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Research Article

On the strongly annihilating-submodule graph of a module

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Abstract

In this paper we continue to study the strongly annihilating-submodule graph. In addition to providing the more properties of this graph, we compare extensively the properties of this graph with the annihilating-submodule graph.

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1. Introduction

Throughout this paper R is a commutative ring with nonzero identity element and M is a unitary right R-module. For a submodule N of M, denoted by $N \leq M$, the ideal $\{r \in R \mid Mr \subseteq N\}$ will be denoted by $(N :_R M)$ (briefly by $(N : M)$). Recall that M is *indecomposable* if it is nonzero and cannot be written as a direct sum of two nonzero submodules. M is called *uniform* if the intersection of any two nonzero submodules is nonzero. Also a submodule N of M is called an *essential submodule* of M, denoted by $N \leq_e M$, if for any nonzero submodule K of M, $K \cap N \neq 0$. For $X \subseteq M$, the annihilator of X in R is the ideal $\text{ann}_R(X) = \{r \in R \mid Xr = 0\}$. We say that M has *uniform dimension* n (written u.dim $M = n$) if there exists an essential submodule $N \leq_e M$ that is a direct sum of n uniform submodules, i.e., u.dim M is the supremum of the set $\{k \mid M$ contains a direct sum of k nonzero submodules$\}$, for more details see [21].

There are many papers on assigning graphs to groups, rings or modules, for example see [1–4, 9, 18, 25]. For any ring R with the set of zero-divisors $Z(R)$, *the zero-divisor graph* of R, denoted by $\Gamma(R)$, is a simple graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. The concept of a *zero-divisor graph* of a commutative ring was introduced in [13], and it was mainly concerned with coloring of rings. The above definition first appeared in the work of Anderson and Livingston [7]. This definition, unlike the earlier work of Anderson and Naseer [8] and Beck [13], dose not take zero to be a vertex of $\Gamma(R)$. The zero-divisor graph of a ring has been studied by several authors, see for example [3, 4, 6–8]. An ideal I of a commutative ring R is called an *annihilating ideal* if $IJ = 0$, for some nonzero ideal J of R. Also the set of all annihilating ideals of R is denoted by $\mathbb{A}(R)$. The notion of annihilating-ideal

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graph was first introduced and studied in [14]. *The annihilating ideal graph* of R, denoted by $\mathbb{AG}(R)$, is a simple graph with vertices $\mathbb{A}(R)^* = \mathbb{A}(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. Later, it was modified and further studied by many authors, see for example [1, 2, 5, 23]. Recently, the notions of *zero divisor graph* and *annihilating-ideal graph* have been extended from rings to modules in different ways. For instance, we can refer to [9] and [25]. In [9], the authors introduced and studied the annihilating-submodule graph. By *the annihilating-submodule graph* of M, denoted by $\mathbb{AG}(M)$, we mean the simple graph with vertices $\{0 \neq N \leq M \mid M(N : M)(K : M) = 0,$ for some nonzero submodule K of $M\}$ and two distinct vertices N and K are adjacent if and only if $M(N : M)(K : M) = 0$. The authors in [10, 11] and [9], investigated the basic properties of this graph and presented some related results.

In this paper, we continue to study the strongly annihilating-submodule graph of a module introduced in [15]. *The strongly annihilating-submodule graph* of M, denoted by $\mathbb{SAG}(M)$, is an undirected (simple) graph in which a nonzero submodule N of M is a vertex if $N(K : M) = 0$ or $K(N : M) = 0$, for some nonzero submodule $K \leq M$ and two distinct vertices N and K are adjacent if and only if $N(K : M) = 0$ or $K(N : M) = 0$. It is clear that if $M = R$, then $\mathbb{SAG}(R) = \mathbb{AG}(R)$ and if M is a multiplication R-module, then $\mathbb{SAG}(M) = \mathbb{AG}(M)$. We compare extensively the properties of this graph with the annihilating-submodule graph.

We state some definitions and notions of graph theory used throughout this paper. Recall that for a graph G, the *degree* of a vertex x in G is the number of edges of G incident with x. A graph G is *connected* if there is a path between any two vertices of G. The *diameter* of G is $\text{diam(G)} = \sup\{d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } G\}$, where $d(x, y)$ is the length of the shortest path from x to y in G and if there is no such path, we write $d(x, y) = \infty$. The *girth* of a graph G, denoted by $\text{gr}(G)$, is the smallest size of the length of cycles of G and if G has no cycles, we write $\text{gr}(G) = \infty$. A *bipartite graph* G is a graph whose vertices can be partitioned into two subsets U and V such that every edge connects a vertex in U to one in V. Vertex sets U and V usually are called the parts of G. A *complete bipartite graph* is one in which every vertex in U is joined to every vertex in V. A complete bipartite graph with parts U and V is called *star graph* if $|U| = 1$ or $|V| = 1$. In a graph G, if all the vertices of G have the same degree r, then G is called *r-regular*, or simply *regular*. A graph in which each pair of distinct vertices is connected by an edge is called a *complete graph*. A connected graph is called a *tree* if it has no cycles. For a graph G, a complete subgraph of G is called a *clique*. The *clique number*, $cl(G)$, is the greatest integer $n \geq 1$ such that G contains a complete subgraph with n vertices, and $cl(G)$ is infinite if for any n, G contains a complete subgraph with n vertices. By $\chi(G)$, we denote the chromatic number of G, i.e., the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. For every graph G, a subset D of $V(G)$ is called a *dominating set* if every vertex of G is either in D or adjacent to at least one vertex in D. The *domination number* of G is the number of vertices in a smallest dominating set of G. A *total dominating set* of a graph G is a subset S of $V(G)$ such that every vertex is adjacent to a vertex in S. The *total domination number* of G is the minimum cardinality of a total dominating set. The notations of graph theory used in the sequel can be found in [19].

The organization of this paper is as follows: In Section 2, we give more properties of $\mathbb{SAG}(M)$ and remind that $\mathbb{SAG}(M)$ can be a strict subgraph of $\mathbb{AG}(M)$. It is shown that if M is not a prime R-module, then $\mathbb{SAG}(M)$ has ACC (resp. DCC) on vertices if and only if M is a Noetherian (resp. an Artinian) module (Theorem 2.8). In Section 3, we compare extensively the properties of the two graphs $\mathbb{SAG}(M)$ and $\mathbb{AG}(M)$; in particular when $\mathbb{SAG}(M)$ (or $\mathbb{AG}(M)$) is a *path*, *bipartite*, *tree*, *star*, *regular* or *complete graph*. For instance, we show that $\text{gr}(\mathbb{SAG}(M)) = 4$ if and only if $\text{gr}(\mathbb{AG}(M)) = 4$; moreover in this case $\mathbb{SAG}(M) = \mathbb{AG}(M)$ (Proposition 3.3 and Theorem 3.8). Also, if $\mathbb{SAG}(M)$ is a tree, then either $\mathbb{SAG}(M)$ is a star graph or $\mathbb{SAG}(M) = P_4$; moreover, $\mathbb{SAG}(M) = P_4$ if and only if $M = F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule (Theorem 3.10). Finally in the fourth section, we compare the (total) dominating number of $\mathbb{SAG}(M)$ and $\mathbb{AG}(M)$ (Theorem 4.2 and Proposition 4.3). Also the dominating number and the total dominating number of $\mathbb{SAG}(M)$ are investigated (Theorem 4.4).

2. More properties of $\mathbb{SAG}(M)$

Throughout the paper, M is a unitary right R-module and N, K are nonzero submodules of M. In $\mathbb{SAG}(M)$, M itself can be a vertex. In fact M is a vertex if and only if every nonzero submodule is a vertex, if and only if there exists a nonzero proper submodule N of M such that $(N : M) = \mathrm{ann}(M)$. We note that for any R-module M, $\mathbb{SAG}(M)$ is a subgraph of $\mathbb{AG}(M)$ and if $M = R$, then the three graphs $\mathbb{SAG}(M)$, $\mathbb{AG}(M)$ and the annihilating-ideal graph introduced in [14] coincide. However, the following example shows that $\mathbb{SAG}(M)$ is a strict subgraph of $\mathbb{AG}(M)$ even in the case where M is semiprime (defined later) or semisimple (see part (5) in the following example). For a given R-module M, we use the notation $n(M)$ for the number of the submodules of M. Also the degree of a vertex K in graphs $\mathbb{SAG}(M)$ and $\mathbb{AG}(M)$ is denoted by $\deg_S(K)$ and $\deg_A(K)$, respectively. In the following example we consider M as a $\mathbb{Z}$-module.

Example 2.1. (1) Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$, $N_1 = (0) \oplus \mathbb{Z}_4$ and $N_2 = (\bar{1}, \bar{1})\mathbb{Z}$. Then N_1 and N_2 are adjacent in $\mathbb{AG}(M)$, but not adjacent in $\mathbb{SAG}(M)$. Thus $\mathbb{SAG}(M)$ is different from $\mathbb{AG}(M)$.

(2) Let $M = (\oplus_{i=1}^{n} \mathbb{Z}_p) \oplus (\oplus_{i=1}^{m} \mathbb{Z}_q)$ and $K = (\oplus_{i=1}^{n} \mathbb{Z}_p) \oplus (\oplus_{i=1}^{m-1} \mathbb{Z}_q)$, where p and q are two distinct prime numbers and $m \geq 2$. We set $N = T \oplus (\oplus_{i=1}^{m} \mathbb{Z}_q)$, where T is a nonzero proper submodule of $\oplus_{i=1}^{n} \mathbb{Z}_p$. Then $M(N : M)(K : M) = M(p\mathbb{Z})(q\mathbb{Z}) = 0$. However, $N(K : M) = N(q\mathbb{Z}) \neq 0$ and $K(N : M) = K(p\mathbb{Z}) \neq 0$. Thus N and K are adjacent in $\mathbb{AG}(M)$, but not adjacent in $\mathbb{SAG}(M)$ and hence $\deg_A(K) - \deg_S(K) \geq n(\oplus_{i=1}^{n} \mathbb{Z}_p) - 2$.

(3) Let $M = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus (\oplus_{i=1}^{m} \mathbb{Z}_q)$ and $K = \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus (\oplus_{i=1}^{m-1} \mathbb{Z}_q)$, where p and q are two distinct prime numbers and $m \geq 2$. Since by [26, Corollary 4.4], $n(\mathbb{Z}_p \oplus \mathbb{Z}_p) = p + 3$, the part (2) implies that $\deg_A(K) - \deg_S(K) \geq p + 1$.

(4) In $\mathbb{Z}_{16}$, $\mathbb{SAG}(\mathbb{Z}_{16}) = \mathbb{AG}(\mathbb{Z}_{16})$ is the star graph $N_1 - N_3 - N_2$, where $N_1 = 2\mathbb{Z}_{16}$, $N_2 = 4\mathbb{Z}_{16}$ and $N_3 = 8\mathbb{Z}_{16}$.

(5) Let $M = (\oplus_{i=1}^{2} \mathbb{Z}_2) \oplus (\oplus_{i=1}^{2} \mathbb{Z}_3)$, $N = \mathbb{Z}_2 \oplus (\oplus_{i=1}^{2} \mathbb{Z}_3)$ and $K = (\oplus_{i=1}^{2} \mathbb{Z}_2) \oplus \mathbb{Z}_3$. Clearly N and K are adjacent in $\mathbb{AG}(M)$, but not adjacent in $\mathbb{SAG}(M)$.

An R-module M is called *prime* if $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$, for any nonzero submodule N of M. Also M is called *weakly prime* (resp. *semiprime*), if $\mathrm{ann}_R(N)$ is a prime (resp. semiprime) ideal of R, for any nonzero submodule N of M. The dual of notions prime and weakly prime for modules are *second* and *weakly second*, respectively. Indeed M is called *second* if $\mathrm{ann}_R(M) = \mathrm{ann}_R(M/N)$, for any proper submodule N of M. Also M is called *weakly second*, if $\mathrm{ann}_R(M/N)$ is a prime ideal of R, for any proper submodule N of M. Clearly, any prime module is weakly prime and also any second module is weakly second. For more details about these notions, the reader is referred to [12, 16, 17].

Example 2.2. (1) If $M = \oplus_{i=1}^{n} S_i$, where S_i's are isomorphic simple R-modules, then $\mathbb{SAG}(M)$ is a complete graph such that every nonzero submodule of M is a vertex and so $\mathbb{SAG}(M) = \mathbb{AG}(M)$.

(2) It is easy to see that $\mathbb{SAG}(M)$ is the empty graph if and only if M is a prime module and not vertex.

Proposition 2.3. *If one of the following conditions holds, then the two graphs $\mathbb{SAG}(M)$ and $\mathbb{AG}(M)$ coincide.*
(1) *M is prime.*
(2) *M is weakly prime.*
(3) *M is second.*
(4) *M is weakly second.*
(5) *M is a multiplication R-module (i.e., for every submodule N of M there exists an ideal I of R such that $N = MI$).*
(6) *M is a cyclic R-module.*

Proof. Clear. □

The following useful lemmas will be used frequently in this paper.

Lemma 2.4. *[15, Lemma 2.1] (1) If N and K are adjacent in $\mathbb{SAG}(M)$, then N_1 and K_1 are also adjacent in $\mathbb{SAG}(M)$ for every $0 \neq N_1 \leq N$ and $0 \neq K_1 \leq K$ with $N_1 \neq K_1$.*
(2) *If $N \cap K = 0$, then N and K are adjacent in $\mathbb{SAG}(M)$.*
(3) *If N is not a vertex in $\mathbb{SAG}(M)$, then $N \leq_e M$.*

The converse of (3) in the above lemma is not true. Indeed, if we consider $\mathbb{Z}_{12}$ as a $\mathbb{Z}$-module, then $2\mathbb{Z}_{12} \leq_e \mathbb{Z}_{12}$. However, $2\mathbb{Z}_{12}$ and $6\mathbb{Z}_{12}$ are adjacent and so $2\mathbb{Z}_{12}$ is a vertex in $\mathbb{SAG}(\mathbb{Z}_{12})$.

The following lemma shows that $V(\mathbb{SAG}(M)) = V(\mathbb{AG}(M))$.

Lemma 2.5. *[15, Lemma 2.2] If N and K are adjacent in $\mathbb{AG}(M)$, then the following statements hold.*
(1) *N and K are adjacent in $\mathbb{SAG}(M)$ or there exists a nonzero submodule of $N \cap K$ such that is adjacent to both N and K in $\mathbb{SAG}(M)$.*
(2) *There exists a nonzero submodule of N that is adjacent to K in $\mathbb{SAG}(M)$.*

Corollary 2.6. *Let N and K be adjacent in $\mathbb{AG}(M)$ and N be a minimal submodule of M. Then N and K are adjacent in $\mathbb{SAG}(M)$.*

Proposition 2.7. *Let M be an R-module and I be an ideal of R.*
(1) *If $V(\mathbb{SAG}(M)) \neq \emptyset$, then every minimal submodule of M is a vertex.*
(2) *If $MI - N$ is an edge in $\mathbb{AG}(M)$, then $MI - N$ is an edge in $\mathbb{SAG}(M)$.*
(3) *If $\mathbb{AG}(M)$ is a triangle-free graph or contains no cycle, then $\mathbb{SAG}(M) = \mathbb{AG}(M)$.*

Proof. (1). Let N be a minimal submodule of M. Then for every nonzero submodule K of M, $N \cap K = 0$ or $N \subseteq K$. If $N \cap K = 0$, then N and K are adjacent, and we are done. Thus we may assume that $N \subseteq K$, for any nonzero submodule K of M. Now let $K \in V(\mathbb{SAG}(M))$. Then there exists $0 \neq K' \leq M$ such that $K(K' : M) = 0$ or $K'(K : M) = 0$. Thus $N(K' : M) = 0$ or $K'(N : M) = 0$, as desired.
(2). Since $MI - N$ is an edge in $\mathbb{AG}(M)$, we have $M(MI : M)(N : M) = 0$. Clearly $M(MI : M) = MI$. Thus $MI(N : M) = 0$, as desired.
(3). It is clear by Lemma 2.5. □

Theorem 2.8. *If M is not a prime R-module, then $\mathbb{SAG}(M)$ has* ACC (resp. DCC) *on vertices if and only if M is a Noetherian* (resp. *an Artinian*) *module.*

Proof. Suppose that $\mathbb{SAG}(M)$ has ACC (resp. DCC) on vertices. Since M is not a prime module, there exists $r \in R$ and $m \in M$ such that $mr = 0$ but $m \neq 0$ and $r \notin \text{ann}(M)$. Since $Mr(\text{ann}_M(r) : M) \subseteq \text{ann}_M(r)r = 0$, every nonzero submodule in $\text{ann}_M(r)$ and in Mr is a vertex. This implies that the R-modules $\text{ann}_M(r)$ and Mr have ACC (resp. DCC) on submodules. Now $Mr \cong M/\text{ann}_M(r)$ implies that M is a Noetherian (resp. an Artinian) module. The converse is clear. □

Corollary 2.9. *[15, Theorem 2.4] For any R-module M, $\mathbb{SAG}(M)$ is a connected graph with* $\operatorname{diam}(\mathbb{SAG}(M)) \leq 3$.

Theorem 2.10. *[15, Theorem 2.5] For any R-module M, if $\mathbb{SAG}(M)$ contains a cycle, then* $\operatorname{gr}(\mathbb{SAG}(M)) \leq 4$.

3. Comparison of $\mathbb{SAG}(M)$ and $\mathbb{AG}(M)$

At the beginning of this section, we compare the girth of two graphs $\mathbb{AG}(M)$ and $\mathbb{SAG}(M)$.

Proposition 3.1. *Let M be an R-module with* $\operatorname{u.dim}(M) \geq 2$. *Then*

$$\operatorname{gr}(\mathbb{SAG}(M)) = 3 \Longleftrightarrow \operatorname{gr}(\mathbb{AG}(M)) = 3.$$

Proof. Let $\operatorname{gr}(\mathbb{AG}(M)) = 3$ and $N_1 - N_2 - N_3 - N_1$ be a cycle in $\mathbb{AG}(M)$. If this cycle is also a cycle in $\mathbb{SAG}(M)$, there is nothing to prove. Thus without loss of generality, we may assume that N_1 and N_2 are not adjacent in $\mathbb{SAG}(M)$. By Lemma 2.5, there exists $L \leq N_1 \cap N_2$ such that $N_1 - L - N_2$ is a path in $\mathbb{SAG}(M)$. If there exists $0 \neq L_1 \lneq L$, then $L - L_1 - N_1 - L$ is a cycle in $\mathbb{SAG}(M)$. Now assume that L is minimal and $L_2 \leq M$ is a complement of L in M. Then $L \oplus L_2 \leq_e M$ and since M is not uniform, $L_2 \neq 0$. If $N_1 \cap L_2 = 0$, then $L - L_2 - N_1 - L$ is a cycle in $\mathbb{SAG}(M)$. Thus we assume that $N_1 \cap L_2 \neq 0$ and consider the following cases.
Case 1: $N_1 \cap L_2 \neq L_2$. Then since $N_2 - N_1 \cap L_2$ is an edge in $\mathbb{AG}(M)$, by Lemma 2.5 $L - N_2 - N_1 \cap L_2 - L$ is a cycle in $\mathbb{SAG}(M)$ or there exists $K \leq N_2 \cap (N_1 \cap L_2)$ such that $N_2 - K - N_1 \cap L_2$ is a path in $\mathbb{SAG}(M)$. In the latter, $L - N_2 - K - L$ is a cycle in $\mathbb{SAG}(M)$.
Case 2: $N_1 \cap L_2 = L_2$. Then $L_2 \subsetneq N_1$ and since $N_2 - L_2$ is an edge in $\mathbb{AG}(M)$, by Lemma 2.5 $L - N_2 - L_2 - L$ in $\mathbb{SAG}(M)$ or there exists $K \leq N_2 \cap L_2$ such that $N_2 - K - L_2$ is a path in $\mathbb{SAG}(M)$. In the latter, $L - N_2 - K - L$ is a cycle in $\mathbb{SAG}(M)$. The converse is clear. □

We have not found any example of a module that $\operatorname{gr}(\mathbb{AG}(M)) = 3$ and $\operatorname{gr}(\mathbb{SAG}(M)) \neq 3$. The lack of such counterexample together with the above proposition motivates the following fundamental conjecture.

Conjecture 3.2. *For any R-module M,* $\operatorname{gr}(\mathbb{SAG}(M)) = 3$ *if and only if* $\operatorname{gr}(\mathbb{AG}(M)) = 3$.

Proposition 3.3. *For any R-module M,* $\operatorname{gr}(\mathbb{SAG}(M)) = 4$ *if and only if* $\operatorname{gr}(\mathbb{AG}(M)) = 4$.

Proof. Let $\operatorname{gr}(\mathbb{AG}(M)) = 4$ and $N_1 - N_2 - N_3 - N_4 - N_1$ be a cycle in $\mathbb{AG}(M)$. We claim that $N_1 \cap N_2 = N_2 \cap N_3 = N_3 \cap N_4 = N_4 \cap N_1 = 0$ and this implies that $N_1 - N_2 - N_3 - N_4 - N_1$ is also a cycle in $\mathbb{SAG}(M)$. On the contrary, assume that $N_1 \cap N_2 \neq 0$. Then the following cases can occur.
Case 1: $N_1 \cap N_2 \notin \{N_1, N_2\}$. Then $N_1 - N_1 \cap N_2 - N_2 - N_1$ is a cycle in $\mathbb{AG}(M)$, a contradiction.
Case 2: $N_1 \cap N_2 = N_1$. Then $N_1 \subseteq N_2$ and so $N_1 - N_2 - N_3 - N_1$ is a cycle in $\mathbb{AG}(M)$, a contradiction.
Case 3: $N_1 \cap N_2 = N_2$. Then $N_2 \subseteq N_1$ and so $N_1 - N_2 - N_4 - N_1$ is a cycle in $\mathbb{AG}(M)$, a contradiction. Thus the claim is proved and so $\operatorname{gr}(\mathbb{SAG}(M)) = 4$.
Conversely, let $\operatorname{gr}(\mathbb{SAG}(M)) = 4$ and $N_1 - N_2 - N_3 - N_4 - N_1$ be a cycle in $\mathbb{SAG}(M)$. By [9, Theorem 3.4], $\operatorname{gr}(\mathbb{AG}(M)) \leq 4$. First, we claim that N_1 and N_3 are not adjacent in $\mathbb{AG}(M)$. On the contrary, assume that N_1 and N_3 adjacent in $\mathbb{AG}(M)$. By Lemma 2.5, there exists $L \leq N_1 \cap N_3$ such that L is adjacent both N_1 and N_3 in $\mathbb{SAG}(M)$. If $L \notin \{N_2, N_4\}$, then $N_3 - L - N_4 - N_3$ is a cycle in $\mathbb{SAG}(M)$, a contradiction. If $L = N_2$ or $L = N_4$, then $N_1 - N_2 - N_4 - N_1$ is a cycle in $\mathbb{SAG}(M)$, a contradiction. Similarly,

we can show that N_2 and N_4 are not adjacent in $\mathbb{AG}(M)$. Now if $\mathrm{gr}(\mathbb{AG}(M)) = 3$, then by Proposition 3.1, M must be uniform and as in proof of that proposition, we will have a triangle $N_5 - N_6 - N_7 - N_5$ in $\mathbb{AG}(M)$ such that N_5 is a minimal essential submodule of M and so for every vertex K, $N_5 \subseteq K$. If $N_5 \neq N_i$ for $i \in \{1, 2, 3, 4\}$, then $N_5 - N_1 - N_2 - N_5$ is a cycle in $\mathbb{SAG}(M)$, because $N_5 \subseteq N_4$ and also $N_5 \subseteq N_3$, a contradiction. If $N_5 = N_i$ for some $i \in \{1, 2, 3, 4\}$, without loss of generality, we may assume that $N_5 = N_2$. Then $N_1 - N_2 - N_4 - N_1$ is a cycle in $\mathbb{SAG}(M)$, because $N_2 \subseteq N_1$, which again is a contradiction. □

Corollary 3.4. *Let M be an R-module with* $\mathrm{u.dim}(M) \geq 2$. *Then $\mathbb{AG}(M)$ is a tree if and only if $\mathbb{SAG}(M)$ is a tree.*

Proof. Let $\mathbb{SAG}(M)$ be a tree. If $\mathbb{AG}(M)$ contains a cycle, then by [9, Theorem 3.4], $\mathrm{gr}(\mathbb{AG}(M)) \leq 4$. Now by Proposition 3.1 and Proposition 3.3, we have $\mathrm{gr}(\mathbb{SAG}(M)) = 4$ or $\mathrm{gr}(\mathbb{SAG}(M)) = 3$, a contradiction. Thus $\mathbb{AG}(M)$ is also tree. Clearly, if $\mathbb{AG}(M)$ is tree, then so is $\mathbb{SAG}(M)$. □

Proposition 3.5. (1) *If $\mathbb{AG}(M)$ is a bipartite graph, then so is $\mathbb{SAG}(M)$.*
(2) *If $\mathbb{SAG}(M)$ is a bipartite graph with parts V_1 and V_2 such that $|V_1| \geq 2$ and $|V_2| \geq 2$, then $\mathbb{AG}(M)$ is also bipartite.*
(3) *If $\mathbb{SAG}(M)$ is a bipartite graph with* $\mathrm{u.dim}(M) \geq 2$, *then $\mathbb{AG}(M)$ is also bipartite.*

Proof. (1) is clear.
(2). Let $\mathbb{SAG}(M)$ be a bipartite graph with the conditions said in (2). Suppose, on the contrary, there exist N and K in V_1 such that $N - K$ is an edge in $\mathbb{AG}(M)$. By Lemma 2.5, there exists $L \leq N \cap K$ such that both N and K are adjacent to L in $\mathbb{SAG}(M)$. Clearly $L \in V_2$. By hypothesis, we can choose $L \neq T \in V_2$ and consider $d_S(T, K)$ and $d_S(T, N)$, where $d_S(T, N)$ is the length of the shortest path from T to N in $\mathbb{SAG}(M)$. If $d_S(T, N) = 1$, then $L \subseteq N \cap K$ implies that $T - L$ is an adge in $\mathbb{SAG}(M)$, a contradiction. Similarly, $d_S(T, K) \neq 1$. Since $\mathbb{SAG}(M)$ is connected with diameter at most 3 and it is bipartite, we conclude that $d_S(T, N) = d_S(T, K) = 3$. Thus $T - T_1 - P - K$ is a path in $\mathbb{SAG}(M)$, for some $T_1 \in V_1$ and $P \in V_2$. If $P \neq L$, then P is adjacent to L in $\mathbb{SAG}(M)$, a contradiction, since P is adjacent to K in $\mathbb{SAG}(M)$. Thus $P = L$. Clearly $N \cap T_1 \neq 0$ and L is adjacent to $N \cap T_1$ in $\mathbb{SAG}(M)$, so $N \cap T_1 \in V_1$. Now one of the following cases may occur.
Case 1: $N \cap T_1 = N$. Then $N \subseteq T_1$ and since T is adjacent to T_1 in $\mathbb{SAG}(M)$, so T is adjacent to N in $\mathbb{SAG}(M)$, a contradiction.
Case 2: $N \cap T_1 = K$. Then $K \subseteq T_1$ and since T is adjacent to T_1 in $\mathbb{SAG}(M)$, so T is adjacent to K in $\mathbb{SAG}(M)$, a contradiction.
Case 3: $N \cap T_1 = T_1$. Then $T_1 \subseteq N$ and since K is adjacent to N in $\mathbb{AG}(M)$, we conclude that K is adjacent to T_1 in $\mathbb{AG}(M)$. Now by Lemma 2.5, there exists $T_2 \lneq T_1$ such that K is adjacent to T_2 in $\mathbb{SAG}(M)$. Thus $T_2 \in V_2$ and it is easy to see that $T_2 = L$. This means $L \subseteq T_1$ and hence T is adjacent to L in $\mathbb{SAG}(M)$, because T is adjacent to T_1 in $\mathbb{SAG}(M)$, a contradiction.
Case 4: $N \cap T_1 \notin \{N, K, T_1\}$. By replacing T_1 with $N \cap T_1$ in Case 3, we get a contradiction. Thus $\mathbb{AG}(M)$ is a bipartite graph, as desired.
(3). If $\mathbb{AG}(M)$ is not bipartite, then by Lemma 2.5, it contains a triangle and so $\mathrm{gr}(\mathbb{AG}(M)) = 3$. Now by Proposition 3.1, $\mathrm{gr}(\mathbb{SAG}(M)) = 3$, a contradiction. □

Corollary 3.6. *Suppose that one of the two graphs $\mathbb{AG}(M)$ and $\mathbb{SAG}(M)$ is bipartite with parts V_1 and V_2. If one of the following holds, then $\mathbb{AG}(M)$ and $\mathbb{SAG}(M)$ coincide.*
(1) $|V_1| \geq 2$ *and* $|V_2| \geq 2$.
(2) $\mathrm{u.dim}(M) \geq 2$.

Proof. Follows from Proposition 2.7(3) and Proposition 3.5 parts (2) and (3). □

A similar result of the following has appeared in [24, Lemma 3.3]. Here, we give a shorter proof of this fact.

Lemma 3.7. *If $\mathbb{SAG}(M)$ contains a cycle of odd length, then it contains a triangle.*

Proof. Using induction, we show that for every cycle of odd length $2n+1 \geq 5$, there exists a cycle with length $2k+1$ such that $k < n$. Assume that $N_1 - N_2 - \cdots - N_{2n+1} - N_1$ is a cycle with odd length $2n+1$. If two distinct non consecutive N_i and N_j are adjacent, the proof is complete. Otherwise, we set $0 \neq L = N_1 \cap N_3$. Then by Lemma 2.4, $L \neq N_i$ for all $1 \leq i \leq 2n+1$ and L is adjacent to both N_4 and N_{2n+1}. Hence we have the cycle $N_{2n+1} - L - N_4 - N_5 - \cdots - N_{2n+1}$, which is the desired cycle. □

Theorem 3.8. *If $\mathrm{gr}(\mathbb{SAG}(M)) = 4$ or $\mathrm{gr}(\mathbb{AG}(M)) = 4$, then $\mathbb{AG}(M)$ and $\mathbb{SAG}(M)$ coincide.*

Proof. Follows from Propositions 3.3 and 2.7(3). □

For any R-module M, since $\mathrm{diam}(\mathbb{AG}(M)) \leq 3$ and $\mathrm{diam}(\mathbb{SAG}(M)) \leq 3$, if either $\mathbb{AG}(M) = P_n$ or $\mathbb{SAG}(M) = P_n$, then $n \leq 4$. Now we have the following interesting proposition.

Proposition 3.9. *Let M be an R-module. Then*
(1) *For $n \neq 3$, $\mathbb{AG}(M) = P_n$ if and only if $\mathbb{SAG}(M) = P_n$.*
(2) *If M is not uniform, then $\mathbb{AG}(M) = P_3$ if and only if $\mathbb{SAG}(M) = P_3$*

Proof. (1). For $n = 1$ and $n = 2$, the proof is clear. Let $\mathbb{SAG}(M)$ be a path, say, $N_1 - N_2 - N_3 - N_4$. If $\mathbb{AG}(M) \neq \mathbb{SAG}(M)$, since the set of all vertices of $\mathbb{AG}(M)$ is equal to the set of all vertices of $\mathbb{SAG}(M)$, $\mathbb{AG}(M)$ must contain a cycle. Thus by [9, Theorem 3.4], $\mathrm{gr}(\mathbb{AG}(M)) \leq 4$. If $\mathrm{gr}(\mathbb{AG}(M)) = 4$, then by Proposition 3.3, $\mathrm{gr}(\mathbb{SAG}(M)) = 4$, a contradiction. If $\mathrm{gr}(\mathbb{AG}(M)) = 3$, then without loss of generality, we may assume N_1 and N_3 are adjacent in $\mathbb{AG}(M)$. Then by Lemma 2.5, there exists $L \subseteq N_3$ such that $N_1 - L$ is an edge in $\mathbb{SAG}(M)$. Since $\mathbb{SAG}(M) = P_4$, we will have $L = N_2$ and so $N_2 \subseteq N_3$. Now N_4 and N_2 must be adjacent in $\mathbb{SAG}(M)$, a contradiction. The converse is clear.
(2). Let $\mathbb{SAG}(M)$ be a path, say, $N_1 - N_2 - N_3$ and $\mathbb{AG}(M) \neq \mathbb{SAG}(M)$. Then $\mathbb{AG}(M)$ must be a triangle. Thus $\mathrm{gr}(\mathbb{AG}(M)) = 3$ and so by Proposition 3.1, $\mathrm{gr}(\mathbb{SAG}(M)) = 3$, a contradiction. The converse is clear. □

Theorem 3.10. *If $\mathbb{SAG}(M)$ is a tree, then either $\mathbb{SAG}(M)$ is a star graph or $\mathbb{SAG}(M) = P_4$. Moreover, $\mathbb{SAG}(M) = P_4$ if and only if $M = F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule.*

Proof. If M is a vertex of $\mathbb{SAG}(M)$, then there exists nonzero submodule $N \leq M$ such that $M(N : M) = 0$ and so $K(N : M) = 0$ for every nonzero submodule $K \leq M$. Thus every vertex is adjacent to N and since $\mathbb{SAG}(M)$ is tree, it must be a star graph. Now we assume that M is not a vertex of $\mathbb{SAG}(M)$ and $\mathbb{SAG}(M)$ is not star. Now by [19, Proposition 1.6.1], $\mathbb{SAG}(M)$ is a bipartite graph with parts V_1 and V_2 such that $|V_1| \geq 2$ and $|V_2| \geq 2$. By Proposition 3.5, $\mathbb{AG}(M)$ is a bipartite graph and so by Corollary 3.6, $\mathbb{SAG}(M)$ and $\mathbb{AG}(M)$ coincide. Now by [10, Theorem 2.7], $\mathbb{AG}(M) = P_4$ and hence $\mathbb{SAG}(M) = P_4$. For the latter assertion, if $\mathbb{SAG}(M) = P_4$, then by Proposition 3.9, $\mathbb{AG}(M) = P_4$ and so by [10, Theorem 2.7], the proof is complete. □

Theorem 3.11. *Let R be an Artinian ring and $\mathbb{SAG}(M)$ be a bipartite graph. Then $\mathbb{SAG}(M)$ is a star graph or $\mathbb{SAG}(M) = P_4$. Moreover, if R is an Artinian local ring, then $\mathbb{SAG}(M)$ is a star graph.*

Proof. Suppose that $\mathbb{SAG}(M)$ is not a star graph, then $\mathbb{SAG}(M)$ is a bipartite graph with parts V_1 and V_2 such that $|V_1| \geq 2$ and $|V_2| \geq 2$. By Corollary 3.6, $\mathbb{SAG}(M) = \mathbb{AG}(M)$. Now by [10, Theorem 2.8], $\mathbb{SAG}(M) = P_4$. Let $\mathfrak{m}$ be a unique maximal ideal of R. Since R

is an Artinian ring, there exists a natural number k such that $Mm^k = 0$ and $Mm^{k-1} \neq 0$. Clearly $Mm^{k-1}(N : M) = 0$ for each submodule N of M and so Mm^{k-1} is adjacent to every other vertex of $\mathbb{SAG}(M)$. Thus $\mathbb{SAG}(M)$ is a star graph. □

Proposition 3.12. *Let $M = M_1 \times M_2$ where $M_1 = Me \neq 0$, $M_2 = M(1-e) \neq 0$ and e be an idempotent element of R. If $\mathbb{SAG}(M)$ is a triangle-free graph, then one of the following statements holds.*
(1) *Both M_1 and M_2 are prime R-modules.*
(2) *One M_i is a prime module for $i = 1, 2$ and the other one is a module with unique non-trivial submodule. Moreover, $\mathbb{SAG}(M)$ has no cycle if and only if $M = F \times S$ or $M = F \times D$, where F is a simple module, S is a module with a unique non-trivial submodule and D is a prime module.*

Proof. Clearly M is not a uniform module. If $\mathbb{SAG}(M)$ is a triangle-free graph, then by Proposition 3.1, $\mathbb{AG}(M)$ is a triangle-free graph. Thus by [10, Theorem 2.6], (1) or (2) holds. Now suppose that $\mathbb{SAG}(M)$ has no cycle. Then by Theorem 3.10, $\mathbb{SAG}(M)$ is a star graph or $\mathbb{SAG}(M) = P_4$ and also $\mathbb{SAG}(M) = P_4$ if and only if $M = F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule. If $\mathbb{SAG}(M)$ is a star graph, then by Proposition 3.1, $\mathbb{AG}(M)$ is a star graph and so $\mathbb{SAG}(M) = \mathbb{AG}(M)$. Now by [10, Theorem 2.6], we are done. The converse is trivial. □

Lemma 3.13. *If $\mathbb{SAG}(M)$ is a regular graph of degree r, then so is $\mathbb{AG}(M)$; in particular $\mathbb{AG}(M) = \mathbb{SAG}(M)$.*

Proof. Let $\mathbb{SAG}(M)$ be a regular graph of degree r. If $\mathbb{SAG}(M)$ is a complete graph, then it is clear that $\mathbb{AG}(M)$ is also complete. Now assume that $\mathbb{SAG}(M)$ is not complete. Then we show that $\mathbb{AG}(M) = \mathbb{SAG}(M)$. Suppose, on the contrary, there exist two vertices N and K that are adjacent in $\mathbb{AG}(M)$ but are not adjacent in $\mathbb{SAG}(M)$. Then by Lemma 2.5, there exists $L \leq N \cap K$ such that both N and K are adjacent to L in $\mathbb{SAG}(M)$. On the other hand since $L \subseteq N$, every vertex that is adjacent to N, is also adjacent to L. Thus we conclude that $\deg_S(L) \geq r - 1 + 2 = r + 1$, a contradiction. □

Theorem 3.14. *Let* $\mathrm{ann}(M)$ *be a nil ideal of R. If $\mathbb{SAG}(M)$ is a regular graph of finite degree, then $\mathbb{SAG}(M)$ is a complete graph; in particular $\mathbb{AG}(M) = \mathbb{SAG}(M)$.*

Proof. Suppose that $\mathbb{SAG}(M)$ is a regular graph of degree r. Then by Lemma 3.13, $\mathbb{AG}(M)$ is also a regular graph of degree r and $\mathbb{AG}(M) = \mathbb{SAG}(M)$. Now by [10, Theorem 2.9], $\mathbb{AG}(M)$ is a complete graph and so is $\mathbb{SAG}(M)$. □

In the following theorem for any vertex K in the graph $\mathbb{AG}(M)$, we denote by $N_A(K)$, the set of all vertices of G adjacent to K.

Theorem 3.15. *Let $\mathbb{AG}(M)$ be a regular graph of degree r. If $|V(\mathbb{AG}(M))| \geq r + 2$, then $\mathbb{SAG}(M)$ is also regular; in particular $\mathbb{SAG}(M) = \mathbb{AG}(M)$.*

Proof. Clearly $r \neq 1$. Suppose that N and K are adjacent in $\mathbb{AG}(M)$. We claim that $N \cap K = 0$ and so N and K are adjacent in $\mathbb{SAG}(M)$. Suppose, on the contrary, $N \cap K \neq 0$. One of the following cases holds.
Case 1: $N \cap K \notin \{N, K\}$. Then we may assume $N_A(K) = \{N, N \cap K, K_1, K_2, \cdots, K_{r-2}\}$. As $N_A(K) \setminus \{N \cap K\} \subseteq N_A(N \cap K)$ and $\mathbb{SAG}(M)$ is a regular graph of degree r, we have $N_A(N \cap K) = \{N, K, K_1, K_2, \cdots, K_{r-2}\}$. This implies that $N_A(N) = \{K, N \cap K, K_1, K_2, \cdots, K_{r-2}\}$. Now, since $|V(\mathbb{AG}(M))| \geq r + 2$, we consider a vertex L such that $L \notin \{N, K, N \cap K, K_1, K_2, \cdots, K_{r-2}\}$. Clearly L is not adjacent to any of the vertices N, K and $N \cap K$. Thus there exists a subset $\{L_1, L_2\} \subseteq V(\mathbb{AG}(M)) \setminus \{N, K, N \cap K, K_1, K_2, \cdots, K_{r-2}\}$ such that L_i is adjacent to L in $\mathbb{AG}(M)$, for $1 \leq i \leq 2$. It is easy to check that $0 \neq N \cap L \notin \{L, N, K, N \cap K, K_1, K_2, \cdots, K_{r-2}\}$. Now since K is adjacent to N, K is also adjacent to $N \cap L$ and so $\deg_A(K) \geq r + 1$, a contradiction.

Case 2: $N \cap K = N$. Then $N \subseteq K$. Suppose $N_A(K) = \{N, K_1, K_2, \cdots, K_{r-1}\}$. Then $N_A(N) = \{K, K_1, K_2, \cdots, K_{r-1}\}$. Now since $|V(\mathbb{AG}(M))| \geq r + 2$, there exists a vertex L such that $L \notin \{N, K, K_1, K_2, \cdots, K_{r-1}\}$. Clearly L is not adjacent to any of the vertices N and K. Thus there exists a vertex L_1 such that is adjacent to L in $\mathbb{AG}(M)$ and $L_1 \notin \{N, K, K_1, K_2, \cdots, K_{r-1}\}$. It is easy to check that $0 \neq N \cap L \notin \{L, N, K, K_1, K_2, \cdots, K_{r-1}\}$. Clearly $N \cap L$ is adjacent to K and so $\deg_A(K) \geq r + 1$, a contradiction.
Case 3: $N \cap K = K$. It is similar to Case 2. □

For any $N \leq M$, the prime radical $\mathrm{rad}_M(N)$ or simply $\mathrm{rad}(N)$ is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule, $\mathrm{rad}_M(N)$ is defined to be M. Also the set of all minimal prime submodules of M is denoted by $\mathrm{Min}(M)$.

Proposition 3.16. *Let M be a finitely generated module, $\mathrm{ann}(M)$ be a nil ideal and $|Min(M)| = 1$. If $\mathbb{SAG}(M)$ is a triangle-free graph, then $\mathbb{SAG}(M)$ is a star graph.*

Proof. Let $\mathbb{SAG}(M)$ be a triangle-free graph. By Lemma 3.7, $\mathbb{SAG}(M)$ contains no odd cycle. Now by [19, Proposition 1.6.1] $\mathbb{SAG}(M)$ is a bipartite graph. Suppose, on the contrary, $\mathbb{SAG}(M)$ is not star. Thus $\mathbb{SAG}(M)$ is a bipartite graph with parts V_1, V_2 such that $|V_1| \geq 2$ and $|V_2| \geq 2$. By Proposition 3.5, $\mathbb{AG}(M)$ is also bipartite and hence $\mathbb{AG}(M)$ is triangle-free. Now by [10, Theorem 2.13], $\mathbb{AG}(M)$ is a star graph and so is $\mathbb{SAG}(M)$, a contradiction. □

Corollary 3.17. *Let M be a finitely generated module, $\mathrm{ann}(M)$ a nil ideal and $|\mathrm{Min}(M)| = 1$. If $\mathbb{SAG}(M)$ is a bipartite graph, then $\mathbb{SAG}(M)$ is a star graph.*

Remark 3.18. Let $\mathrm{u.dim}M = n$, where M is an R-module. Then we have $U_1 \oplus U_2 \oplus \ldots \oplus U_n \leq M$, where $U_i \neq 0$ for $1 \leq i \leq n$. By Lemma 2.4, U_i and U_j are adjacent for each $i \neq j$ and hence $\mathrm{u.dim}M \leq cl(\mathbb{SAG}(M))$.

Proposition 3.19. *For every module M, $cl(\mathbb{SAG}(M)) = 2$ if and only if $\chi(\mathbb{SAG}(M)) = 2$; in particular $\mathbb{SAG}(M)$ is bipartite if and only if $\mathbb{SAG}(M)$ is triangle-free.*

Proof. Suppose that $cl(\mathbb{SAG}(M)) = 2$ and $\mathbb{SAG}(M)$ is not bipartite. Then $\mathbb{SAG}(M)$ contains an odd cycle and so by Lemma 3.7, the graph contains a triangle, a contradiction. Thus $\mathbb{SAG}(M)$ is bipartite and so $\chi(\mathbb{SAG}(M)) = 2$. The converse is clear. □

If M is a cyclic module, then clearly M is a multiplication module and so by Proposition 2.3, the two graphs $\mathbb{SAG}(M)$ and $\mathbb{AG}(M)$ coincide. Now by [10], we have the following results.

Proposition 3.20. *Let M be a cyclic module.*
(1) *If $\{P_1, \cdots, P_n\}$ is a finite set of distinct minimal prime submodules of M, then $\mathbb{SAG}(M)$ has a clique of size n.*
(2) *$cl(\mathbb{SAG}(M)) \geq |\mathrm{Min}(M)|$ and if $|\mathrm{Min}(M)| \geq 3$, then $\mathrm{gr}(\mathbb{SAG}(M)) = 3$.*
(3) *If $\mathrm{rad}_M(0) = (0)$, then $\chi(\mathbb{SAG}(M)) = cl(\mathbb{SAG}(M)) = |\mathrm{Min}(M)|$.*

Proposition 3.21. *Let M be a cyclic module and $\mathrm{ann}(M)$ be a nil ideal of R.*
(1) *If $\mathrm{rad}_M(0) \neq (0)$ and $|\mathrm{Min}(M)| = 2$, then either $\mathbb{SAG}(M)$ contains a cycle or $\mathbb{SAG}(M) = P_4$.*
(2) *If $|\mathrm{Min}(M)| \geq 3$, then $\mathbb{SAG}(M)$ contains a cycle.*

4. Dominating set and total dominating set

For every graph G, the *dominating number* of G and the *total dominating number* of G are denoted by $\gamma(G)$ and $\gamma_t(G)$, respectively. A dominating set of cardinality $\gamma(G)$ ($\gamma_t(G)$) is called a *γ-set* (*γ_t-set*). Several authors studied the (total) domination number in the zero-divisor graphs and the annihilating-ideal graphs, see for example [20, 22, 23]. In this section we compare $\gamma(\mathbb{SAG}(M))$ with $\gamma(\mathbb{AG}(M))$ and also $\gamma_t(\mathbb{SAG}(M))$ with $\gamma_t(\mathbb{AG}(M))$. Some results are similar to some of the results for $\gamma(\mathbb{AG}(R))$ in [23]. In the following example we consider M as a $\mathbb{Z}$-module. We remind that $V(\mathbb{AG}(M)) = V(\mathbb{SAG}(M))$.

Example 4.1. (1) If $M = \mathbb{Z}_p \oplus \mathbb{Z}_p$, where p is prime, then $\gamma(\mathbb{SAG}(M)) = 1$ and $\gamma_t(\mathbb{SAG}(M)) = 2$, because $\mathbb{SAG}(M)$ is a complete graph.
(2) If $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ and $N = \mathbb{Z}_p \oplus (0)$, then since $(N : M) = p^2\mathbb{Z}$, every vertex of $\mathbb{SAG}(M)$ is adjacent to N. Thus $\gamma(\mathbb{SAG}(M)) = 1$.
(3) If $M = \mathbb{Z}_p \oplus \mathbb{Z}_q$, where p and q are distinct primes, then $\gamma(\mathbb{SAG}(M)) = 1$ and $\gamma_t(\mathbb{SAG}(M)) = 2$.
(4) Let $M = \mathbb{Z}_{pqr}$, where p, q, r are distinct primes. Then $\{N_1, N_2, N_3\}$ is both a γ-*set* and γ_t-*set*, where $N_1 = pq\mathbb{Z}_{pqr}$, $N_2 = pr\mathbb{Z}_{pqr}$ and $N_3 = qr\mathbb{Z}_{pqr}$. Thus $\gamma(\mathbb{SAG}(M)) = \gamma_t(\mathbb{SAG}(M)) = 3$.
(5) Let $M_1 = M_2 = \mathbb{Z}_p$ and $M = M_1 \oplus M_2$. Then we have $\gamma(\mathbb{SAG}(M)) = 1$ and $\gamma(\mathbb{SAG}(M_1)) = \gamma(\mathbb{SAG}(M_2)) = 0$. Hence $\gamma(\mathbb{SAG}(M_1 \oplus M_2)) \neq \gamma(\mathbb{SAG}(M_1)) + \gamma(\mathbb{SAG}(M_2))$.

Notice that for any R-module M, $\gamma(\mathbb{AG}(M)) \leq \gamma(\mathbb{SAG}(M))$. However, we have the following interesting result.

Theorem 4.2. *Let M be an R-module. Then*
(1) $\gamma(\mathbb{SAG}(M)) = 1$ *if and only if* $\gamma(\mathbb{AG}(M)) = 1$.
(2) *If* $\gamma(\mathbb{SAG}(M)) = 2$*, then* $\gamma(\mathbb{AG}(M)) = 2$.
(3) *If* $\gamma(\mathbb{AG}(M)) = n > 1$*, then* $n \leq \gamma(\mathbb{SAG}(M)) \leq 2n$

Proof. (1). Let $\gamma(\mathbb{SAG}(M)) = 1$ and $\{N\}$ be a γ-*set*. Then since every edge in $\mathbb{SAG}(M)$ is also an edge in $\mathbb{AG}(M)$, $\{N\}$ is a γ-*set* in $\mathbb{AG}(M)$ and so $\gamma(\mathbb{SAG}(M)) = 1$. Conversely, suppose that $\{N\}$ is a γ-*set* in $\mathbb{AG}(M)$ and we set $N_1 = M(N : M)$. If $N_1 = 0$, or $N_1 = N$, then it is easy to see that $\{N\}$ is also a γ-*set* in $\mathbb{SAG}(M)$ and we are done. Now assume that $0 \neq N_1 \subsetneq N$. Then for each vertex $K \neq N$ of $\mathbb{SAG}(M)$, we have $M(N : M)(K : M) = 0$. Thus $N_1 - N$ is an edge in $\mathbb{AG}(M)$ and so $M(N_1 : M)(N : M) = 0$. We set $N_2 = M(N_1 : M)$. Again if $N_2 = 0$, then clearly $\{N_1\}$ is a γ-*set* in $\mathbb{SAG}(M)$. Now if $N_2 \neq 0$, then we claim that $\{N_2\}$ is a γ-*set* in $\mathbb{SAG}(M)$. Since $N_2(N : M) = 0$ and $N_2 \neq N$, $N - N_2$ is an edge in $\mathbb{SAG}(M)$. Suppose that $K \neq N$ is another vertex of $\mathbb{SAG}(M)$. Then K must be adjacent to N in $\mathbb{AG}(M)$ and so $M(N : M)(K : M) = 0$. Since $N_1 \subseteq N$, we have $N_2(K : M) = M(N_1 : M)(K : M) \subseteq M(N : M)(K : M) = 0$. Thus $K = N_2$ or K is adjacent to N_2 in $\mathbb{SAG}(M)$ and the proof is complete.
(2). Let $\gamma(\mathbb{SAG}(M)) = 2$. Then $\gamma(\mathbb{AG}(M)) \leq 2$. If $\gamma(\mathbb{AG}(M)) = 1$, then by (1) we have $\gamma(\mathbb{SAG}(M)) = 1$, a contradiction. Thus $\gamma(\mathbb{AG}(M)) = 2$.
(3). Let $\gamma(\mathbb{AG}(M)) = n$ and $\{N_1, \cdots, N_n\}$ be a γ-*set* in $\mathbb{AG}(M)$. We set $K_i = M(N_i : M)$ for $i = 1, 2, \cdots, n$. If $K_i = M(N_i : M) = 0$, for some i, then $\{N_i\}$ is a γ-*set* in $\mathbb{AG}(M)$ and hence $\gamma(\mathbb{AG}(M)) = 1$, a contradiction. We claim that $\{N_1, \cdots, N_n, K_1, \cdots, K_n\}$ is a dominating set in $\mathbb{SAG}(M)$. Let $K \notin \{N_1, \cdots, N_n\}$ be a vertex of $\mathbb{SAG}(M)$. Then K is adjacent to N_i in $\mathbb{AG}(M)$ for some $i \in \{1, \cdots, n\}$. Thus $M(N_i : M)(K : M) = 0$ and hence $K_i(K : M) = 0$. This means that $K = K_i$ or K is adjacent to K_i in $\mathbb{SAG}(M)$. Thus we have $\gamma(\mathbb{SAG}(M)) \leq 2n$. □

Proposition 4.3. *For any R-module M,* $\gamma_t(\mathbb{SAG}(M)) \leq \gamma_t(\mathbb{AG}(M))$.

Proof. If $\gamma_t(\mathbb{AG}(M)) = \infty$, there is no thing to prove. Let $\gamma_t(\mathbb{AG}(M)) = n$ and $\{N_1, \cdots, N_n\}$ be a γ_t-*set* in $\mathbb{AG}(M)$. Set $K_i = M(N_i : M)$ for $i = 1, 2, \cdots, n$. If $K_i = 0$ for some i, then $\{N_i, N\}$ is a γ_t-*set* in both the graph $\mathbb{AG}(M)$ and the graph $\mathbb{SAG}(M)$, for every vertex $N \neq N_i$ and we are done. Now, suppose that $K_i \neq 0$ for every $1 \leq i \leq n$. We may assume the indexing is arranged such that $K_1, K_2, \cdots, K_r$ are pairwise distinct ($r \leq n$). Let K be a vertex in $\mathbb{SAG}(M)$. If $K = K_i$, for some $1 \leq i \leq n$, then since there exists $1 \leq j \leq n$ such that N_i is adjacent to N_j in $\mathbb{AG}(M)$ we have K_i is adjacent to K_j in $\mathbb{SAG}(M)$. Now if $K \neq K_i$, for any $1 \leq i \leq n$, then there exists $1 \leq i \leq n$ such that K is adjacent to N_i in $\mathbb{AG}(M)$. Thus $K_i(K : M) = M(N_i : M)(K : M) = 0$ and so K is adjacent to K_i. This means that $\{K_1, \cdots, K_r\}$ is a total dominating set in $\mathbb{SAG}(M)$. Therefore $\gamma_t(\mathbb{SAG}(M)) \leq n$. □

Theorem 4.4. *For any R-module M,* $\gamma_t(\mathbb{SAG}(M)) = \gamma(\mathbb{SAG}(M))$ *or* $\gamma_t(\mathbb{SAG}(M)) = \gamma(\mathbb{SAG}(M)) + 1$.

Proof. Let $\gamma_t(\mathbb{SAG}(M)) \neq \gamma(\mathbb{SAG}(M))$ and D be a γ-*set* in $\mathbb{SAG}(M)$. If $\gamma(\mathbb{SAG}(M)) = 1$, then it is clear that $\gamma_t(\mathbb{SAG}(M)) = 2$. So let $\gamma(\mathbb{SAG}(M)) \gneq 1$ and set $m = max\{n \mid \cap_{i=1}^{n} N_i \neq 0$, for some $N_1, \cdots, N_n \in D\}$. Since $\gamma_t(\mathbb{SAG}(M)) \neq \gamma(\mathbb{SAG}(M))$, we have $m \geq 2$. Suppose that $\cap_{i=1}^{m} N_i \neq 0$, for some $N_1, \cdots, N_m \in D$. Since D is a γ-*set* in $\mathbb{SAG}(M)$, there exist distinct vertices $K_1, \cdots, K_m$ such that K_i is adjacent to N_i for $1 \leq i \leq m$. As $\cap_{i=1}^{m} N_i \subseteq N_i$, we conclude that K_i is adjacent to $\cap_{i=1}^{m} N_i$, for each i. Now we claim that $S = \{\cap_{i=1}^{m} N_i, K_1, \cdots, K_m\} \cup D \setminus \{N_1, \cdots, N_m\}$ is a γ_t-*set* in $\mathbb{SAG}(M)$. Let L be a vertex of $\mathbb{SAG}(M)$. Then one of the following cases holds.
Case 1: $L \in D$. If $L \in \{N_1, \cdots, N_m\}$, then L is adjacent to K_i, for some $1 \leq i \leq m$. If $L \notin \{N_1, \cdots, N_m\}$, then by the maximality of m, $\cap_{i=1}^{m} N_i \cap L = 0$ and hence L is adjacent to $\cap_{i=1}^{m} N_i$.
Case 2: $L \notin D$. If L is adjacent to one of the N_i's, then L is adjacent to $\cap_{i=1}^{m} N_i$. Otherwise L is adjacent to one of the element of $D \setminus \{N_1, \cdots, N_m\}$. This means that S is a γ_t-*set* in $\mathbb{SAG}(M)$. Thus $\gamma_t(\mathbb{SAG}(M)) = \gamma(\mathbb{SAG}(M)) + 1$. □

Example 4.5. (1) Let $R = \mathbb{Z}$, $M = \mathbb{Z}_4$, $N = 2\mathbb{Z}_4$. Then $\{N\}$ is both a γ-*set* and γ_t-*set* in $\mathbb{SAG}(M)$. Thus $\gamma_t(\mathbb{SAG}(M)) = \gamma(\mathbb{SAG}(M)) = 1$.
(2) Let $R = \mathbb{Z}$, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $N_1 = \mathbb{Z}_2 \oplus (0)$, and $N_2 = (0) \oplus \mathbb{Z}_2$. Then $\{N_1\}$ is a γ-*set* and $\{N_1, N_2\}$ is a γ_t-*set* in $\mathbb{SAG}(M)$. Thus $\gamma(\mathbb{SAG}(M)) = 1$ and $\gamma_t(\mathbb{SAG}(M)) = 2$.

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