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Research Article

Sharp upper bounds of A_α-spectral radius of cacti with given pendant vertices

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Abstract

For $\alpha \in [0,1]$, let $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$ be A_α-matrix, where $A(G)$ is the adjacent matrix and $D(G)$ is the diagonal matrix of the degrees of a graph G. Clearly, $A_0(G)$ is the adjacent matrix and $2A_{\frac{1}{2}}$ is the signless Laplacian matrix. A connected graph is a cactus graph if any two cycles of G have at most one common vertex. We first propose the result for subdivision graphs, and determine the cacti maximizing A_α-spectral radius subject to fixed pendant vertices. In addition, the corresponding extremal graphs are provided. As consequences, we determine the graph with the A_α-spectral radius among all the cacti with n vertices; we also characterize the n-vertex cacti with a perfect matching having the largest A_α-spectral radius.

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1. Introduction

Throughout this paper, we consider finite simple connected graph G with vertex set $V(G)$ and edge set $E(G)$. The order of a graph is the number of vertices $|V(G)| = n$ and the size is the number of edges $|E(G)|$. Let $v \in V(G)$ be a vertex of G, $N(v) = N_G(v) = \{w \in V(G), vw \in E(G)\}$ be the neighborhood of v , and $d_G(v)$ (or briefly d_v) be the degree of v with $d_G(v) = |N(v)|$. If e is an edge of G and $G - e$ contains at least two components, then e is a cut edge of G. If $P_k = v_1v_2 \cdots v_k$ is a subgraph of G such that v_1 is a cut vertex of degree at least 3, $d(v_k) = 1$ and $d(v_i) = 2$ for $i \in [2, k-1]$, then P_k is called a pendant path in G. For other undefined notations and terminologies, refer to [2].

It's known that $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal matrix of the degrees of G. The signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. For $\alpha \in [0,1]$, the A_α-matrix

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$$

is given by Nikiforov [15]. Clearly, $A_0(G)$ is the adjacent matrix and $2A_{\frac{1}{2}}$ is the signless Laplacian matrix of G, respectively.

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The studies of the (adjacency, signless Laplacian) spectral radius are interesting and meaningful [7, 10–12, 19–23]. As examples, the spectral radius of trees are proposed by Lovász and J. Pelikán [14]. Feng et al.[10] studied the minimal Laplacian spectral radius of trees with given matching number. Chen [4] found the properties of spectra of graphs and their line graphs. Cvetković [8] explored the signless Laplacian spectra of graphs and a spectral theory in graphs. The bounds of signless Laplacian spectral radius and its hamiltonicity are studied by Zhou [24]. Lin and Zhou [13] obtained graphs with at most one signless Laplacian eigenvalue larger than three. In addition to the successful considerations of these spectral radius, A_α-spectral radius is provided as a general version of adjacency and signless Laplacian radius, and this area would be challenging. For the A_α-spectral radius, Nikiforov et al. [15, 16]introduced some properties of this spectral radius and provided the upper bounds on trees.

It is known that a tree is a noncyclic graph. If some vertices in a tree are replaced by cycles, then this graph has some cycles. The trees are extended as the definition that a cactus graph is a connected graph such that any two cycles have at most one common vertex. Denoted by $\mathcal{C}_n^k$ the set of all cacti with n vertices and k pendant vertices.

The cactus graphs have attracted many interests among the mathematical literature including algebra and graph theory. For instance, the properties of cacti with n vertices [3] are explored by Borovićanin and Petrović. Chen and Zhou [5] investigated some upper bounds of the signless Laplacian spectral radius of cactus graphs. The signless Laplacian spectral radius of cacti with given matching number are obtained by Shen et al. [17]. Some results for spectral radius on cacti with k pendant vertices are studied Wu et al. [18]. Ye et al. [22] gave the maximal adjacency or signless Laplacian spectral radius of graphs subject to fixed connectivity.

Motivated by the above results, in this paper, we generalize the results of A_α-spectra from the trees to the cacti subject to fixed pendant vertices. For $\alpha \in [0, 1]$, we first propose the result for subdivision graphs, and determine the cacti maximizing A_α-spectral radius subject to fixed pendant vertices. In addition, the corresponding extremal graphs are determined. As consequences, we determine the graph with the A_α-spectral radius among all the cacti with n vertices; we also characterize the n-vertex cacti with a perfect matching having the largest A_α-spectral radius.

2. Preliminary

In this section, we provide some important concepts and lemmas that will be used in the main proofs.

If G is a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G)$, then the $A_\alpha-$matrix $A_\alpha(G)$ of G has the (i, j)-entry of $A_\alpha(G)$ is $1 - \alpha$ if $v_iv_j \in E(G)$; $\alpha d(v_i)$ if $i = j$, and otherwise 0. For $\alpha \in [0, 1]$, let $\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \cdots \geq \lambda_n(A_\alpha(G))$ be the eigenvalues of $A_\alpha(G)$. The A_α-spectral radius of G is considered as the maximal eigenvalue $\rho(G) := \lambda_1(A_\alpha(G))$. Let $X = (x_{v_1}, x_{v_2}, \cdots, x_{v_n})^T$ be a real vector of $\rho(G)$. By $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$, we have the quadratic formula of $X^TA_\alpha(G)X$ can be expressed that

$$X^TA_\alpha(G)X = \alpha \sum_{v_i \in V(G)} x_{v_i}^2 d_{v_i} + 2(1 - \alpha) \sum_{v_iv_j \in E(G)} x_{v_i}x_{v_j}.$$

Because $A_\alpha(G)$ is a real symmetric matrix, and by Rayleigh principle, we have the formula $\rho(G) = max_{X \neq 0} \frac{X^TA_\alpha(G)X}{X^TX}$. Furthermore, if X is a unit eigenvector of$A_\alpha(G)$ corresponding to $\rho(G)$, then we have the formula $\rho(G) = X^TA_\alpha(G)X$.

As we know that once X is an eigenvector of $\rho(G)$ for a connected graph G, X should be unique and positive. The corresponding eigenequations for $A_\alpha(G)$ is rewritten as

$$\rho(G)x_{v_i} = \alpha d_{v_i} x_{v_i} + (1-\alpha) \sum_{v_iv_j \in E(G)} x_{v_j}. \tag{2.1}$$

As $A_1(G) = D(G)$, we study the $A_\alpha-$matrix for $\alpha \in [0,1)$ below. Based on the definition of A_α-spectral radius, we have

Lemma 2.1 ([16, 21])**.** Denote by $A_\alpha(G)$ the A_α-matrix of a connected graph G with $\alpha \in [0,1)$, $v, w \in V(G)$, $u \in S \subset V(G)$ such that $S \subset N(v) \setminus (N(w) \cup \{w\})$. Let H be a graph with vertex set $V(G)$ and edge set $E(G) \setminus \{uv, u \in S\} \cup \{uw, u \in S\}$, and X a unit eigenvector to $\rho(A_\alpha(G))$. If $x_w \geq x_v$ and $|S| \neq 0$, then $\rho(H) \geq \rho(G)$.

Lemma 2.2 ([22])**.** Let $A_\alpha(G)$ the A_α-matrix of a connected graph G with $\alpha \in [0,1)$, $s, t, u, v \in V(G)$, $st, uv \in E(G)$, $sv, tu \notin E(G)$. Let H be a graph with vertex set $V(G)$ and edge set $E(G) \setminus \{uv, st\} \cup \{sv, ut\}$, and X a unit eigenvector to $\rho(A_\alpha(G))$. If $(x_s - x_u)(x_v - x_t) \geq 0$, then $\rho(H) \geq \rho(G)$.

If G is a connected graph, then $A_\alpha(G)$ is a nonnegative irreducible symmetric matrix. By the results of [1, 6, 15], if we add some edges to a connected graph, then A_α-spectral radius will increase and the following lemma is straightforward.

Lemma 2.3. If H is a proper subgraph of a connected graph G, and ρ is the A_α-spectral radius, then $\rho(H) < \rho(G)$.

Let $P_t = v_0v_1v_2\cdots v_t$ be a subgraph of G. If v_0 is a cut vertex of degree at least 3, $d(v_t) = 1$ and $d(v_j) = 2$ with $j \in [1, t-1]$, then P_t is called a pendant path in G. The following lemma is useful below.

Lemma 2.4. Let $G \in \mathcal{C}_n^k$. If $\rho(G)$ is maximal, then all pendant paths share a common vertex.

Proof. Assume that G is a cactus graph with k pendant vertices and contains at least two pendant paths $P_t = v_0v_1\cdots v_t$ and $P_s = u_0u_1\cdots u_s$. Note that $d(u_0), d(v_0) \geq 3$. Without loss of generality, let $x_{v_0} \geq x_{u_0}$. Suppose that u_0 is a vertex in a cycle and this cycle contains at least one edge of the shortest path $P[u_0, v_0]$ between u_0 and v_0. Set G_1 to be a new graph with vertex set $V(G)$ and edge set $E(G)\setminus\{u_0v, v \in N\} \cup \{v_0v, v \in N\}$ with $N = N(u_0) \setminus \{w_1, w_2\}$, where w_1 is in $P[u_0, v_0]$, and v_0, w_1, w_2 are in the same cycle; if u_0 is not in any cycle, then let G_2 be a new graph with vertex set $V(G)$ and edge set $E(G) - \{u_0v, v \in N\} \cup \{v_0v, v \in N\}$ with $N = N(u_0) \setminus \{w_1, w_2\}$, where w_1 is in the shortest path between v_0 and u_0, and w_2 is another neighbor of u_0.

Note that both G_1 and G_2 are cacti with k pendant vertices. By Lemma 2.1, we have $\rho(G_1) \geq \rho(G)$ and $\rho(G_2) \geq \rho(G)$. We can continue this process and move all pendant paths to a common vertex such that $\rho(G)$ is increasing. Then this lemma is proved. □

Lemma 2.5. Let $G \in \mathcal{C}_n^k$. If $\rho(G)$ is maximal, then the length of any pendant path is at most 2, and there is at most one pendant path of the length 2.

Proof. First we prove the length of any pendant path is at most 2. We prove it by a contradiction. Assume there are have a pendent path P, $P = v_0v_1\cdots v_m$, $m \geq 3$. Let G_1 be a new graph with vertex set $V(G)$ and $E(G) + v_1v_{m-1}$, then G_1 is a cactus with k pendent vertices and $\rho(G_1) > \rho(G)$ (by Lemma 2.3). Then there exists a contradicted graph. Thus, if $\rho(G)$ is maximal, then the length of any pendant path is at most 2. Next we prove there is at most one pendant path of length 2. Suppose there are r, $(r > 1)$ pendent path of the length 2. Without loss of generality $P_i = v_0v_{i1}v_{i2}; (i = 1, 2, \cdots, r)$. Let G_2 be a new graph with vertex set $V(G)$ and $E(G) \cup \{v_{11}v_{21}, v_{31}v_{41}, \cdots, v_{(2\lfloor \frac{r}{2} \rfloor - 1)1}v_{(2\lfloor \frac{r}{2} \rfloor)1}\}$, then G_2 is a cactus with k pendent vertices and $\rho(G_2) > \rho(G)$ (by Lemma 2.3). Then there exists a contradicted graph. Thus, if $\rho(G)$ is maximal, there is at most one pendant path of the length 2. This completes the proof. $\square$

Lemma 2.6. Let $G \in \mathcal{C}_n^k$. If $\rho(G)$ is maximal, then there does not exist an internal path such that it is built by cut edges.

Proof. We prove it by a contradiction. Note that $d(v_0), d(v_t) \geq 3$. Let $P_t = v_0v_1 \cdots v_t$ be an internal path of G such that every edge of P_t is an cut edge. If $t \geq 2$, then let $G_1 = G + v_0v_t$. Then G_1 is a cactus with k pendant vertices and G is a proper subgraph of G_1. By Lemma 2.3, we have $\rho(G_1) > \rho(G)$, which is a contradiction. Next we consider $t = 1$. Without loss of generality, let $x_0 \geq x_1$ and $w \in N(v_1) \setminus \{v_0, v_1'\}$ such that v_1' is a neighbor except for v_0. Denote a new graph G_2 with vertex set $V(G_2) = V(G)$ and edge set $E(G_2) = E(G) \setminus \{v_1w, w \in N(v_1) \setminus \{v_0, v_1'\}\} \cup \{v_0w, w \in N(v_1) \setminus \{v_0, v_1'\}\}$. Then G_2 is a cactus with k pendant vertices and $\rho(G_2) \geq \rho(G)$ (by Lemma 2.1). These are contradictions and this lemma is proved. $\square$

Lemma 2.7. Let $G \in \mathcal{C}_n^k$. If $\rho(G)$ is maximal, then all cycles share a common vertex.

Proof. Suppose that there are two cut vertices v_0, v_1 in G such that not all cycles contain them. If there are only two cycles, then it is proved by Lemma 2.6: there does not exist an internal path such that it is built by cut edges. If there are more 3 cycles, then choose such v_0 and v_1 having the longest distance. Then $d(v_0), d(v_1) \geq 4$. Without loss of generality, let $x_{v_0} \geq x_{v_1}$ and $w \in N(v_1)\setminus\{v_0\}$. Denote a new graph G_1 with vertex set $V(G_1) = V(G)$ and edge set $E(G_1) = E(G) \setminus \{v_1w, w \in N(v_1) \setminus \{v_l, v_l'\}\} \cup \{v_0w, w \in N(v_1) \setminus \{v_l, v_l'\}\}$, where v_l, v_l' are neighbors of v_1 and on a same cycle. Then G_2 is a cactus with k pendant vertices and $\rho(G_1) \geq \rho(G)$ (by Lemma 2.1). We can continue this method to increase $\rho(G)$ until there exist a unique cut vertex sharing with all cycles. So, the result is proved. $\square$

Lemma 2.8. Let $G \in \mathcal{C}_n^k$. If $\rho(G)$ is maximal, then the length of any cycle is at most 4, and there is at most one cycle of length 4.

Proof. Let $C_t = v_1v_2 \cdots v_tv_1$ be a cycle of length t in G and v_1 is a cut vertex. If $x_{v_1} \geq x_{v_3}$, we build a new graph G_1 such that $V(G_1) = V(G)$ and $E(G_1) = E(G) \setminus \{v_3v_4\} \cup \{v_1v_4\}$. Then $\rho(G) \leq \rho(G_1)$ (by Lemma 2.1). In addition, G_1 is a subgraph of $G_2 = G_1 \cup \{v_1v_3\}$, which yields that $\rho(G_1) < \rho(G_2)$ (by Lemma 2.3). If $x_{v_1} \leq x_{v_3}$, then we set up a graph G_3 such that $V(G_3) = V(G)$ and $E(G_3) = E(G) \setminus \{v_tv_1\} \cup \{v_tv_3\}$. We have $\rho(G) \leq \rho(G_3)$ (by Lemma 2.1). G_4 is a graph by connecting v_1 and v_3 from G_3. So, G_3 is a subgraph of G_4. By Lemma 2.3, we have $\rho(G_4) > \rho(G_3)$. Thus, if G contains a cycle of length at least 5, then there exists a contradicted graph.

Next we show that there is at most one cycle of length 4. Suppose that there at at least two 4-cycles C_1 and C_2 in G. By Lemma 2.7, these two cycles share a common cut vertex. Let $C_1 = v_0v_1v_2v_3v_0$ and $C_2 = v_0u_1u_2u_3v_0$. If $x_{v_0} \geq min\{x_{v_1}, x_{v_3}\}$ and $x_{v_0} \geq min\{x_{u_1}, x_{u_3}\}$, say $x_{v_0} \geq x_{v_1}, x_{v_0} \geq x_{u_1}$, then we set a new graph H_1 such that $V(H_1) = V(G)$ and $E(H_1) = E(G) \setminus \{v_2v_1, u_2u_1\} \cup \{v_2v_0, u_2v_0\}$. By Lemma 2.1, we have $\rho(G) \leq \rho(H_1)$. Let H_2 be a graph from H_1 by connecting u_1v_1. Since H_2 is a proper subgraph of H_1, then $\rho(H_1) < \rho(H_2)$. This is a contradiction to the assumption that $\rho(G)$ is maximal.

If $x_{v_0} \leq min\{x_{v_1}, x_{v_3}\}$ and $x_{v_0} \leq min\{x_{u_1}, x_{u_3}\}$, say $x_{v_0} \leq x_{v_1}, x_{v_0} \leq x_{u_1}$, then we set new graphs H_3 with vertex set $V(H_3) = V(G)$ and $E(H_3) = E(G) \setminus \{v_3v_0, u_3u_0\} \cup \{v_3v_1, u_3u_1\}$, H_4 from H_3 by connecting v_1u_1. By Lemmas 2.1,2.3, we have $\rho(G) < \rho(H_3) < \rho(H_4)$. We can use Lemma 2.7 to find a graph in $\mathcal{C}_n^k$ with only one common vertex among cycles. This is a contradiction to the choice of G.

Lastly, without loss of generality, we consider the case of $max\{x_{u_1}, x_{u_3}\} \leq x_{v_0} \leq min\{x_{v_1}, x_{v_3}\}$, say $x_{u_1} \leq x_{v_0}$ and $x_{v_0} \leq x_{v_1}$. Let H_5 be a graph with $V(H_5) = V(G)$

and $E(H_5) = E(G) \setminus \{u_2u_1, v_3v_0\} \cup \{u_2v_0, v_3v_1\}$. By Lemma 2.1, $\rho(G) \le \rho(H_5)$. We build a new graph H_6 by adding v_1u_1. Then H_5 is a proper subgraph of H_6 and $\rho(H_5) < \rho(H_6)$. We can use Lemma 2.7 to find a graph in $\mathcal{C}_n^k$ with only one common vertex among cycles. This is a contradiction to the choice of G. So, this lemma is true. □

3. Main results

In this section, we determine the cacti maximizing A_α-spectral radius subject to fixed pendant vertices. In addition, we find the graph with the A_α-spectral radius among all the cacti with n vertices, and we also characterize the n-vertex cacti with a perfect matching having the largest A_α-spectral radius.

Since $\mathcal{C}_n^k$ is the set of all cacti with $n > 0$ vertices and $k > 0$ pendant vertices, then let C^e be a cactus graph in $\mathcal{C}_n^k$ such that $n-k-1$ is even and all cycles (if any) have length 3, that is, C^e contains $\frac{n-k-1}{2}$ cycles $vv_1v_1'v, vv_2v_2'v, \cdots,$
$vv_{\frac{n-k-1}{2}}v'_{\frac{n-k-1}{2}}v$ and k pendant edges (if any) $vu_1, vu_2, \cdots, vu_k$. Similarly, let C^o be a cactus graph in $\mathcal{C}_n^k$ such that $n-k-1$ is odd and all cycles (if any) have length 3, that is C^o contains $\frac{n-k-2}{2}$ cycles $vv_1v_1'v, vv_2v_2'v, \cdots, vv_{\frac{n-k-2}{2}}v'_{\frac{n-k-2}{2}}v$, $k-1$ pendant edges (if any) $vu_1, vu_2, \cdots, vu_{k-1}$ and 1 pendant path $vu_k'u_k$.

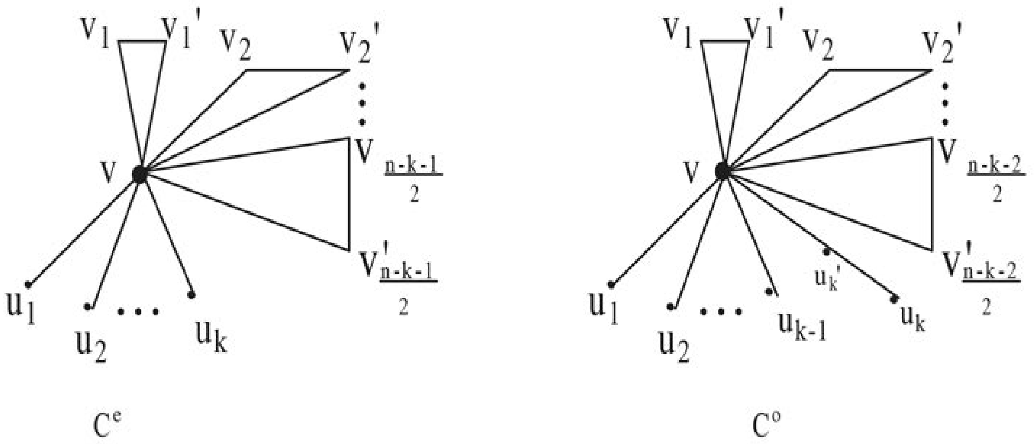

Figure 1. C^e: $n-k-1$ is even, contains $\frac{n-k-1}{2}$ cycles and k pendant edges (if any); C^o: $n-k-1$ is odd, contains $\frac{n-k-2}{2}$ cycles,$k-1$ pendant edges (if any) and 1 pendant path.

Theorem 3.1. (i) *If $n-k$ is odd and G is a graph with the maximum $A_\alpha-$spectral radius in $\mathcal{C}_n^k$, then $G \cong C^e$;*

(ii) *If $n-k$ is even and G is a graph with the maximum $A_\alpha-$spectral radius in $\mathcal{C}_n^k$, then $G \cong C^o$.*

Proof. Choose a cactus graph $G \in \mathcal{C}_n^k$ such that $\rho(G)$ is maximal. Assume $V(G) = \{v_0, v_2, \cdots, v_{n-1}\}$. By Lemma 2.4, we have all pendant paths share a common vertex. By Lemma 2.5 implies that the length of any pendant path is at most 2 and there is at most one pendant path of length 2. By Lemma 2.6 yields that there does not exist an internal path such that it is built by cut edges. By Lemma 2.8 all cycles share a common vertex. By Lemma 2.8 we have the length of any cycle is at most 4, and there is at most one cycle of length 4. In order to find the main results, we need the following two claims.
Claim 1. The pendant paths and cycles share a common vertex.
Proof. Suppose that all cycles share a vertex v and all pendant paths share a vertex u, $u, v \in \{v_0, v_1, \cdots, v_{n-1}\}$. Clearly, u and v is in a same cycle C'. Let $N'(u) = N(u) \setminus V(C')$

and $N'(v) = N(v) \setminus V(C')$. If $x_u \geq x_v$, then set a new graph G_1 with vertex set $V(G_1) = V(G) \setminus \{wv, \in w \in N'(v)\} \cup \{wu, \in w \in N'(v)\}$; Otherwise, if $x_u \leq x_v$, let a new graph G_2 with vertex set $V(G_2) = V(G) \setminus \{wu, \in w \in N'(u)\} \cup \{wv, \in w \in N'(u)\}$. By Lemma 2.1, we have $\rho(G) \leq \rho(G_1)$ or $\rho(G) \leq \rho(G_2)$. A contradiction yields this claim.

Claim 2. If there is a pendant path P with the length at most 2, then there is no cycle of length 4.

Proof. Let $v_0v_1v_2v_3v_0$ be a cycle of length 4 and P is a pendant path in G. By lemma 2.5 we know the length of P is 1 or 2. Next we prove $x_{v_0} \geq max\{x_{v_1}, x_{v_2}, x_{v_3}\}$. Assume $x_{v_1} > x_{v_0}$. Let $S = N(v_0) \setminus \{v_1, v_3\}$, set a new graph H with vertex set $V(G)$, $E(G) \setminus \{wv_0, w \in S\} \cup \{wv_1, w \in S\}$. Note that H is a cactus graph with k pendent vertices. By Lemma 2.1, we have $\rho(G) < \rho(H)$. It contradicts that $\rho(G)$ is maximal, thus, $x_{v_0} \geq x_{v_1}$. Similarity, we have $x_{v_0} \geq x_{v_2}$ and $x_{v_0} \geq x_{v_3}$. Thus, $x_{v_0} \geq max\{x_{v_1}, x_{v_2}, x_{v_3}\}$.

Case 1. $|P| = 2$. Assume $P = v_0v_4v_5$.
Let H_1 be a new graph with vertex set $V(G)$, $E(G)\setminus\{v_2v_3\} \cup \{v_0v_2\}$. Since $x_{v_0} \geq x_{v_3}$, then $\rho(G) \leq \rho(H_1)$ (by Lemma 2.1). Let H_2 be a new graph with vertex set $V(G)$, $E(H_1) + v_3v_4$. H_1 is proper subgraph of H_2 . By Lemma 2.3, we have $\rho(H_1) < \rho(H_2)$. Then, $\rho(G) < \rho(H_2)$. Note that H_2 is a cactus graph with k pendent vertices.

Case 2. $|P| = 1$. Assume $P = v_0v_6$.

Subcase 2.1. $x_{v_2} \leq x_{v_6}$.
Let H_3 be a new graph with vertex set $V(G)$, $E(G)\setminus\{v_2v_3\} \cup \{v_3v_6\}$. Note that H_3 is a cactus graph with k pendent vertices. By Lemma 2.1, we have $\rho(G) \leq \rho(H_3)$.

Subcase 2.2. $x_{v_2} > x_{v_6}$.
Let H_4 be a new graph with vertex set $V(G)$, $E(G)\setminus\{v_2v_3, v_0v_6\} \cup \{v_0v_2, v_3v_6\}$. Note that H_4 is a cactus graph with k pendent vertices. Since $x_{v_0} \geq x_{v_3}$ and $x_{v_2} > x_{v_6}$, then $(x_{v_2} - x_{v_6})(x_{v_0} - x_{v_3}) \geq 0$. By Lemma 2.2, we have $\rho(G) \leq \rho(H_4)$. Note that H_4 is a cactus graph with k pendent vertices. It is a contradiction and this claim is proved.

Therefore, if $n - k$ is odd, then $\rho(G) \leq \rho(C^e)$; if $n - k$ is even, then $\rho(G) \leq \rho(C^o)$. So, this theorem is proved. □

Lemma 3.2 ([9])**.** Given a partition $\{1, 2, \cdots, n\} = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$ with $|\Delta_i| = n_i > 0$, A be any matrix partitioned into blocks A_{ij}, where A_{ij} is an $n_i \times n_j$ block. Suppose that the block A_{ij} has constant row sums b_{ij} , and let $B = (b_{ij})$. Then the spectrum of B is contained in the spectrum of A (taking into account the multiplicities of the eigenvalues).

Next we provide all eigenvalues of C^e and C^o in the proposition.

Proposition 3.3. *Let* $\alpha \in [0, 1)$. *The following statements hold.* (i) *The maximum eigenvalues of* $A_\alpha(C^e)$ *satisfy the equation:* $f(\rho) = (\alpha - \rho)^3 + (n\alpha - 2\alpha + 1)(\alpha - \rho)^2 + [(1 - n)\alpha^2 + (3n - 4)\alpha + 1 - n](\alpha - \rho) - k(1 - \alpha)^2 = 0$. (ii) *The maximum eigenvalues of* $A_\alpha(C^o)$ *satisfy the equation:* $g(\rho) = (n\alpha - 2\alpha - \rho)(\alpha - \rho)(\alpha - \rho + 1)(\rho^2 - 3\alpha\rho + \alpha^2 + 2\alpha - 1) - (k - 1)(1 - \alpha)^2(\alpha - \rho + 1)(\rho^2 - 3\alpha\rho + \alpha^2 + 2\alpha - 1) - (n - k - 2)(1 - \alpha)^2(\alpha - \rho)(\rho^2 - 3\alpha\rho + \alpha^2 + 2\alpha - 1) - (1 - \alpha)^2(\alpha - \rho)^2(\alpha - \rho + 1) = 0$.

Proof. Since the matrix $A_\alpha = \alpha D + (1 - \alpha)A$, where D has on the diagonal the vector $(n-1, 2, 1)$ and A consists of the following three row-vectors, in the order: $(0, n-k-1, k)$; $(1, 1, 0)$; $(1, 0, 0)$. By Lemma 3.2, thus, the eigenvector x of $\rho(A_\alpha(C^e))$ (C^e, see Figure 1)is a constant value β_2 on the vertex set $\{v_1, v_1', v_2, v_2', \cdots, v_{\frac{n-k-1}{2}}, v'_{\frac{n-k-1}{2}}\}$, and constant value β_3 on the vertex set $\{u_1, u_2, \cdots, u_k\}$. Defining $x(v) =: \beta_1$, $\rho(C^e) =: \rho$, also by (1), we get $(\rho - (n-1)\alpha)\beta_1 = (1-\alpha)((n-k-1)\beta_2 + k\beta_3)$, $(\rho - 2\alpha)\beta_2 = (1-\alpha)(\beta_1 + \beta_2)$ and $(\rho - \alpha)\beta_3 = (1 - \alpha)\beta_1)$.

Then we get:

$$f(\rho) = (\alpha-\rho)^3+(n\alpha-2\alpha+1)(\alpha-\rho)^2+[(1-n)\alpha^2+(3n-4)\alpha+1-n](\alpha-\rho)-k(1-\alpha)^2 = 0.$$

Next we consider $A_\alpha(C^o)$ (C^o, see Figure 1), since the matrix $A_\alpha = \alpha D + (1-\alpha)A$, where D has on the diagonal the vector $(n-2, 2, 1, 2, 1)$ and A consists of the following five row-vectors, in the order: $(0, n-k-2, k-1, 1, 0)$; $(1, 1, 0, 0, 0)$; $(1, 0, 0, 0, 0)$; $(1, 0, 0, 0, 1)$ $(0, 0, 0, 1, 0)$. By Lemma 3.2, thus, the eigenvector x of $\rho(A_\alpha(C^o))$ is a constant value β_2 on the vertex set $\{v_1, v_1', \cdots, v_{\frac{n-k-2}{2}}, v'_{\frac{n-k-2}{2}}\}$, and constant value β_3 on the vertex set $\{u_1, u_2, \cdots, u_{k-1}\}$. Defining $x(v) =: \beta_1$, and $x(u_k') =: \beta_4$, and $x(u_k) =: \beta_5$. $\rho(C^e) =: \rho$, also by (1), similarly as above the computation of $A_\alpha(C^e)$, we obtain:

$g(\rho) = (n\alpha - 2\alpha - \rho)(\alpha - \rho)(\alpha - \rho + 1)(\rho^2 - 3\alpha\rho + \alpha^2 + 2\alpha - 1) - (k-1)(1-\alpha)^2(\alpha - \rho + 1)(\rho^2 - 3\alpha\rho + \alpha^2 + 2\alpha - 1) - (n-k-2)(1-\alpha)^2(\alpha - \rho)(\rho^2 - 3\alpha\rho + \alpha^2 + 2\alpha - 1) - (1-\alpha)^2(\alpha-\rho)^2(\alpha - \rho + 1) = 0.$

Thus, our proof is finished. □

Denote by $\mathcal{C}_n^*$ be the set of all cacti with n vertices. Let C_n^{*1} be a cactus graph in $\mathcal{C}_n^*$ such that n is odd and C_n^{*1} contains $\frac{n-1}{2}$ cycles of length 3 (if any). Let C_n^{*2} be a cactus graph in $\mathcal{C}_n^*$ such that n is even and C_n^{*2} contains $\frac{n-2}{2}$ cycles of length 3 (if any) and one pendant edge.

Theorem 3.4. (i) *If n is odd and G is a graph with the maximum A_α−spectral radius in $\mathcal{C}_n^*$, then $G \cong C_n^{*1}$;*

(ii) *If n is even and G is a graph with the maximum A_α−spectral radius in $\mathcal{C}_n^*$, then $G \cong C_n^{*2}$.*

Proof. By the proof of Theorem 3.1, we have the sharp upper bounds of A_α−spectral radius attain at C^e and C^o. We can set up a new graph by connecting any two pendant vertices and the original graph is the proper subgraph of this new graph. By Lemma 2.2, we have $\rho(G)$ is increasing by this operation. Therefore, $\rho(G) \le \rho(C^{*1})$ if n is odd, and $\rho(G) \le \rho(C^{*2})$ if n is even. Since C^{*1} is the cactus graph C^e when $k = 0$, and C^{*2} is the cactus graph C^o when $k = 1$. Thus, this theorem is proved.

By Proposition 3.3, and letting $k = 0, 1$, we can also obtain their corresponding eigenvalues. □

Based on the above outcomes, we can determine the sharp upper bound for the A_α-spectral radius of cacti with a perfect matching. Let $\mathcal{C}_{2k}^m$ be the set of all $2k$-vertex cacti with a perfect matching.

Theorem 3.5. *If G is a graph with the maximum A_α−spectral radius in $\mathcal{C}_{2k}^m$, then $G \cong C_{2k}^{*2}$.*

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