Covering morphisms of topological internal groupoids

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Abstract

Let X be a topological group with operations whose underlying space has a universal cover. Then the fundamental groupoid πX becomes a topological internal groupoid, i.e., an internal groupoid in the category of topological groups. In this paper, we prove that the slice category $\mathsf{Cov_{sTC}}/X$ of covering morphisms $p \colon \widetilde{X} \to X$ of topological groups with operations in which $\widetilde{X}$ has also a universal cover and the category $\mathsf{Cov_{Gpd(TC)}}/\pi X$ of covering morphisms $q \colon \widetilde{G} \to \pi X$ of topological internal groupoids based on πX are equivalent. We also prove that for a topological internal groupoid G, the category $\mathsf{Cov_{Gpd(TC)}}/G$ of covering morphisms of topological internal groupoids based on G and the category $\mathsf{Act_{Gpd(TC)}}/G$ of topological internal groupoid actions of G on topological groups with operations are equivalent.

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1. Introduction

In the theory of covering groupoids there is a classical equivalence of categories for a groupoid G of covering morphisms of the groupoid G and actions of the groupoid G on sets. In [10, Theorem 2] it was proved that for a topological groupoid G, the category $\mathsf{Cov_{TGd}}/G$ of topological groupoid coverings of G and the category $\mathsf{Act_{TGd}}(G)$ of topological actions of G are equivalent.

On the other hand there are two important results on group-groupoids, which are internal groupoids in the category of groups. One is that if X is a topological group whose topology has a universal cover, then the category $\mathsf{Cov_{TGr}}/X$ of covering groups of X and the category $\mathsf{Cov_{GrGd}}/\pi X$ of group-groupoid covers of πX are equivalent [8, Proposition 2.3]. The other is that if G is a group-groupoid, then the category $\mathsf{Cov_{GrGd}}/G$ of group-groupoid covers of G is equivalent to the category $\mathsf{Act_{GpGd}}(G)$ of group-groupoid actions of G on groups [8, Proposition 3.1]. The former equivalence is also strengthened in the local case of topological groups in [1]. These two results in [8] recently have been generalized in [2] to a wide class of algebraic structures, which include groups, rings without identity, R-modules, Lie algebras, Jordan algebras, and many others. These are conveniently handled

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by working in a category C of *groups with operations* in the sense of Orzech [22]. Mucuk and Şahan in [19] give the relations between the coverings of an internal groupoid within topological groups with operations and associated crossed module.

In this paper we deal with the topological cases of the results in [2] working in the category TC of *topological groups with operations*. So for the first one we prove that for a topological group with operations X whose underlying space has a universal cover, the category $\mathsf{Cov}_{\mathsf{sTC}}/X$ of covering morphisms $p\colon \widetilde{X} \to X$ of topological group with operations in which $\widetilde{X}$ has also a universal cover, is equal to the category $\mathsf{Cov}_{\mathsf{Gpd(TC)}}/\pi X$ of covering morphisms $q\colon \widetilde{G} \to \pi X$ of topological internal groupoids in which the space $\mathrm{Ob}(\widetilde{G})$ has a universal covering. For the second one we prove that if G is a topological internal groupoid, then the category $\mathsf{Cov}_{\mathsf{Gpd(TC)}}/G$ of covering morphisms of topological internal groupoid G is equivalent to the category $\mathsf{Act}_{\mathsf{Gpd(TC)}}/G$ of topological internal groupoid actions of G on topological groups with operations.

2. Preliminaries on groupoids and coverings

A *groupoid* is a (small) category in which each morphism is an isomorphism [5, p.205]. So a groupoid G has a set G of morphisms , which we call just *elements* of G, a set $\mathrm{Ob}(G)$ of *objects* together with maps $s, t\colon G \to \mathrm{Ob}(G)$ and $\epsilon\colon \mathrm{Ob}(G) \to G$ such that $s\epsilon = t\epsilon = 1_{\mathrm{Ob}(G)}$. The maps s, t are called *initial* and *final* point maps respectively and the map ϵ is called *object inclusion*. If $a, b \in G$ and $t(a) = s(b)$, then the *composite* $a \circ b$ exists such that $s(a \circ b) = s(a)$ and $t(a \circ b) = t(b)$. So there exists a partial composition defined by $G_t \times_s G \to G, (a, b) \mapsto a \circ b$, where $G_t \times_s G$ is the pullback of t and s. Further, this partial composition is associative, for $x \in \mathrm{Ob}(G)$ the element $\epsilon(x)$ denoted by 1_x acts as the identity, and each element a has an inverse a^{-1} such that $s(a^{-1}) = t(a)$, $t(a^{-1}) = s(a)$, $a \circ a^{-1} = (\epsilon s)(a)$, $a^{-1} \circ a = (\epsilon t)(a)$. The map $G \to G$, $a \mapsto a^{-1}$ is called the *inversion*.

In a groupoid G for $x, y \in \mathrm{Ob}(G)$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. We say G is *transitive* if for all $x, y \in \mathrm{Ob}(G)$, the set $G(x, y)$ is not empty. For $x \in \mathrm{Ob}(G)$ we denote the star $\{a \in G \mid s(a) = x\}$ of x by G_x.

Let $p\colon \widetilde{G} \to G$ be a morphism of groupoids. Then p is called a *covering morphism* and $\widetilde{G}$ a *covering groupoid* of G if for each $\widetilde{x} \in \mathrm{Ob}(\widetilde{G})$ the restriction $\widetilde{G}_{\widetilde{x}} \to G_{p(\widetilde{x})}$ is bijective.

Assume that $p\colon \widetilde{G} \to G$ is a covering morphism. Then we have a lifting function $S_f\colon G_s \times_{\mathrm{Ob}(p)} \mathrm{Ob}(\widetilde{G}) \to \widetilde{G}$ assigning to the pair (a, x) in the pullback $G_s \times_{\mathrm{Ob}(p)} \mathrm{Ob}(\widetilde{G})$ the unique element b of $\widetilde{G}_x$ such that $f(b) = a$. Clearly S_f is inverse to $(p, s)\colon \widetilde{G} \to G_s \times_{\mathrm{Ob}(p)} \mathrm{Ob}(\widetilde{G})$. So it is stated that $p\colon \widetilde{G} \to G$ is a covering morphism if and only if (p, s) is bijective.

A covering morphism $p\colon \widetilde{G} \to G$ is called *transitive* if both $\widetilde{G}$ and G are transitive. A transitive covering morphism $p\colon \widetilde{G} \to G$ is called *universal* if $\widetilde{G}$ covers every cover of G, i.e., if for every covering morphism $q\colon \widetilde{H} \to G$ there is a unique morphism of groupoids $\widetilde{p}\colon \widetilde{G} \to \widetilde{H}$ such that $q\widetilde{p} = p$ (and hence $\widetilde{p}$ is also a covering morphism), this is equivalent to that for $\widetilde{x}, \widetilde{y} \in O_{\widetilde{G}}$ the set $\widetilde{G}(\widetilde{x}, \widetilde{y})$ has not more than one element.

A morphism $p\colon (\widetilde{G}, \widetilde{x}) \to (G, x)$ of pointed groupoids is called a *covering morphism* if the morphism $p\colon \widetilde{G} \to G$ is a covering morphism.

It was proved in that [5, 10.6.1] that If X is a topological space which has a universal cover, then the category $\mathsf{Cov}_{\mathsf{Top}}/X$ of covering maps $p\colon \widetilde{X} \to X$ is equivalent to the category $\mathsf{Cov}_{\mathsf{Gd}}/\pi_1 X$ of covering morphisms $q\colon \widetilde{G} \to \pi_1 X$ of groupoids.

Let G be a groupoid. An *action of G on a set* consists of a set X, a function $\theta\colon X \to \mathrm{Ob}(G)$ and a function $\varphi\colon X_\theta \times_s G \to X, (x, a) \mapsto xa$ defined on the pullback $X_\theta \times_s G$ of θ and s such that

(1) $\theta(xa) = t(a)$ for $(x,a) \in X_\theta \times_s G$;

(2) $x(ab) = (xa)b$ for $(a,b) \in G_t \times_s G$ and $(x,a) \in X_\theta \times_s G$;

(3) $x1_{\theta(x)} = x$ for $x \in X$.

Given such an action, *semidirect product groupoid* $G \ltimes X$ is defined to be the groupoid with object set X and elements of $(G \ltimes X)(x,y)$ the pairs (a,x) such that $a \in G(\theta(x), \theta(y))$ and $xa = y$. The groupoid composite is defined to be

$$(a,x) \circ (b,y) = (ab,x)$$

when $y = xa$

A *topological groupoid* is a groupoid such that the sets G of morphisms and the set $\mathrm{Ob}(G)$ of objects are topological spaces and source, target, inclusion, inverse and product maps are continuous (see for example [15, Section 2] and [7]). Let G and $\widetilde{G}$ be two topological groupoids. Then a morphism $p \colon \widetilde{G} \to G$ of topological groupoids is a morphism of groupoids such that $p \colon \widetilde{G} \to G$ and $\mathrm{Ob}(p) \colon \mathrm{Ob}(\widetilde{G}) \to \mathrm{Ob}(G)$ are continuous.

Definition 2.1. A morphism $p \colon \widetilde{G} \to G$ of topological groupoids is called *covering morphism of topological groupoids* if for each $x \in \mathrm{Ob}(\widetilde{G})$ the restriction $p_{\widetilde{x}} \colon \widetilde{G}_{\widetilde{x}} \to G_{p(\widetilde{x})}$ of p is a homeomorphism.

Example 2.2 ([10, Example 3]). If $p \colon \widetilde{X} \to X$ be a covering morphism of topological groups in which both underlying spaces of X and $\widetilde{X}$ have universal covers, then $\pi(p) \colon \pi\widetilde{X} \to \pi X$ becomes a covering morphism of topological groupoids.

As it is stated in [12, Section 2], equivalently a covering morphism of topological groupoids can be restated as follows:

Theorem 2.3. *Let $p \colon \widetilde{G} \to G$ be a morphism of topological groupoid groupoids. Then p is a covering morphism of topological groupoids if and only if*

$$(p,s) \colon \widetilde{G} \to G_s \times_{\mathrm{Ob}(p)} \mathrm{Ob}(\widetilde{G})$$

is a homeomorphism.

The topological action of a topological groupoid on a topological space is defined in [10, p.144] as follow:

Let G be a topological groupoid, X a toplogical space and $G \ltimes X$ a groupoid action on X. If the functions $\theta \colon X \to \mathrm{Ob}(G)$ and $\varphi \colon X_\theta \times_s G \to X, (x,a) \mapsto xa$ are continuous, then we say that G *acts as topologically* on X.

Example 2.4 ([10, Theorem 2]). Let $p \colon \widetilde{G} \to G$ be a covering morphism of topological groupoids. Then G acts as topologically on $\mathrm{Ob}(\widetilde{G})$.

Recall that a covering map $p \colon \widetilde{X} \to X$ of connected spaces is called *universal* if it covers every covering of X in the sense that if $q \colon \widetilde{Y} \to X$ is another covering of X then there exists a map $r \colon \widetilde{X} \to \widetilde{Y}$ such that $p = qr$ (hence r becomes a covering). A covering map $p \colon \widetilde{X} \to X$ is called *simply connected* if $\widetilde{X}$ is simply connected. So a simply connected covering is a universal covering.

The following result was proved in [6, Theorem 1] and we need it in the proof of Theorem 3.11.

Proposition 2.5. *If X is a locally path connected and semi-locally simply connected space, then the fundamental groupoid πX may be given a topology making it a topological groupoid.*

3. Topological groups with operations and topological internal categories

The idea of the definition of categories of groups with operations comes from Higgins [13] and Orzech [22]; and the definition below is from Porter [23, Section 1] and Datuashvili [11, p.21], which is adapted from Orzech[22].

Definition 3.1. The notion of *a group with a set of operations* consists of a pair (Ω, E) where E is a set of identities including the group laws and Ω of operations which includes the group operations, and the following conditions hold: If Ω_i is the set of i-ary operations in Ω, then

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) The group operations written additively $0, -$ and $+$ are the elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$ and assume that if $\star \in \Omega_2'$, then $\star^\circ$ defined by $a \star^\circ b = b \star a$ is also in Ω_2'. Also assume that $\Omega_0 = \{0\}$;

(c) For each $\star \in \Omega_2'$, E includes the identity $a \star (b + c) = a \star b + a \star c$;

(d) For each $\omega \in \Omega_1'$ and $\star \in \Omega_2'$, E includes the identities $\omega(a + b) = \omega(a) + \omega(b)$ and $\omega(a) \star b = \omega(a \star b)$.

From now on C will be a category of groups with operations satisfying the conditions (a)-(d).

A *morphism* $f \colon (E, \Omega) \to (E', \Omega)$ consists of a function $f \colon E \to E'$ is a group homomorphism, which preserves the operations.

Remark 3.2. The set Ω_0 contains exactly one element, the group identity; hence for instance the category of associative rings with unit is not a category of groups with operations.

Example 3.3. The categories of groups, rings generally without identity, R-modules, associative, associative commutative, Lie, Leibniz, alternative algebras are examples of categories of groups with operations.

The category of topological groups with operations are defined in [2] as follows:

Definition 3.4. A category TC of topological groups with a set Ω of continuous operations and with a set E of identities such that E includes the group laws such that the conditions (a)-(d) of Definition 3.1 are satisfied, is called a *category of topological groups with operations*.

In the rest of the paper TC will denote the category of topological groups.

A *morphism* between any two objects of TC is a continuous group homomorphism, which preserves the operations in Ω_1' and Ω_2'.

The categories of topological groups, topological rings and topological R-modules are examples of categories of topological groups with operations.

Definition 3.5 ([23, Section 3])**.** Let C be the category of groups with operations. An *internal category* in C is a category C in which $\mathrm{Ob}(C)$ and C are the objects of C such that the initial and final point maps $s, t \colon C \rightrightarrows \mathrm{Ob}(C)$, the object inclusion map $\epsilon \colon \mathrm{Ob}(C) \to C$ and the partial composition $\circ \colon C_t \times_s C \to C, (a, b) \mapsto a \circ b$ are morphisms in the category C.

Note that since ϵ is a morphism in C, $\epsilon(0) = 0$ and that the operation $\circ$ being a morphism implies that for all $a, b, c, d \in C$ and $\star \in \Omega_2$,

$$(a \star c) \circ (b \star d) = (a \circ b) \star (c \circ d) \tag{3.1}$$

whenever one side makes sense. This is called the *interchange law* [23] .

We also note from [23] that any internal category in C is an internal groupoid since, given $a \in C$, $a^{-1} = \epsilon t(a) - a + \epsilon s(a)$ satisfies $a^{-1} \circ a = \epsilon t(a)$ and $a \circ a^{-1} = \epsilon s(a)$. So we

use the term *internal groupoid* rather than internal category and write G for an internal groupoid. For the category of internal groupoids in C we use the notation $\mathsf{Gpd}(\mathsf{C})$. Here a *morphism* $f\colon H \to G$ in $\mathsf{Gpd}(\mathsf{C})$ is the morphism of underlying groupoids and a morphism in C.

In particular if C is the category of groups, then an internal groupoid G in C becomes a group object in the category of groupoids, which is quite often called 2-group, see for example [4], *group-groupoid* or $\mathcal{G}$-groupoid [9, p.297], [16]. The advantages of the group-groupoid term are: (i) there is no confusion with the group theoretic term "2-group" as a group all of whose elements are of order 2; (ii) it is a clear statement of the structure involved. Recently the notion of monodromy for topological group-groupoids was developed in [21, Theorem 3.10]; and the normality and quotient in group-groupoids were developed in [20]. In the case where C is the category of rings, an internal groupoid is a ring object in the category of groupoids [17, Definition 1.1] (see also [3, Definition 4.1] for topological R-module case).

Example 3.6 ([2])**.** Let X be an object of TC. Then the groupoid $G = X \times X$ is an internal groupoid.Here a pair (x, y) is a morphism from x to y with inverse morphism (y, x). The groupoid composition is defined by $(x, y) \circ (u, z) = (x, z)$ whenever $y = u$. The binary operations in G are defined by $(x, y)\widetilde{\star}(u, v) = (x \star u, y \star v)$ for $\star \in \Omega_2$ and the unary operations by $\widetilde{\omega}(x, y) = (\omega(x), \omega(y))$ for $\omega \in \Omega_1$. For the interchange law if $a = (x, y)$, $b = (y, z)$, $c = (u, v)$ and $d = (v, w)$ are the morphisms in G so that the compositions $a \circ b$ and $c \circ d$ are defined, then we have $(a \circ b)\widetilde{\star}(c \circ d) = (x \star u, z \star w)$ and $(a\widetilde{\star}c) \circ (b\widetilde{\star}d) = (x \star u, z \star w)$ and therefore we have the interchange law

$$(a\widetilde{\star}c) \circ (b\widetilde{\star}d) = (a \circ b)\widetilde{\star}(c \circ d).$$

Remark 3.7 ([2, 3.7])**.** The following are immediate from Definition 3.5:
(i) By Definition 3.5 we know that in an internal groupoid G in C, the initial and final point maps s and t, the object inclusion map ϵ are the morphisms in C and the interchange law (3.1) is satisfied. Therefore in an internal groupoid G, the unary operations are endomorphisms of the underlying groupoid of G and the binary operations are morphisms from the underlying groupoid of $G \times G$ to the one of G.
(ii) Let G be an internal groupoid in C and $0 \in \mathrm{Ob}(G)$ the identity element. Then $\mathsf{Ker}s = \mathsf{St}_G 0$, called in [5, p.3] *transitivity component* or *connected component* of 0, is also an internal groupoid.

Let H and G be two internal groupoids in C. A *morphism* of internal groupoids is a morphism $f\colon H \to G$ of underlying groupoids which is also a morphism of groups with operations. A morphism $f\colon H \to G$ of internal groupoids is called *covering* (resp. *universal covering*) if it is a covering (resp. universal covering) morphism on the underlying groupoids.

Definition 3.8. An internal groupoid in the category TC of topological groups with operations is called a *topological internal groupoid.*

So a topological internal groupoid is a topological groupoid G in which the set of morphisms and the set $\mathrm{Ob}(G)$ of objects are objects of TC and all structural maps of G, i.e, the source and target maps $s, t\colon G \to \mathrm{Ob}(G)$, the object inclusion map $\epsilon\colon \mathrm{Ob}(G) \to G$ and the composition map $\circ\colon G_t \times_s G \to G$, are morphisms of TC.

If TC is the category of topological groups, then an internal topological groupoid becomes a topological group-groupoid which was previously defined in [14, Definition 1] by the requirement that is a topological groupoid which is also a group-groupoid with continuous structural maps of groups.

Example 3.9. A topological group with operations which is abelian according to all binary operations $\star \in \Omega_2$ can be thought as a topological internal groupoid.

Example 3.10 ([18, Example 3.10]). Let X be an object of TC. Then the groupoid $G = X \times X$ is a topological internal groupoid. By Example 3.6 it is seen that $G = X \times X$ is an internal groupoid. In addition since the maps s, t, ϵ and the groupoid composition $\circ \colon G_t \times_s G \to G, (a, b) \mapsto a \circ b$ are continuous then the structural maps of G are continuous. Hence G becomes a topological internal groupoid.

Theorem 3.11. *Let X be an object of TC such that the underlying space is locally path connected and semi-locally simply connected. Then the fundamental groupoid πX is a topological internal groupoid.*

Proof. Let X be a topological group with operations as assumed. By Theorem 2.5, πX has a topology such that it is a topological groupoid. We know by [6, Proposition 3] that when X and Y are endowed with such topologies, for a continuous map $f \colon X \to Y$, the induced morphism $\pi(f) \colon \pi X \to \pi Y$ is also continuous. Hence the continuous binary operations $\star \colon X \times X \to X$ for $\star \in \Omega_2$ and the unary operations $\omega \colon X \to X$ for $\omega \in \Omega_1$ respectively induce continuous binary operations $\widetilde{\star} \colon \pi X \times \pi X \to \pi X$ and unary operations $\widetilde{\omega} \colon \pi X \to \pi X$. So the set of morphisms becomes a topological group with operations. Now we show that the groupoid structural maps are morphisms of groups with operations, i.e., preserve the operations.

For the morphism $s \colon \pi X \to X, s([a]) = a(0)$, the operations $\star \in \Omega_2$ and $\omega \in \Omega_1$ we have the following:

$$\begin{aligned} s([a \star b]) &= (a \star b)(0) \\ &= a(0) \star b(0) \\ &= s[a] \star s[b] \end{aligned}$$

and

$$\begin{aligned} s(\omega[a]) &= s[\omega(a)] \\ &= (w(a))(0) \\ &= w(a(0)) \\ &= ws[a]. \end{aligned}$$

For the final map $t \colon \pi X \to X, t([a]) = a(1)$, the operations $\star \in \Omega_2$, $\omega \in \Omega_1$ and for $[x] \in \pi X$ the following are satisfied:

$$\begin{aligned} t([a \star b]) &= (a \star b)(1) \\ &= a(1) \star b(1) \\ &= t[a] \star t[b] \end{aligned}$$

and

$$\begin{aligned} t(\omega[a]) &= t[\omega(a)] \\ &= (w(a))(1) \\ &= w(a(1)) \\ &= wt[a]. \end{aligned}$$

Since X is a topological group with operations, then for the partial composition $\pi X_t \times_s \pi X \to \pi X, ([x], [y]) \mapsto [x \circ y]$ and $x, y, z, t \in X$,

$$[(x \star z) \circ (y \star t)] = [(x \circ y) \star (t \circ z)],$$

where $(x \circ y)$ and $(t \circ z)$ are defined. This equivalence shows us that the partial composition is a morphism of groups with operations.

Therefore πX becomes a topological internal groupoid. $\qquad\square$

Let sTC be the full subcategory of TC on those objects whose underlying spaces are locally path connected and semi-locally simply connected; and let Gpd(TC) be the category of topological internal groupoids in C. Then we have a functor

$$\pi \colon \mathsf{sTC} \to \mathsf{Gpd}(\mathsf{TC}).$$

4. Covering morphisms of topological internal groupoids

We now give the definition of a covering morphism of topological internal groupoids:

Definition 4.1. Let $\widetilde{G}$ and G be topological internal groupoids and let $p \colon \widetilde{G} \to G$ be a morphism of topological internal groupoids. Then p is called a *covering morphism of topological internal groupoids* if it is also a covering morphism of topogical groupoids.

Example 4.2. Let $p \colon \widetilde{X} \to X$ be a covering morphism of topological groups with operations in which X and $\widetilde{X}$ has universal covers. We know from Theorem 3.11 that $\pi\widetilde{X}$ and πX are topological internal groupoids. Then the induced morphism $\pi(p) \colon \pi\widetilde{X} \to \pi X$ is a covering morphism of topological internal groupoids in C.

Let X be a topological group with operations which has a universal cover. Let $\mathsf{Cov}_{\mathsf{sTC}}/X$ be the category of covering morphisms $p \colon \widetilde{X} \to X$ of topological groups with operations in which $\widetilde{X}$ has also a universal cover. Let $\mathsf{Cov}_{\mathsf{Gpd}(\mathsf{TC})}/\pi X$ be the category of covering morphisms $p \colon \widetilde{G} \to \pi X$ of topological internal groupoids based on πX such that $\mathrm{Ob}(\widetilde{G})$ has a universal cover. We now prove the following equivalence of the categories.

Theorem 4.3. *The categories* $\mathsf{Cov}_{\mathsf{sTC}}/X$ *and* $\mathsf{Cov}_{\mathsf{Gpd}(\mathsf{TC})}/\pi X$ *are equivalent.*

Proof. Define a functor

$$\pi \colon \mathsf{Cov}_{\mathsf{sTC}}/X \to \mathsf{Cov}_{\mathsf{Gpd}(\mathsf{TC})}/\pi X$$

as follows: Suppose that $p \colon \widetilde{X} \to X$ is a covering morphism of topological groups with operations in which both underlying spaces X and $\widetilde{X}$ have also a universal covers. From [2, Theorem 3.1] the induced morphism $\pi(p) \colon \pi\widetilde{X} \to \pi X$ is a morphism of internal groupoids and a covering morphism on the underlying groupoids. By Theorem 3.11, πX and $\pi\widetilde{X}$ are topological internal groupoids. Since the map $(\pi(p), s)$ in the sense of Theorem 2.3 is a homeomorphism and so $\pi(p)$ is a covering morphism of topological groupoids [10, Example 3]. Therefore $\pi(p) \colon \pi\widetilde{X} \to \pi X$ is a covering morphism of topological internal groupoids. In this way we have a functor as required.

We now define a functor

$$\Gamma \colon \mathsf{Cov}_{\mathsf{Gpd}(\mathsf{TC})}/\pi_1 X \to \mathsf{Cov}_{\mathsf{sTC}}/X$$

as follows: Let $q \colon \widetilde{G} \to \pi_1 X$ be a covering morphism of topological internal groupoids in which $\mathrm{Ob}(\widetilde{G}) = \widetilde{X}$ and X have a universal cover. By the lifted topology on $\widetilde{X}$ [5, 10.5.5] we have a covering morphism $p \colon \widetilde{X} \to X$ of topological spaces where $p = \mathrm{Ob}(q)$ and $\widetilde{X} = \mathrm{Ob}(\widetilde{G})$. Since q is a covering morphism of topological internal groupoid, q and $\mathrm{Ob}(q) = p$ are morphisms of topological groups with operations.

Since the category $\mathsf{Cov}_{\mathsf{Top}}/X$ of topological covering maps of X is equivalent to the category $\mathsf{Cov}_{\mathsf{Gd}}/\pi(X)$ of groupoid covering morphisms of πX, by the following diagram the proof is completed.

$$
\begin{array}{ccc}
\mathsf{Cov}_{\mathsf{sTC}}/X & \xrightarrow{\ \pi\ } & \mathsf{Cov}_{\mathsf{Gpd}(\mathsf{TC})}/\pi(X) \\
\downarrow & & \downarrow \\
\mathsf{Cov}_{\mathsf{Top}}/X & \xrightarrow[\ \pi\ ]{} & \mathsf{Cov}_{\mathsf{Gd}}/\pi(X)
\end{array}
$$

$\square$

We define an action of a topological internal groupoid on a topological group with operation as follows:

Definition 4.4. Let G be a topological internal groupoid in TC and X an object of TC. If the underlying groupoid of G acts on the underlying set of X such that $\theta\colon X \to \mathrm{Ob}(G)$ and $\varphi\colon X_\theta \times_s G \to X, (x,a) \mapsto xa$ are morphisms in TC, then we say that the topological internal groupoid G *acts as topologically* on the topological group with operations X via θ.

We write (X, θ, φ) for an action.

Here note that $\varphi\colon X_\theta \times_s G \to X, (x,a) \mapsto xa$ is a morphism in TC, then we have

$$(x \star y)(a \star b) = (xa) \star (yb)$$

for $x, y \in X$ and $a, b \in G$ whenever one side is defined.

Example 4.5. Let G and $\widetilde{G}$ be topological internal groupoids in TC and let $p\colon \widetilde{G} \to G$ be a covering morphism of of topological internal groupoids as defined in Definition 4.1. Then the topological internal groupoid G acts topologically on the topological group with operations $\widetilde{X} = \mathrm{Ob}(\widetilde{G})$ via $\mathrm{Ob}(p)$ assigning to $\widetilde{x} \in \widetilde{X}$ and $a \in G_{p(\widetilde{x})}$ the target of the unique lifting of a with source $\widetilde{x}$: In fact $p\colon \widetilde{G} \to G$ is a covering morphism of topological groupoids and therefore by Example 2.4 the topological groupoid G act topologically on the space $\widetilde{X} = \mathrm{Ob}(\widetilde{G})$ via $\mathrm{Ob}(p)$. The condition

$$(\widetilde{x} \star \widetilde{y})(a \star b) = (\widetilde{x}a) \star (\widetilde{y}b)$$

is satisfied for $\widetilde{x}, \widetilde{y} \in \widetilde{X}$ and $a, b \in G$ whenever one sight is defined. The other conditions related to the group operations are satisfied.

We know from [10, p.145] that if G is a topological groupoid and X is a topological space such that G acts topologically on X, then the semidirect product groupoid $G \ltimes X$ becomes a topological groupoid. The internal groupoid case of the following theorem was given in [2, Theorem 2.15]. So we obtain the following result for topological internal groupoids.

Theorem 4.6. *Let G be a topological internal category in TC and X an object of TC. Suppose that the topological internal groupoid G acts as topologically on the topological group with operations X. Then the semidirect product groupoid $G \ltimes X$ becomes a topological internal groupoid in TC such that the projection $p\colon G \ltimes X \to G$ is a morphism of topological internal groupoids.*

Let G be a topological internal groupoid in the category TC of topological groups with operations. Then we have a category $\mathsf{Cov}_{\mathsf{Gpd}(\mathsf{TC})}/G$ of covering morphisms of topological internal groupoids based on G. So the objects are the covering morphisms $p\colon \widetilde{G} \to G$ of topological internal groupoids and a morphism say from $p\colon \widetilde{G} \to G$ to $q\colon \widetilde{H} \to G$ is a morphism $f\colon \widetilde{G} \to \widetilde{H}$ of topological internal groupoids which becomes also a covering morphism, such that $qf = p$.

Let $\mathsf{Act}_{\mathsf{Gpd}(\mathsf{TC})}(G)$ be the category of topological internal groupoid actions on topological groups with operations. So an object of $\mathsf{Act}_{\mathsf{Gpd}(\mathsf{TC})}(G)$ is an topological internal groupoid action (X, θ, φ) of G on a topological group with operations X and a morphism say from (X, θ, φ) to (X, θ, φ) is a morphism of topological groups with operations $f\colon X \to X'$ such that $\theta = \theta' f$ and $f(xa) = (fx)a$ whenever xa is defined.

In [10, Theorem 2] it was proved that for a topological groupoid G, the category $\mathsf{Act}_{\mathsf{TGd}}(G)$ of topological groupoid actions of G and the category $\mathsf{Cov}_{\mathsf{TGd}}/G$ of covering morphisms of topological groupoids based on G are equivalent. We now generalize this equivalence to the topological internal groupoids as follows:

Theorem 4.7. *The categories* $\mathsf{Cov}_{\mathsf{Gpd(TC)}}/G$ *and* $\mathsf{Act}_{\mathsf{Gpd(TC)}}/G$ *are equivalent.*

Proof. We define a functor

$$\Gamma\colon \mathsf{Cov}_{\mathsf{Gpd(TC)}}/G \to \mathsf{Act}_{\mathsf{Gpd(TC)}}(G)$$

as follows: Let $p\colon \widetilde{G} \to G$ be a covering morphism of topological internal groupoid based on G. Then by Example 4.5 topological internal groupoid G acts as topologically on topological group with operations $\mathrm{Ob}(\widetilde{G})$ via θ.

We now define a functor

$$\Phi\colon \mathsf{Act}_{\mathsf{Gpd(TC)}}(G) \to \mathsf{Cov}_{\mathsf{Gpd(TC)}}/G$$

as follows: Let (X, θ, φ) be a topological internal groupoid action on a topological group with operation. So by [8] the projection $p\colon G \ltimes X \to G$ is covering morphism of topological groupoids and by [2, Theorem 3.14] is a covering morphism of internal groupoids. By compounding these we have that p is a covering morphism of topological internal groupoids.

The natural equivalences $\Gamma\Phi \simeq 1$ and $\Phi\Gamma \simeq 1$ follow. $\qquad\square$

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