2 - 2g_2) + (2(2 - 2g_1) + n_2)(2 - 2g_2) \\ &\quad - 2(2 - 2g_1)(2 - 2g_2) \\ &= 8(1 - g_1)(1 - g_2) + 2n_1(1 - g_2) + 2n_2(1 - g_1) \end{aligned}$$

Considering $c_2(X_i) = e(X_i) = 12\chi_h(X_i) - c_1^2(X_i)$, we can write c_3 in terms of χ_h and c_1^2 .

$$\begin{aligned} c_3(M) &= 2(12\chi_h(X_1) - c_1^2(X_1))(1 - g_2) \\ &\quad + 2(12\chi_h(X_2) - c_1^2(X_2))(1 - g_1) - 8(1 - g_1)(1 - g_2) \end{aligned}$$

Calculation of c_1^3 :

$$\begin{aligned} c_1^3(M) &= c_1^3(X_1 \times \Sigma_2) + c_1^3(X_2 \times \Sigma_1) - 6c_1^2(\Sigma_1 \times \Sigma_2) \\ &= (c_1(X_1) + c_1(\Sigma_2))^3 + (c_1(X_2) + c_1(\Sigma_1))^3 - 12c_1(\Sigma_1)c_1(\Sigma_2) \\ &= c_1^3(X_1) + 3c_1^2(X_1)c_1(\Sigma_2) + 3c_1(X_1)c_1^2(\Sigma_2) + c_1^3(\Sigma_2) \\ &\quad + c_1^3(X_2) + 3c_1^2(X_2)c_1(\Sigma_1) + 3c_1(X_2)c_1^2(\Sigma_1) + c_1^3(\Sigma_1) \\ &\quad - 12c_1(\Sigma_1)c_1(\Sigma_2) \\ &= 6(1 - g_2)c_1^2(X_1) + 6(1 - g_1)c_1^2(X_2) - 48(1 - g_1)(1 - g_2) \end{aligned}$$

Calculation of c_1c_2 :

$$\begin{aligned} c_1c_2(M) &= c_1c_2(X_1 \times \Sigma_2) + c_1c_2(X_2 \times \Sigma_1) - 2c_1^2(\Sigma_1 \times \Sigma_2) - 2c_2(\Sigma_1 \times \Sigma_2) \\ &= (c_1(X_1) + c_1(\Sigma_2))(c_2(X_1) + c_1(X_1)c_1(\Sigma_2)) \\ &\quad + (c_1(X_2) + c_1(\Sigma_1))(c_2(X_2) + c_1(X_2)c_1(\Sigma_1)) \\ &\quad - 2(c_1(\Sigma_1) + c_1(\Sigma_2))^2 - 2c_1(\Sigma_1)c_1(\Sigma_2) \\ &= c_1^2(X_1)c_1(\Sigma_2) + c_2(X_1)c_1(\Sigma_2) + c_1^2(X_2)c_1(\Sigma_1) + c_2(X_2)c_1(\Sigma_1) \\ &\quad - 4c_1(\Sigma_1)c_1(\Sigma_2) - 2c_1(\Sigma_1)c_1(\Sigma_2) \\ &= (2 - 2g_2)(c_1^2(X_1) + c_2(X_1)) + (2 - 2g_1)(c_1^2(X_2) + c_2(X_2)) \\ &\quad - 6(2 - 2g_1)(2 - 2g_2) \\ &= (2 - 2g_2)(3\sigma(X_1) + 3e(X_1)) + (2 - 2g_1)(3\sigma(X_2) + 3e(X_2)) \\ &\quad - 6(2 - 2g_1)(2 - 2g_2) \end{aligned}$$

If we summarize, the Chern numbers of M are as follows.

2.1. Lemma. *The Chern numbers of the symplectic 6-manifold M constructed by fiber summing $X_1 \times \Sigma_2$ and $X_2 \times \Sigma_1$ along $\Sigma_1 \times \Sigma_2$ are:*

$$\begin{aligned} c_3(M) &= 2(12\chi_h(X_1) - c_1^2(X_1))(1 - g_2) \\ &\quad + 2(12\chi_h(X_2) - c_1^2(X_2))(1 - g_1) - 8(1 - g_1)(1 - g_2), \\ c_1^3(M) &= 6(1 - g_2)c_1^2(X_1) + 6(1 - g_1)c_1^2(X_2) - 48(1 - g_1)(1 - g_2), \\ c_1c_2(M) &= 24(1 - g_2)\chi_h(X_1) + 24(1 - g_1)\chi_h(X_2) - 24(1 - g_1)(1 - g_2), \end{aligned}$$

where $\chi_h(X_i)$ is the holomorphic Euler characteristic of X_i . □

3. Examples

In this section we give two families of symplectic 6-manifolds which are constructed using fiber summing as above.

3.1. Example. Let X_1 be the elliptic surface $E(m)$ with the Lefschetz fibration structure whose generic fibers are tori. $\chi_h(X_1)$ is m and $c_1^2(X_1)$ is 0. Let X_2 be the ruled surface $S^2 \times S^2$ and consider the sphere fibration structure on it with sphere sections of self-intersection zero. We may think of it as a Lefschetz fibration with no singular fibers. Genus g_2 is zero. $\chi_h(X_2)$ is 1 and $c_1^2(X_2)$ is 8. Apply the construction explained in Section 2 to the manifolds X_1, X_2 . Then the Chern numbers of the constructed symplectic manifold M are $c_3(M) = c_1c_2(M) = 24m$ and $c_1^3(M) = 0$.

3.2. Example. Let Y_1 be the elliptic surface $E(m)$ ($m \geq 1$). Let Y_2 be the manifold $E(k)_K$ ($k \geq 1$) which is obtained by applying the knot surgery of Fintushel-Stern [2] to the elliptic surface $E(k)$ over the fibered knot K . For a fixed k , $E(k)$ and $E(k)_K$ are homeomorphic to each other which means that $\chi_h(Y_2)$ is k and $c_1^2(Y_2)$ is zero. Y_2 has an Lefschetz fibration with generic fiber genus $2g + k - 1$, where $g \geq 0$ is the genus of the knot [3]. If we apply the construction to manifolds Y_1, Y_2 , then the Chern numbers of the constructed symplectic manifold M are $c_3(M) = c_1c_2(M) = 24m(2 - 2g - k)$ and $c_1^3(M) = 0$.

3.3. Remark. All examples given above are simply connected. Throughout these families, although c_1^3 is zero, the only manifold with c_1 equal to zero is when we choose $m = k = 2$ and $g = 0$ in Example 3.2. See [1] for further discussion on the symplectic Calabi-Yau manifolds.

3.4. Remark. Many similar examples may be produced by applying the procedure above. Still there will be triples $(2r, 2s, 24t)$ which we cannot realize as Chern numbers due to the fact that c_1^3 is divisible by six for the product manifold of this operation. This is behind the result of Halic in [7]. However, potentially all of the manifolds constructed in Example 3.2 are diffeomorphic for a fixed k . It may be interesting to see if they are diffeomorphic.

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