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RELATED FIXED POINTS FOR TWO PAIRS OF SET VALUED MAPPINGS ON TWO METRIC SPACES

Vijendra K. Chourasia* and Brian Fisher†

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Abstract

A related fixed point theorem for two pairs of set valued mappings on two complete metric spaces is proved..

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1. Introduction

In the following we let (X, d) be a complete metric space and $B(X)$ the set of all nonempty subsets of X. As in [1] and [2] we define the function $\delta(A, B)$ with A and B in $B(X)$ by $\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\}$. If A consists of a single point a we write $\delta(A, B) = \delta(a, B)$. If B also consists of single point b we write $\delta(A, B) = \delta(a, B) = d(a, b)$. It follows immediately that $\delta(A, B) = \delta(B, A) \geq 0$, $\delta(A, B) = 0$ implies $A = B$ and this set is a singleton, and $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ for all A, B in $B(X)$.

If now $\{A_n : n = 1, 2, \ldots\}$ is a sequence of sets in $B(X)$, we say that it *converges to the closed set A in $B(X)$* if

(i) each point $a \in A$ is the limit of some convergent sequence $\{a_n \in A_n : n = 1, 2, \ldots\}$, and

(ii) for arbitrary $\epsilon > 0$, there exists an integer N such that $A_n \subset A_\epsilon$ for $n > N$, where A_ϵ is the union of all open spheres with centres in A and radius ϵ.

The set A is then said to be the *limit* of the sequence $\{A_n\}$.

The following lemma was proved in [2].

1.1. Lemma. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space (X, d) which converge to the bounded subsets A and B, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

*Department of Mathematics, R. D. Govt. College, Mandla, M. P. 481661, India.

†Department of Mathematics and Computer Science, University of Leicester, LE1 7RH, U.K. E-mail: fbr@le.ac.uk

Now let F be a mapping of X into $B(X)$. We say that the mapping F is *continuous at a point* x if whenever $\{x_n\}$ is a sequence of points in X converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to Fx in $B(X)$.

We say that F is a *continuous mapping of* X *into* $B(X)$ if F is continuous at each point x in X. We say that a point z in X is a *fixed point of* F if z is in Fz.

If A is in $B(X)$ we define the set $FA = \bigcup_{a\in A} Fa$.

The following theorem was proved in [4].

1.2. Theorem. *Let* (X, d_1) *and* (Y, d_2) *be complete metrics spaces, let* F *be a mapping of* X *into* $B(Y)$ *and* G *a mapping of* Y *into* $B(X)$ *satisfying the inequalities*

$$\delta_1(GFx, GFx') \le c\max\{d_1(x, x'),\ \delta_1(x, GFx),\ \delta_1(x', GFx'),\ \delta_2(Fx, Fx')\},$$
$$\delta_2(FGy, FGy') \le c\max\{d_2(y, y'),\ \delta_2(y, FGy),\ \delta_2(y', FGy'),\ \delta_1(Gy, Gy')\}$$

for all x, x' *in* X *and* y, y' *in* Y*, where* $0 \le c < 1$*. If* F *is continuous, then* GF *has a unique fixed point* z *in* X *and* FG *has a unique fixed point* w *in* Y*.*

2. Results

We now prove the following generalization of Theorem 1.2.

2.1. Theorem. *Let* (X, d_1) *and* (Y, d_2) *be complete metrics spaces, let* F *and* G *be mappings of* X *into* $B(Y)$ *and* P *and* Q *mappings of* Y *into* $B(X)$ *satisfying the inequalities*

$$\delta_1(PFx, QGx') \le c\max\{d_1(x, x'),\ \delta_1(x, PFx),\ \delta_1(x', QGx'),\ \delta_2(Fx, Gx')\}, \tag{1}$$
$$\delta_2(GPy, FQy') \le c\max\{d_2(y, y'),\ \delta_2(y, GPy),\ \delta_2(y', FQy'),\ \delta_1(Py, Qy')\} \tag{2}$$

for all x, x' *in* X *and* y, y' *in* Y*, where* $0 \le c < 1$*. If* F *and* G *are continuous,then* PF *and* QG *have a unique fixed point* z *in* X *and* GP *and* FQ *have a unique fixed point* w *in* Y*.*

Proof. Let x_1 be an arbitrary point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively as follows. Choose a point y_1 in Fx_1, a point x_2 in Py_1, a point y_2 in Gx_2 and then a point x_3 in Qy_2. In general, having chosen x_n in X and y_n in Y, choose a point y_{2n-1} in Fx_{2n-1}, a point x_{2n} in Py_{2n-1}, a point y_{2n} in Gx_{2n} and then a point x_{2n+1} in Qy_{2n} for $n = 1, 2, \dots$. Then, using inequality (1), we have

$$\begin{aligned} d_1(x_{2n+2}, x_{2n+1}) &\le \delta_1(PFx_{2n+1}, QGx_{2n}) \\ &\le c\max\{d_1(x_{2n+1}, x_{2n}),\ \delta_1(x_{2n+1},\ PFx_{2n+1}), \\ &\qquad \delta_1(x_{2n}, QGx_{2n}),\ \delta_2(Fx_{2n+1}, Gx_{2n})\} \\ &\le c\max\{\delta_1(QGx_{2n}, PFx_{2n-1}),\ \delta_1(QGx_{2n}, PFx_{2n+1}), \\ &\qquad \delta_1(PFx_{2n-1}, QGx_{2n}),\ \delta_2(Fx_{2n+1}, Gx_{2n})\} \\ &= c\max\{\delta_1(PFx_{2n-1}, QGx_{2n}),\ \delta_2(GPy_{2n-1}, FQy_{2n})\}, \end{aligned} \tag{3}$$

since

$$\delta_2(Fx_{2n+1}, Gx_{2n}) \le \delta_2(GPy_{2n-1}, FQy_{2n}).$$

Similarly, using inequality (1) again, we have

$$\begin{aligned} d_1(x_{2n+2}, x_{2n+3}) &\le \delta_1(PFx_{2n+1}, QGx_{2n+2}) \\ &\le c\max\{\delta_1(PFx_{2n+1}, QGx_{2n}),\ \delta_2(GPy_{2n+1}, FQy_{2n})\}. \end{aligned} \tag{4}$$

Using inequality (2), we have

$$\begin{aligned} d_2(y_{2n+1}, y_{2n+2}) &\le \delta_2(FQy_{2n}, GPy_{2n+1}) \\ &\le c\max\{d_2(y_{2n}, y_{2n+1}), \delta_2(y_{2n+1}, GPy_{2n+1}), \\ &\qquad \delta_2(y_{2n}, FQy_{2n}),\ \delta_1(Py_{2n+1}, Qy_{2n})\} \\ &\le c\max\{\delta_2(GPy_{2n-1}, FQy_{2n}),\ \delta_2(FQy_{2n}, GPy_{2n+1}), \\ &\qquad \delta_2(GPy_{2n-1}, FQy_{2n}),\ \delta_1(Py_{2n+1}, Qy_{2n})\} \\ &\le c\max\{\delta_2(GPy_{2n-1}, FQy_{2n}),\ \delta_1(PFx_{2n+1}, QGx_{2n})\}, \qquad (5) \end{aligned}$$

since

$$\delta_1(Py_{2n+1}, Qy_{2n}) \le \delta_1(PFx_{2n+1}, QGx_{2n}).$$

Similarly, using inequality (2) again, we have

$$\begin{aligned} d_2(y_{2n+2}, y_{2n+3}) &\le \delta_2(GPy_{2n+1}, FQy_{2n+2}) \\ &\le c\max\{\delta_2(GPy_{2n+1}, FQy_{2n}),\ \delta_1(PFx_{2n+1}, QGx_{2n+2})\}. \qquad (6) \end{aligned}$$

We will now prove that

$$\delta_1(PFx_{2n+1}, QGx_{2n}) \le c^n K, \qquad (7)$$

$$\delta_1(PFx_{2n+1}, QGx_{2n+2}) \le c^n K, \qquad (8)$$

$$\delta_2(FQy_{2n}, GPy_{2n+1}) \le c^n K, \qquad (9)$$

$$\delta_2(GPy_{2n+1}, FQy_{2n+2}) \le c^n K, \qquad (10)$$

where

$$\begin{aligned} K = \max\{&\delta_1(PFx_1, QGx_2),\ \delta_1(PFx_3, QGx_2), \delta_1(PFx_3, QGx_4), \\ &\delta_2(GPy_1, FQy_2),\ \delta_2(GPy_3, FQy_2)\}, \end{aligned}$$

for $n = 1, 2, \ldots$.

Inequalities (7) to (10) clearly hold when $n = 1$. Suppose inequalities (7) to (10) hold for some n. Then it follows from inequality (3) that

$$\begin{aligned} \delta_1(PFx_{2n+3}, QGx_{2n+2}) &\le c\max\{\delta_1(PFx_{2n+1}, QGx_{2n+2}),\ \delta_2(GPy_{2n+1}, FQy_{2n+2})\} \\ &\le c^{n+1}K \end{aligned}$$

on using our assumptions on inequalities (8) and (10). Inequality (7) now follows by induction.

Using inequality (5), we have

$$\begin{aligned} \delta_2(FQy_{2n+2}, GPy_{2n+3}) &\le c\max\{\delta_2(GPy_{2n+1}, FQy_{2n+2}),\ \delta_1(PFx_{2n+3}, QGx_{2n+2})\} \\ &\le c^{n+1}K, \end{aligned}$$

on using inequality (7) and our assumption on inequality (10). Inequality (9) now follows by induction.

Using inequality (4), we have

$$\begin{aligned} \delta_1(PFx_{2n+3}, QGx_{2n+4}) &\le c\max\{\delta_1(PFx_{2n+3}, QGx_{2n+2}),\ \delta_2(GPy_{2n+3}, FQy_{2n+2})\} \\ &\le c^{n+1}K, \end{aligned}$$

on using inequalities (7) and (9). Inequality (8) now follows by induction.

Finally, using inequality (6), we have

$$\begin{aligned} \delta_2(GPy_{2n+3}, FQy_{2n+4}) &\le c\max\{\delta_2(GPy_{2n+3}, FQy_{2n+2}),\ \delta_1(PFx_{2n+3}, QGx_{2n+4})\} \\ &\le c^{n+1}K, \end{aligned}$$

on using inequalities (8) and (9). Inequality (10) now follows by induction.

It follows that, for $r = 1, 2, \ldots$,

$$\begin{aligned} d_1(x_{2n+1}, x_{2n+r+1}) &\leq d_1(x_{2n+1}, x_{2n+2}) + d_1(x_{2n+2}, x_{2n+3}) + \ldots \\ &\qquad + d_1(x_{2n+r}, x_{2n+r+1}) \\ &\leq \delta_1(QGx_{2n}, PFx_{2n+1}) + \delta_1(PFx_{2n+1}, QGx_{2n+2}) + \cdots \\ &\leq (c^n + c^n + c^{n+1} + c^{n+1} + \cdots)K \\ &< \epsilon, \end{aligned}$$

for n greater than some N, since $c < 1$. The sequence $\{x_n\}$ is therefore a Cauchy sequence in the complete metric space X, and so has a limit z in X. Similarly the sequence $\{y_n\}$ is a Cauchy sequence in the complete metric space Y and so has a limit w in Y.

Further, with $m > n$, we have

$$\begin{aligned} \delta_1(QGx_{2n}, PFx_{2m+1}) &\leq \delta_1(QGx_{2n}, PFx_{2n+1}) + \delta_1(PFx_{2n+1}, QGx_{2n+2}) + \\ &\qquad + \cdots + \delta_1(QGx_{2m}, PFx_{2m+1}) \\ &\leq (c^n + c^n + c^{n+1} + c^{n+1} + \cdots)K \\ &< \epsilon \end{aligned} \tag{11}$$

for $n > N$. Next, we have

$$\begin{aligned} \delta_1(z, QGx_{2n}) &\leq d_1(z, x_{2m+2}) + \delta_1(x_{2m+2}, QGx_{2n}) \\ &\leq d_1(z, x_{2m+2}) + \delta_1(PFx_{2m+1}, QGx_{2n}), \end{aligned}$$

since $x_{2m+2} \in PFx_{2m+1}$. Thus, on using inequality (11), we have

$$\delta_1(z, QGx_{2n}) \leq d_1(z, x_{2m+2}) + \epsilon$$

for $m > n > N$. Letting m tend to infinity it follows that

$$\delta_1(z, QGx_{2n}) \leq \epsilon$$

for $n > N$, and so

$$\lim_{n\to\infty} QGx_{2n} = \{z\}, \tag{12}$$

since ϵ is arbitrary.

Similarly,

$$\lim_{n\to\infty} PFx_{2n+1} = \{z\}, \tag{13}$$

$$\lim_{n\to\infty} GPy_{2n+1} = \{w\} = \lim_{n\to\infty} FQy_{2n}. \tag{14}$$

From the continuity of F of G, we have

$$\lim_{n\to\infty} Fx_{2n+1} = Fz = \{w\}, \tag{15}$$

$$\lim_{n\to\infty} Gx_{2n} = Gz = \{w\}. \tag{16}$$

Using inequality (1), we now have

$$\delta_1(PFz, QGx_{2n}) \leq c\max\{d_1(z, x_{2n}),\ \delta_1(z, PFz), \delta_1(x_{2n}, QGx_{2n}),\ \delta_2(Fz, Gx_{2n})\}.$$

Letting n tend to infinity, and using equations (12) and (16), we have

$$\delta_1(PFz, z) \leq c\,\delta_1(PFz, z).$$

Since $c < 1$, we must have

$$PFz = \{z\} = Pw, \tag{17}$$

on using equation (15), proving that z is a fixed point of PF.

Using inequality (1) again, we now have

$$\begin{aligned}\delta_1(x_{2n+2}, QGz) &\le \delta_1(PFx_{2n+1}, QGz)\\ &\le c\max\{d_1(x_{2n+1}, z), \delta_1(x_{2n+1}, PFx_{2n+1}),\\ &\qquad \delta_1(z, QGz),\ \delta_2(Fx_{2n+1}, Gz)\}.\end{aligned}$$

Letting n tend to infinity, and using equations (13), (15) and (16), we have

$$\delta_1(z, QGz) \le c\,\delta_1(z, QGz).$$

Since $c < 1$, we must have

(18) $\quad QGz = \{z\} = Qw,$

on using equation (16), proving that z is also a fixed point of QG.

It now follows from equations (15) and (18) that

$$FQw = Fz = \{w\},$$

and it follows from equations (16) and (17) that

$$GPw = Gz = \{w\}.$$

Therefore, w is a fixed point of FQ and GP.

To prove uniqueness, suppose that PF and QG have a second common fixed point z'. Then using inequalities (1) and (2), we have

$$\begin{aligned}&\max\{\delta_1(z', QGz'),\ \delta_1(z', PFz')\} \le \delta_1(PFz', QGz')\\ &\le c\max\{d_1(z', z'),\ \delta_1(z', PFz'),\ \delta_1(z', QGz'),\ \delta_2(Fz', Gz')\}\\ &= c\delta_2(Fz', Gz')\\ &\le c\delta_2(GPFz', FQGz')\\ &\le c^2\max\{\delta_2(Fz', Gz'),\ \delta_2(Fz', GPFz'),\ \delta_2(Gz', FQGz'),\ \delta_1(PFz', QGz')\}\\ &\le c^2\max\{\delta_2(GPFz', FQGz')\},\ \delta_1(PFz', QGz')\\ &= c^2\delta_2(PFz', QGz')\end{aligned}$$

and it follows that

$$\max\{\delta_1(z', QGz'),\ \delta_1(z', PFz')\} = \delta_1(PFz', QGz') = \delta_2(Fz', Gz') = 0,$$

since $c < 1$. Thus Fz' and Gz' are singletons and

$$PFz' = QGz' = \{z'\}.$$

Using inequalities (1) and (2) again, we have

$$\begin{aligned}d_1(z, z') &= \delta_1(PFz, QGz')\\ &\le c\max\{d_1(z, z'),\ \delta_1(z, PFz),\ \delta_1(z', QGz'), \delta_2(Fz, Gz')\}\\ &= cd_2(Fz, Gz')\\ &= cd_2(Gz, Fz')\\ &\le c\delta_2(GPFz, FQGz')\\ &\le c^2\max\{d_2(Fz, Gz'),\ \delta_2(Fz, GPFz),\ \delta_2(Gz', FQGz'),\ \delta_1(PFz, QGz')\}\\ &= c^2\max\{d_2(Fz, Fz'), d_1(z, z')\}\\ &= c^2 d_1(z, z').\end{aligned}$$

Since $c < 1$, the uniqueness of z follows.

Similarly, w is the unique fixed point of GP and FQ. This completes the proof of the theorem. □

If we let F and G be single valued mappings of X into Y and let P and Q be single valued mappings of Y into X, we obtain the following corollary, which generalizes a result given in [3].

2.2. Corollary. *Let (X, d_1) and (Y, d_2) be complete metric spaces. If F and G are continuous mappings of X into Y and P and Q are mappings of Y into X satisfying the inequalities*

$$d_1(PFx, QGx') \leq c \max\{d_1(x, x'),\ d_1(x, PFx),\ d_1(x', QGx'),\ d_2(Fx, Gx')\},$$
$$d_2(GPy, FQy') \leq c \max\{d_2(y, y'),\ d_2(y, GPy),\ d_2(y', FQy'),\ d_1(Py, Qy')\}$$

for all x, x' in X and y, y' in Y, where $0 \leq c < 1$, then PF and QG have a unique fixed point z in X and GP and FQ have a unique fixed point w in Y.

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