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Orlicz-Lorentz spaces and their multiplication operators

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Abstract

The boundedness, closed range, invertibility, compactness and closedness of multiplication operators on Orlicz-Lorentz spaces are characterized in this paper.

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1. Introduction

Let f a complex-valued measurable function defined on a σ-finite measure space $(X, \mathcal{A}, \mu)$. For $\lambda \geq 0$, define $D_f(\lambda)$ the distribution function of f as

(1.1) $$D_f(\lambda) = \mu\left(\{x \in X : |f(x)| > \lambda\}\right).$$

Observe that D_f depends only on the absolute value $|f|$ of the function f and D_f may assume the value $+\infty$.

The distribution function D_f provides information about the size of f but not about the behavior of f itself near any given point. For instance, a function on $\mathbb{R}^n$ and each of its translates have the same distribution function. It follows from (1.1) that D_f is a decreasing function of λ (not necessarily strictly) and continuous from the right.

Let (X, μ) be a measurable space and f and g be a measurable functions on (X, μ) then D_f enjoy the following properties for all $\lambda_1, \lambda_2 \geq 0$:

(1) $|g| \leq |f|$ μ-a.e. implies that $D_g \leq D_f$;
(2) $D_{cf}(\lambda) = D_f\left(\frac{\lambda}{|c|}\right)$ for all $c \in \mathbb{C} \setminus \{0\}$;

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(3) $D_{f+g}(\lambda_1+\lambda_2) \leq D_f(\lambda_1) + D_g(\lambda_2)$;
(4) $D_{fg}(\lambda_1\lambda_2) \leq D_f(\lambda_1) + D_g(\lambda_2)$.

For more details on distribution function see [7].

By f^* we mean the non-increasing rearrangement of f given as

$$f^*(t) = \inf\{\lambda > 0 : D_f(\lambda) \leq t\}, \quad t \geq 0$$

where we use the convention that $\inf \emptyset = \infty$. f^* is decreasing and right-continuous. Notice

$$f^*(0) = \inf\{\lambda > 0 : D_f(\lambda) \leq 0\} = \|f\|_\infty,$$

since

$$\|f\|_\infty = \inf\{\alpha \geq 0 : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\}.$$

Also observe that if D_f is strictly decreasing, then

$$f^*(D_f(t)) = \inf\{\lambda > 0 : D_f(\lambda) \leq D_(f)t\} = t.$$

This fact demonstrates that f^* is the inverse function of the distribution function D_f. Let $\mathcal{F}(X, \mathcal{A})$ denote the set of all $\mathcal{A}$-measurable functions on X. Let $(X, \mathcal{A}_0, \mu)$ and $(Y, \mathcal{A}_1, \nu)$ be two measure spaces.

Two functions $f \in F(X, \mathcal{A}_0)$ and $g \in F(X, \mathcal{A}_1)$ are said to be equimeasurable if they have the same distribution function, that is, if

$$\mu\left(\{x \in X : |f(x)| > \lambda\}\right) = \nu\left(\{y \in Y : |g(y)| > \lambda\}\right), \quad \text{for all } \lambda \geq 0. \tag{1.2}$$

So then there exists only one right-continuous decreasing function f^* equimeasurable with f. Hence the decreasing rearrangement is unique.

In what follows, we gather some useful properties of the decreasing rearrangement function:

a) f^* is decreasing.
b) $f^*(t) > \lambda$ if and only if $D_f(\lambda) > t$.
c) f and f^* are equimeasurables, that is

$$D_f(\lambda) = D_{f^*}(\lambda) \quad \text{for all } \lambda \geq 0.$$

d) If $|f| \leq \liminf_{n\to\infty} |f_n|$ then $f^* \leq \liminf_{n\to\infty} f_n^*$.
e) If $E \in \mathcal{A}$, then $(\chi_E)^*(t) = \chi_{[0,\mu(E))}(t)$.
f) If $E \in \mathcal{A}$, then $(f\chi_E)^*(t) \leq f^*(t)\chi_{[0,\mu(E))}(t)$.

A weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0, \infty)$ almost everywhere. Therefore, weights are allowed to be zero or infinite only on a set of Lebesgue measure zero.

Let $\varphi : [0, \infty) \to [0, \infty)$ be a convex function such that

(1) $\varphi(x) = 0$ if and only if $x = 0$;
(2) $\lim_{x\to\infty} \varphi(x) = \infty$.

Such as function is known as a Young function. A Young function is strictly increasing, in fact, let $0 < x < y$ then $0 < \frac{x}{y} < 1$ and hence, we might write

$$x = \left(1 - \frac{x}{y}\right)0 + \frac{x}{y}y.$$

Since φ is convex, we have

$$\begin{aligned}\varphi(x) &= \varphi\left(\left(1-\frac{x}{y}\right)0+\frac{x}{y}y\right)\\ &\le \left(1-\frac{x}{y}\right)\varphi(0)+\frac{x}{y}\varphi(y)\\ &< \varphi(y).\end{aligned}$$

A Young function is said to satisfy the Δ_2-condition if there exists a nonnegative constant x_0 and k such that

$$\varphi(2x) \le k\varphi(x) \quad \text{for } x \ge x_0. \tag{1.3}$$

If $x_0 = 0$, we say that φ satisfy globally the Δ_2-condition. The smaller constant k which satisfy (1.3) is denoted by k_Δ.

1.1. Claim. If φ is a Young function such that satisfy the Δ_2-condition, then for each $r \ge 0$ there exists a constant $k_\Delta(r)$ such that

$$\varphi(rx) \le k_\Delta(r)\varphi(x) \tag{1.4}$$

for $x > 0$ large enough.

Proof of the claim. If $r > 0$, we can choose $n \in \mathbb{N}$ such that $r \le 2^n$. Then we can applied (1.3) n-times and use the fact that φ is increasing to obtain

$$\varphi(rx) \le \varphi(2^n x) \le k^n\varphi(x),$$

and hence we have (1.4). □

1.2. Example. The function $\varphi_1(x) = \frac{x^p}{p}$ with $p > 1$ is a Young function which satisfy globally the Δ_2-condition with $k_\Delta = \frac{2^p}{p}$.

1.3. Example. The function $\varphi_2(t) = t^p \log(1+t)$ with $p \ge 1$ and $t \ge 0$ is a Young function which satisfy the Δ_2-condition, indeed, since

$$\lim_{t\to\infty} \frac{\varphi_2(2t)}{\varphi_2(t)} = \lim_{t\to\infty} \frac{2^p t^p \log(1+2t)}{t^p \log(1+t)} = 2^{p-1}.$$

Also, φ_2 satisfy globally the Δ_2-condition.

In fact, since for each $t \ge 0$ we have $(1+t)^2 \ge 1+2t$, then

$$\begin{aligned}\varphi_2(2t) &= 2^p t^p \log(1+2t)\\ &\le 2^{p+1} t^p \log(1+2t)\\ &\le 2^{p+1}\varphi_2(2t).\end{aligned}$$

1.4. Lemma. A Young function φ satisfy the Δ_2-condition if and only if there exist constants $\lambda > 1$ and $t_0 > 0$ such that

$$\frac{tp(t)}{\varphi(t)} < \lambda$$

for all $t \ge t_0$, where p is the right derivate of φ.

Proof. Suppose that φ satisfy the Δ_2-condition, then there exists a constant $k > 0$ such that

$$k\varphi(t) \ge \varphi(2t) = \int_0^{2t} p(s)\,ds > \int_t^{2t} p(s)\,ds$$

for t large enough, since p is increasing, then we have

$$\int_t^{2t} p(s)\,ds > tp(t);$$

hence, for t large enough, we obtain

$$\frac{tp(t)}{\varphi(t)} \leq k.$$

Conversely, if

$$\frac{tp(t)}{\varphi(t)} < \lambda$$

for all $t \geq t_0$, then

$$\int_t^{2t} \frac{p(s)}{\varphi(s)}\,ds < \lambda \int_t^{2t} \frac{ds}{s} = \lambda \log 2.$$

Since $p(s) = \varphi'(s)$, we have

$$\log\left(\frac{\varphi(2t)}{\varphi(t)}\right) < \lambda \log 2,$$

which implies that

$$\varphi(2t) < 2^{\lambda}\varphi(t).$$

□

The following result show us that the Young functions which satisfy the Δ_2-condition have a cross rate less than the function t^p for some $p > 1$.

1.5. Theorem. If φ is a Young function which satisfy the Δ_2-condition, then there exist constants $\lambda > 1$ and $C > 0$ such that

$$\varphi(t) \leq Ct^{\lambda}$$

for t large enough.

Proof. By (1.4) we can write

$$\int_{t_0}^{t} \frac{p(s)}{\varphi(s)}\,ds < \lambda \int_{t_0}^{t} \frac{ds}{s}$$

where $t \geq t_0$. Then

$$\log\left(\frac{\varphi(t)}{\varphi(t_0)}\right) < \lambda \log\left(\frac{t}{t_0}\right),$$

therefore

$$\varphi(t) < \frac{\varphi(t_0)}{t_0^{\lambda}} t^{\lambda}.$$

And the proof is complete. □

1.6. Example. The following are Young functions:

(1) $\varphi(x) = \frac{|x|^p}{p}$ with $p > 1$.
(2) $\varphi(x) = e^{|x|} - |x| - 1$.
(3) $\varphi(x) = e^{|x|^{\delta}} - 1$ with $\delta > 1$.
(4) $\varphi(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1 \\ +\infty, & \text{otherwise.} \end{cases}$

Related with the Young function φ, we define, for $t \geq 0$ the complementary function of Young function as

$$\psi(t) = \sup\{ts - \varphi(s) : s \geq 0\}.$$

1.7. Example. If $\varphi(t) = \frac{1}{p}t^p$ with $p > 1$ and $t \geq 0$, then its complementary function is $\psi(t) = \frac{1}{q}t^q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Indeed, by definition we have

$$\psi(t) = \sup\left\{ts - \frac{1}{p}s^p : s \geq 0\right\},$$

next, for $t > 0$ fixed, we can consider the function

$$g(s) = ts - \frac{1}{p}s^p, \quad \text{with } s \geq 0.$$

It is not hard to check that g achieved its maximum at $s = t^{\frac{1}{p-1}}$ which is given by

$$g\left(t^{\frac{1}{p-1}}\right) = \frac{1}{q}t^q.$$

Hence

$$\psi(t) = \sup\left\{ts - \frac{1}{p}s^p : s \geq 0\right\} = \frac{1}{q}t^q.$$

1.8. Proposition. If φ is a Young function, then its complementary function ψ is also a Young function.

Proof. It is clear that $\psi(0) = 0$ if and only if $x = 0$. Now, we just need to show that ψ is a convex function. To this end, let us choose $t_1, t_2 \in [0, +\infty)$ and $\lambda \in [0, 1]$. Then, by definition of ψ we have

$$\psi(\lambda t_1 + (1-\lambda)t_2) = \sup\{s(\lambda t_1 + (1-\lambda)t_2) - \varphi(s) : s \geq 0\}.$$

On the other hand

$$\lambda\psi(t_1) = \lambda \sup\{st_1 - \varphi(s) : s \geq 0\} \geq \lambda(st_1 - \varphi(s)) \ \forall\ s \geq 0$$

and

$$(1-\lambda)\psi(t_2) = (1-\lambda)\sup\{st_2 - \varphi(s) : s \geq 0\} \geq (1-\lambda)(st_2 - \varphi(s)) \ \forall\ s \geq 0.$$

From the last two inequalities, we have

$$\begin{aligned} s(\lambda t_1 + (1-\lambda)t_2) - \varphi(s) &= \lambda(st_1 - \varphi(s)) + (1-\lambda)(st_2 - \varphi(s)) \\ &\leq \lambda\psi(t_1) + (1-\lambda)\psi(t_2) \end{aligned}$$

for all $s \geq 0$. Which means that $\lambda\psi(t_1) + (1-\lambda)\psi(t_2)$ is an upper bound of the set

$$\{s(\lambda t_1 + (1-\lambda)t_2) - \varphi(s) : s \geq 0\},$$

then

$$\psi(\lambda t_1 + (1-\lambda)t_2)) \leq \psi(t_1) + (1-\lambda)\psi(t_2),$$

and so ψ is convex. □

1.9. Theorem (Young's Inequality)**.** Let ψ be the complementary function of φ. Then

$$ts \leq \varphi(s) + \psi(t)$$

where $t, s \in [0, +\infty)$.

Proof. Let $t, s \in [0, +\infty)$. Then

$$\begin{aligned}\psi(t) &= \sup\{st - \varphi(s) : s \geq 0\} \\ &\geq st - \varphi(s) \quad \forall\ s \geq 0,\end{aligned}$$

then

$$\psi(t) + \varphi(s) \geq st,$$

and the proof is complete. □

For more details on Young functions see [10].

2. Weighted Lorentz-Orlicz Spaces

The aim of this section is to present basic results about Lorentz-Orlicz spaces. We have tried to make the proofs as self-contained and synthetic as possible.

2.1. Definition (Luxemburg norm)**.** Let φ be a Young function. For any measurable function f on X,

$$\|f\|_{\varphi,w} = \inf\left\{\varepsilon > 0 : \int_0^\infty \varphi\left(\frac{f^*(t)}{\varepsilon}\right) w(t)\, dt \leq 1\right\} \in [0, \infty).$$

Where it is understood that $\inf(\emptyset) = +\infty$.

2.2. *Remark.* In this article, we will not always require that the Luxemburg norm actually be a norm. $\|\cdot\|_{\varphi,w}$ is indeed a quasinorm. A quasinorm is a functional that is like a norm except that it does only satisfy the triangle inequality with a constant $C \geq 1$, that is, $\|f + g\| \leq C(\|f\| + \|g\|)$ where $C \geq 1$.

2.3. Lemma. For any measurable function f on X, $\|f\|_{\varphi,w} = 0$ if and only if $f = 0$ μ-almost everywhere.

Proof. Clearly $\|f\|_{\varphi,w} = 0$ if and only if $\int_0^\infty \varphi\left(\frac{f^*(t)}{\varepsilon}\right) w(t)\, dt \leq 1\ \forall\, \varepsilon > 0$. It follows that

$$\begin{aligned}\|f\|_{\varphi,w} = 0 \text{ if and only if } &\int_0^\infty \varphi\left(\alpha f^*(t)\right) w(t)\, dt = 0\ \forall\ \alpha > 0 \\ \text{if and only if } &\varphi\left(\alpha f^*(t)\right) w(t) = 0\ \mu - \text{a.e.}\ \forall\ \alpha > 0 \\ \text{if and only if } &f^*(t) = 0\ \mu - \text{a.e.} \\ \text{if and only if } &D_f(\lambda) = 0\ \mu - \text{a.e.} \\ \text{if and only if } &f = 0\ \mu - \text{a.e.}\end{aligned}$$

□

Identification of almost everywhere equal functions. As with L_p spaces, one identifies the function which are μ-almost everywhere equal. This means that one works with the equivalence classes of the equivalence relation defined by the μ-almost everywhere equality. From now on, this will be done without further mention. Consequently, one write:

$$\|f\|_{\varphi,w} = 0 \text{ if and only if } f = 0. \tag{2.1}$$

2.4. Lemma. If $0 < \|f\|_{\varphi,w} < \infty$ then $\int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\, dt \leq 1$. In particular, $\|f\|_{\varphi,w} \leq 1$ is equivalent to $\int_0^\infty \varphi\left(f^*(t)\right) w(t)\, dt \leq 1$.

Proof. For all $b > \|f\|_{\varphi,w}$, we have

$$\int_0^\infty \varphi\left(\frac{f^*(t)}{b}\right) w(t)\,dt \le 1.$$

Letting b decrease to $\|f\|_{\varphi,w}$, one obtains the first result by monotone convergence. The second statement follows from this and lemma 2.8. □

2.5. Proposition. The gauge $\|\cdot\|_{\varphi,w}$ is a quasinorm on the vector space of all the measurable functions f such that $\|f\|_{\varphi,w} < \infty$.

Proof. It is already seen that (2.1) holds under identification of a.e. equal functions.

It is clear that for all real λ, $\|\lambda f\|_{\varphi,w} = |\lambda|\|f\|_{\varphi,w}$.

It remains to prove the triangle inequality. Let f and g be two measurable functions such that $0 < \|f\|_{\varphi,w} + \|g\|_{\varphi,w} < \infty$. Then

$$\begin{aligned}
&\int_0^\infty \varphi\left(\frac{(f+g)^*(t)}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\right) w(t)\,dt \\
&\le \int_0^\infty \varphi\left(\frac{f^*(t/2)+g^*(t/2)}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\right) w(t)\,dt \\
&= \int_0^\infty \varphi\left(\frac{\|f\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\frac{f^*(t/2)}{\|f\|_{\varphi,w}} + \frac{\|g\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\frac{g^*(t/2)}{\|g\|_{\varphi,w}}\right) w(t)\,dt \\
&\le \frac{\|f\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\int_0^\infty \varphi\left(\frac{f^*(t/2)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
&\quad + \frac{\|g\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}\int_0^\infty \varphi\left(\frac{g^*(t/2)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
&= \frac{\|f\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}2\int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(2t)\,dt \\
&\quad + \frac{\|g\|_{\varphi,w}}{2(\|f\|_{\varphi,w} + \|g\|_{\varphi,w})}2\int_0^\infty \varphi\left(\frac{g^*(t)}{\|f\|_{\varphi,w}}\right) w(2t)\,dt \\
&\le \frac{\|f\|_{\varphi,w}}{\|f\|_{\varphi,w} + \|g\|_{\varphi,w}}\int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
&\quad + \frac{\|g\|_{\varphi,w}}{\|f\|_{\varphi,w} + \|g\|_{\varphi,w}}\int_0^\infty \varphi\left(\frac{g^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \\
&\le 1.
\end{aligned}$$

Where the last but one inequality follows from the convexity of φ and the fact that w is nonincreasing and the last inequality from lemma 2.4. Therefore

$$\|f+g\|_{\varphi,w} \le 2\left(\|f\|_{\varphi,w} + \|g\|_{\varphi,w}\right).$$

As a consequence, the set of all measurable functions f such that $\|f\|_{\varphi,w} < \infty$ is a vector space. □

2.6. Definition. Let φ be a Young function. We define the weighted Lorenz-Orlicz spaces

$$L_{\varphi,w} = \left\{ f : X \to \mathbb{C} \text{ measurable} : \int_0^\infty \varphi(\alpha f^*(t))w(t)\,dt < \infty, \text{ for some } \alpha > 0 \right\}.$$

It follows from proposition 1.8 that if $L_{\varphi,w}$ is a weighted Lorentz-Orlicz space, then $L_{\psi,w}$ is also a weighted Lorenz-Orlicz space.

2.7. Proposition (Hölder's type inequality)**.** For $f \in L_{\varphi,1}$ and $g \in L_{\psi,1}$

$$\int_X |fg|\, d\mu \leq 2\|f\|_{\varphi,1}\|g\|_{\psi,1}.$$

In particular, $fg \in L_1$.

Proof. If $\|f\|_{\varphi,1} = 0$ or $\|g\|_{\psi,1} = 0$, one concludes with lemma 2.8.

Assume now that $0 < \|f\|_{\varphi,1}, \|g\|_{\psi,1}$. Because of Young's inequality: $st \leq \varphi(s) + \varphi(t)$ we have

$$\begin{aligned} \int_X \frac{|fg|}{\|f\|_{\varphi,1}\|g\|_{\psi,1}}\, d\mu &\leq \int_0^\infty \frac{f^*(t)g^*(t)}{\|f\|_{\varphi,1}\|g\|_{\psi,1}}\, dt \\ &\leq \int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,1}}\right) dt + \int_0^\infty \psi\left(\frac{g^*(t)}{\|g\|_{\psi,1}}\right) dt \\ &\leq 2. \end{aligned}$$

Therefore

$$\int_X |fg|\, d\mu \leq 2\|f\|_{\varphi,1}\|g\|_{\psi,1}. \qquad \square$$

2.8. Lemma. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $L_{\varphi,w}$. Then, the following assertions are equivalent:

(a) $\lim_{n\to\infty} \|f_n\|_{\varphi,w} = 0$;
(b) For all $\alpha > 0$, $\limsup_{n\to\infty} \int_0^\infty \varphi(\alpha f_n^*(t))w(t)\, dt \leq 1$;
(c) For all $\alpha > 0$, $\lim_{n\to\infty} \int_0^\infty \varphi(\alpha f_n^*(t))w(t)\, dt = 0$.

Proof. The equivalence $(a) \Leftrightarrow (b)$ is a direct consequence of the definition of $\|\cdot\|_{\varphi,w}$. Off course $(c) \Rightarrow (b)$ is obvious. As φ is convex and $\varphi(0) = 0$ for all $t \geq 0$ and $0 < \varepsilon \leq 1$, we have

$$\varphi(t) = \varphi\left((1-\varepsilon)0 + \varepsilon\frac{t}{\varepsilon}\right) \leq (1-\varepsilon)\varphi(0) + \varepsilon\varphi\left(\frac{t}{\varepsilon}\right),$$

that is

$$\varphi(t) \leq \varepsilon\varphi\left(\frac{t}{\varepsilon}\right) \quad t \geq 0, 0 < \varepsilon \leq 1.$$

From which $(b) \Rightarrow (c)$ follows easily. $\square$

2.9. Theorem. The space $L_{\varphi,w}$ is a quasi-Banach space.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $L_{\varphi,w}$. Let us choose $\tilde{\varepsilon} > 0$ such that $\tilde{\varepsilon}\varphi^{-1}\left(\frac{\varepsilon}{k_0}\right) < \frac{1}{n+m}$ for $n, m \in \mathbb{N}$ and $\varepsilon > 0, k_0 > 0$. For such $\tilde{\varepsilon}$ there exists $n_0 \in \mathbb{N}$ such that

$$\|f_n - f_m\|_{\varphi,w} < \tilde{\varepsilon}.$$

If $n, m \geq n_0$. By the definition of the Luxemburg quasi-norm we can use $k_0 > 0$ in such a way that $k_0 < \tilde{\varepsilon}$ and

$$\int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\, dt \leq 1.$$

Let $E = \{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}$, then

$$\varepsilon\chi_E(x) \leq |f_n(x) - f_m(x)|.$$

And hence

$$\varepsilon\chi_E^*(t) \le (f_n - f_m)^*(t),$$
$$\varepsilon\chi_{(0,\mu(E))}(t) \le (f_n - f_m)^*(t).$$

Therefore

$$\int_0^\infty \varphi\left(\frac{\varepsilon}{k_0}\chi_{(0,\mu(E))}(t)\right) w(t)\,dt \le \int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\,dt.$$

Then

$$\begin{aligned}
&\int_0^{\mu(E)} \varphi\left(\frac{\varepsilon}{k_0}\right) w(t)\,dt \le \int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\,dt \\
\Rightarrow\ & \tilde{\varepsilon}\int_0^{D_{f_n - f_m}(\varepsilon)} w(t)\,dt \le \tilde{\varepsilon}\varphi^{-1}\left(\frac{\varepsilon}{k_0}\right)\int_0^\infty \varphi\left(\frac{(f_n - f_m)^*(t)}{k_0}\right) w(t)\,dt \\
\Rightarrow\ & \tilde{\varepsilon}\int_0^{D_{f_n - f_m}(\varepsilon)} w(t)\,dt \le \frac{1}{n+m} \\
\Rightarrow\ & \tilde{\varepsilon}\lim_{n,m\to\infty}\int_0^{D_{f_n - f_m}(\varepsilon)} w(t) = 0.
\end{aligned}$$

Since $w > 0$, we must have $\lim_{n,m\to\infty} D_{f_n - f_m}(\varepsilon) = 0$ which means that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, then some subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ converges almost everywhere to a measurable function f, that is, $f_{n_k} \to f$ μ-a.e.

Let $\alpha > 0$. By lemma 2.8 there exists a large enough integer $n(\alpha)$ such that

$$\int_0^\infty \varphi\left(\alpha(f_n - f_m)^*(t)\right) w(t)\,dt \le 1, \quad \forall\ m, n \ge n(\alpha).$$

With Fatou's lemma this gives

$$\int_0^\infty \varphi\left(\alpha(f_n - f)^*(t)\right) w(t)\,dt \le \liminf \int_0^\infty \varphi\left(\alpha(f_n - f_m)^*(t)\right) w(t)\,dt \le 1$$

$\forall\ m \ge n(\alpha)$. Therefore $f_n - f$ belongs to $L_{\varphi,w}$, but $f_n \in L_{\varphi,w}$, so that $f \in L_{\varphi,w}$.

Moreover, as $\limsup_{m\to\infty} \int_0^\infty \varphi\left(\alpha(f_m - f)^*(t)\right) w(t)\,dt \le 1$ for all $\alpha > 0$, we have $\lim_{m\to\infty} \|f_m - f\|_{\varphi,w} = 0$. This proves that $L_{\varphi,w}$ is complete. □

2.10. Theorem. Simple functions are dense in $L_{\varphi,w}$.

Proof. Suppose $f \in L_{\varphi,w}$. We may assume that $f \ge 0$. Note that if $D_f(\lambda) = \infty$, then $\lim_{t\to\infty} f^*(t) = 0$. It follows that $D_f(\lambda) < \infty$.

Hence, given $\varepsilon, \delta > 0$, we can find a simple function $s_n \ge 0$ such that $s_n(x) = 0$ when $f(x) \le \varepsilon$ and $f(x) - \varepsilon \le s_n(x) \le f(x)$ when $f(x) > \varepsilon$ except on a set of measure less than δ. It follows that

$$\mu\left(\{x \in X : |f(x) - s_n(x)| > \varepsilon\}\right) < \delta.$$

Next, choose $n \in \mathbb{N}$ such that $n \ge \frac{1}{\varepsilon}$, then

$$(f - s_n)^*(t) = \inf\{\varepsilon > 0 : D_{f - s_n}(\varepsilon) < \delta \le t\}.$$

Thus

$$(f - s_n)^*(t) \le \frac{1}{n} \quad \text{for } t \ge \delta,$$

since $s_n \le f$, then $s_n^*(t) \le f^*(t)$, for each $t > 0$. Since $n > \frac{1}{\varepsilon}$, we have

$$(f - s_n)^*(t) \le \frac{1}{n} < \varepsilon,$$

next,

$$\int_0^\infty \varphi\left(\frac{(f-s_n)^*(t)}{k}\right) w(t)\,dt \le \int_0^\infty \varphi\left(\frac{1}{nk}\right) w(t)\,dt.$$

Let $a = \int_0^\infty w(t)\,dt$, then

$$\begin{aligned}\|f-s_n\|_{\varphi,w} &= \inf\left\{k>0 : \int_0^\infty \varphi\left(\frac{(f-s_n)^*(t)}{k}\right) w(t)\,dt \le 1\right\}\\ &= \frac{1}{n\varphi^{-1}\left(\frac{1}{a}\right)} \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

□

3. Multiplication Operator

Let $F(X)$ be a function space on non-empty set X. Let $u : X \to \mathbb{C}$ be a function such that $u \cdot f \in F(X)$ whenever $f \in F(X)$.

Then, the transformation $f \mapsto u \cdot f$ on F is denoted by M_u. In case $F(X)$ is a topological space and M_u is continuos, we call it a multiplication operator induced by u.

Multiplication operators generalize the notion of operator given by a diagonal matrix. More precisely, one of the results of operator theory is a spectral theorem, which states that every self-adjoint operator on a Hilbert space is unitarily equivalent to a multiplication operator on an L_2 space.

These operators received considerable attention over the past several decades specially on L_p spaces and Bergman spaces and they played an important role in the study of operators on Hilbert spaces.

For more details on these operators we refer to Abrahamese [1], Axler [4], Douglas [6], Halmos [8] and Takagi [12].

3.1. Example. Consider the Hilbert space $X = L_2[-1,3]$ of complex-valued square integrable functions on the interval $[-1,3]$. Define the operator

$$M_u(x) = u(x)x^2,$$

for any function $u \in X$. This will be a self-adjoint bounded linear operator with norm 9. Its spectrum will be the interval $[0,9]$ (the range of the function $x \to x^2$ defined on $[-1,3]$). Indeed, for any complex number λ, the operator $M_u - \lambda$ is given by

$$(M_u - \lambda)(x) = u(x)(x^2 - \lambda).$$

It is invertible if and only if λ is not in $[0,9]$, and then its inverse is

$$(M_u - \lambda)^{-1}(x) = \frac{u(x)}{x^2 - \lambda}.$$

which is another multiplication operator.

For a systematic study of the multiplication operators on different spaces we refer to [1, 3, 4, 5, 9, 11].

3.2. *Remark.* In general, the multiplication operators on measurable spaces is not 1–1. Indeed, let $(X, \mathcal{A}, \mu)$ be a measure space and

$$A = X \setminus \operatorname{supp}(u) = \{x \in X : u(x) = 0\}.$$

If $\mu(A) \ne 0$ and $f = \chi_A$ then for any $x \in X$ we have $f(x)u(x) = 0$ which implies that $M_u(f) = 0$, therefore $\ker(M_u) \ne \{0\}$ and hence M_u is not 1–1.

If, on the contrary, M_u is 1–1, then $\mu(X \setminus \operatorname{supp}(u)) = 0$. On the other hand, if $\mu(X \setminus \operatorname{supp}(u)) = 0$ and μ is a complete measure, then $M_u(f) = 0$ implies $f(x)u(x) = 0\ \forall\, x \in X$, then $\{x \in X : f(x) \ne 0\} \subseteq X \setminus \operatorname{supp}(u)$ and so $f = 0$ μ-a.e. on X.

Hence, if $\mu(X \setminus \operatorname{supp}(u)) = 0$ and μ is a complete measure, then M_u is 1–1.

3.3. Proposition. M_u is 1–1 on $Y = L_{\varphi,w}(\operatorname{supp} u)$.

Proof. Let $Y = L_{\varphi,w}(\operatorname{supp} u) = \{f\chi_{\operatorname{supp} u} : f \in L_{\varphi,w}\}$. Indeed, if $M_u(\tilde{f}) = 0$ with $\tilde{f} = f\chi_{\operatorname{supp} u} \in Y$, then $f(x)\chi_{\operatorname{supp} u}(x)u(x) = 0$ for all $x \in X$, and so

$$\begin{aligned} & f(x)u(x) = 0 \quad \forall\, x \in \operatorname{supp}(u), \\ \Rightarrow\, & f(x) = 0 \quad \forall\, x \in \operatorname{supp}(u), \\ \Rightarrow\, & f(x)\chi_{\operatorname{supp} u(x)}(x) = 0 \quad \forall\, x \in X. \end{aligned}$$

Then $\tilde{f} = 0$ and the proof is complete. □

In what follows, boundedness and invertibility of the multiplication M_u are characterized in terms of the boundedness and invertibility of the complex valued measurable function u respectively.

3.4. Theorem. The linear transformation $M_u : f \to u \cdot f$ on the Orlicz-Lorentz space $L_{\varphi,w}$ is bounded if and only if u is essentially bounded. Moreover

$$\|M_u\| = \|u\|_\infty.$$

Proof. Let $u \in L_\infty(\mu)$, note $|(uf)(x)| \le \|u\|_\infty |f(x)|$, thus

$$\begin{aligned} \{x : |(uf)(x)| > \lambda\} &\subseteq \{x : \|u\|_\infty |f(x)| > \lambda\} \\ &= \left\{x : |f(x)| > \frac{\lambda}{\|u\|_\infty}\right\} \end{aligned}$$

then

$$D_{uf}(\lambda) \le D_f\left(\frac{\lambda}{\|u\|_\infty}\right)$$

and so

$$\left\{\lambda > 0 : D_f\left(\frac{\lambda}{\|u\|_\infty}\right) \le t\right\} \subseteq \{\lambda > 0 : D_{uf}(\lambda) \le t\}.$$

From this we have

$$\begin{aligned} \inf\{\lambda > 0 : D_{uf}(\lambda) \le t\} &\le \inf\left\{\lambda > 0 : D_f\left(\frac{\lambda}{\|u\|_\infty}\right) \le t\right\} \\ &\le \inf\{\alpha\|u\|_\infty > 0 : D_f(\alpha) \le t\} \\ &= \|u\|_\infty \inf\{\alpha > 0 : D_f(\alpha) \le t\}. \end{aligned}$$

Hence

$$(uf)^*(t) \le \|u\|_\infty f^*(t).$$

Then

$$\begin{aligned} \int_0^\infty \varphi\left(\frac{(uf)^*(t)}{\|u\|_\infty \|f\|_{\varphi,w}}\right) w(t)\,dt &\le \int_0^\infty \varphi\left(\frac{\|u\|_\infty f^*(t)}{\|u\|_\infty \|f\|_{\varphi,w}}\right) w(t)\,dt \\ &= \int_0^\infty \varphi\left(\frac{f^*(t)}{\|f\|_{\varphi,w}}\right) w(t)\,dt \le 1. \end{aligned}$$

Hence $f \in L_{\varphi,w}$ and

$$\|M_u f\|_{\varphi,w} \le \|u\|_\infty \|f\|_{\varphi,w}. \tag{3.1}$$

Conversely, suppose M_u is a bounded operator. If u is not essentially bounded function, then for every $n \in \mathbb{N}$, the set $E_n = \{x \in X : |u(x)| > n\}$ has a positive measure. Now, we know that

$$\chi^*_{E_n}(t) = \chi_{0,\mu(E_n)}(t),$$

and note

$$\{x : n\chi_{E_n}(x) > \lambda\} \subseteq \{x : |u\chi_{E_n}(x)| > \lambda\},$$

then

$$D_{n\chi_{E_n}}(\lambda) \leq D_{u\chi_{E_n}}(\lambda),$$

from this we have

$$\{\lambda > 0 : D_{u\chi_{E_n}}(\lambda) \leq t\} \subseteq \{\lambda > 0 : D_{n\chi_{E_n}}(\lambda) \leq t\}.$$

Hence

$$\inf\{\lambda > 0 : D_{n\chi_{E_n}}(\lambda) \leq t\} \leq \inf\{\lambda > 0 : D_{u\chi_{E_n}}(\lambda) \leq t\}.$$

That is,

$$(u\chi_{E_n})^*(t) \geq n(\chi_{E_n})^*(t).$$

This gives us

$$\begin{aligned} 1 &\geq \int_0^\infty \varphi\left(\frac{(u\chi_{E_n})^*(t)}{k}\right) w(t)\, dt \\ &\geq \int_0^\infty \varphi\left(\frac{(n\chi_{E_n})^*(t)}{k}\right) w(t)\, dt, \end{aligned}$$

and so

$$\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(u\chi_{E_n})^*(t)}{k}\right) w(t)\, dt \leq 1\right\} \subseteq \left\{k > 0 : \int_0^\infty \varphi\left(\frac{(n\chi_{E_n})^*(t)}{k}\right) w(t)\, dt \leq 1\right\},$$

thus

$$\inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(n\chi_{E_n})^*(t)}{k}\right) w(t)\, dt \leq 1\right\} \leq \inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(u\chi_{E_n})^*(t)}{k}\right) w(t)\, dt \leq 1\right\},$$

which means that

$$\|M_u\chi_{E_n}\|_{\varphi,w} \geq n\|\chi_{E_n}\|_{\varphi,w},$$

this contradicts the boundedness of M_u. Hence u must be essentially bounded.

Next, clearly by (3.1) we obtain

$$\|M_u\| \leq \|u\|_\infty. \tag{3.2}$$

For $\varepsilon > 0$, let $E = \{x \in X : |u(x)| \geq \|u\|_\infty - \varepsilon\}$ (observe that $\mu(E) > 0$), then

$$\{x \in X : (\|u\|_\infty - \varepsilon)\chi_E(x) > \lambda\} \subseteq \{x \in X : |u\chi_E(x)| > \lambda\},$$

then

$$D_{(\|u\|_\infty - \varepsilon)\chi_E}(\lambda) \leq D_{u\chi_E}(\lambda)$$

and so

$$\{\lambda > 0 : D_{u\chi_E}(\lambda) \leq t\} \subseteq \{\lambda > 0 : D_{(\|u\|_\infty - \varepsilon)\chi_E} \leq t\}$$

from this we have

$$\inf\{\lambda > 0 : D_{(\|u\|_\infty - \varepsilon)\chi_E} \leq t\} \leq \inf\{\lambda > 0 : D_{u\chi_E}(\lambda) \leq t\}.$$

Therefore

$$(u\chi_E)^*(t) \geq (\|u\|_\infty - \varepsilon)(\chi_E)^*(t),$$

then

$$\int_0^\infty \varphi\left(\frac{(\|u\|_\infty - \varepsilon)(\chi_E)^*(t)}{\|M_u\chi_E\|_{\varphi,w}}\right) w(t)\,dt \le \int_0^\infty \varphi\left(\frac{(u\chi_E)^*(t)}{\|M_u\chi_E\|_{\varphi,w}}\right) w(t)\,dt \le 1,$$

which implies that

$$\|(\|u\|_\infty - \varepsilon)\chi_E\|_{\varphi,w} \le \|M_u\chi_E\|_{\varphi,w},$$

and

$$(\|u\|_\infty - \varepsilon)\|\chi_E\|_{\varphi,w} \le \|M_u\chi_E\|_{\varphi,w},$$

hence

$$\|u\|_\infty - \varepsilon \le \frac{\|M_u\chi_E\|_{\varphi,w}}{\|\chi_E\|_{\varphi,w}},$$

which provide that

$$\|M_u\| \ge \|u\|_\infty - \varepsilon \quad \forall\, \varepsilon > 0$$

and so

$$\|M_u\| \ge \|u\|_\infty.$$

Therefore

$$\|M_u\| = \|u\|_\infty.$$ □

We will need the following well known result.

3.5. Theorem. Let $T \in B(X, Y)$ where X and Y are Banach spaces. Then T is bounded below if and only if T is 1–1 and has closed range.

For the proof of theorem 3.5 see [2].

3.6. Corollary. $M_u : L_{\varphi,w}(\operatorname{supp} u) \to L_{\varphi,w}(\operatorname{supp} u)$ has closed range if and only if M_u is bounded below on $L_{\varphi,w}(\operatorname{supp} u)$.

This result is clear since M_u is 1–1 on $L_{\varphi,w}(\operatorname{supp} u)$. Moreover, if $u \ne 0$ μ-a.e. on X with μ a complete measure, then we have the following result.

3.7. Corollary. If $\mu \ne 0$ μ-a.e. on X and μ is a complete measure, then

$$M_u : L_{\varphi,w}(X, \mathcal{A}, u) \to L_{\varphi,w}(X, \mathcal{A}, u)$$

has a closed range if and only if M_u is bounded below on $L_{\varphi,w}(X, \mathcal{A}, u)$.

3.8. Theorem. $M_u : L_{\varphi,w}(\operatorname{supp} u) \to L_{\varphi,w}(\operatorname{supp} u)$ has a closed range if and only if there exists $\delta > 0$ such that $|u(x)| > \delta$ μ-a.e. on $\operatorname{supp}\mu$.

Proof. If there exists a $\delta > 0$ such that $|u(x)| \ge \delta$ μ-a.e. on supp(u), then for $f \in L_{\varphi,w}$ and $t > 0$ we have

$$\{x : |\delta f\chi_{\operatorname{supp}(u)}(x)| > \lambda\} \subseteq \{x : |uf\chi_{\operatorname{supp}(u)}(x)| > \lambda\},$$

and so

$$D_{\delta f\chi_{\operatorname{supp}(u)}}(\lambda) \le D_{uf\chi_{\operatorname{supp}(u)}}(\lambda),$$

then

$$\{\lambda > 0 : D_{uf\chi_{\operatorname{supp}(u)}}(\lambda) \le t\} \subseteq \{\lambda > 0 : D_{\delta f\chi_{\operatorname{supp}(u)}}(\lambda) \le t\},$$

from this we have

$$\inf\{\lambda > 0 : D_{\delta f\chi_{\operatorname{supp}(u)}}(\lambda) \le t\} \le \inf\{\lambda > 0 : D_{uf\chi_{\operatorname{supp}(u)}}(\lambda) \le t\},$$

thus

$$(uf\chi_{\mathrm{supp}(u)})^*(t) \geq \delta f\chi^*_{\mathrm{supp}(u)}(t),$$

then we shall note that

$$\left\{k>0:\int_0^\infty \varphi\left(\frac{(uf\chi_{\mathrm{supp}(u)})^*(t)}{k}\right)w(t)\,dt \leq 1\right\} \subseteq \left\{k>0:\int_0^\infty \varphi\left(\frac{(\delta f\chi_{\mathrm{supp}(u)})^*(t)}{k}\right)w(t)\,dt \leq 1\right\}.$$

Hence

$$\inf\left\{k>0:\int_0^\infty \varphi\left(\frac{(\delta f\chi_{\mathrm{supp}(u)})^*(t)}{k}\right)w(t)\,dt \leq 1\right\} \leq \inf\left\{k>0:\int_0^\infty \varphi\left(\frac{(uf\chi_{\mathrm{supp}(u)})^*(t)}{k}\right)w(t)\,dt \leq 1\right\},$$

which means that

$$\|\delta f\chi_{\mathrm{supp}(u)}\|_{\varphi,w} \leq \|M_u f\chi_{\mathrm{supp}(u)}\|_{\varphi,w},$$

thus

$$\|M_u f\chi_{\mathrm{supp}(u)}\|_{\varphi,w} \geq \delta\|f\chi_{\mathrm{supp}(u)}\|_{\varphi,w}.$$

Therefore M_u has closed range.

Conversely, assume that M_u has closed range on $L_{\varphi,w}(\mathrm{supp}(u))$. Since $M_u : L_{\varphi,w}(\mathrm{supp}(u)) \to L_{\varphi,w}(\mathrm{supp}(u))$ is 1–1, then M_u is bounded below, then there exists an $\varepsilon > 0$ such that

$$\|M_u f\|_{\varphi,w} \geq \varepsilon\|f\|_{\varphi,w}$$

for all $f \in L_{\varphi,w}(\mathrm{supp}(u))$. Let $E = \{x \in \mathrm{supp}(u) : |u(x)| < \varepsilon/2\}$.

If $\mu(E) > 0$, then we can find a measurable set $F \subseteq E$ such that $\chi_F \in L_{\varphi,w}(\mathrm{supp}(u))$. Then

$$\{x : |u\chi_F| > \lambda\} \subseteq \left\{x : \left|\frac{\varepsilon}{2}\chi_F\right| > \lambda\right\}$$

and so

$$D_{u\chi_F}(\lambda) \leq D_{\frac{\varepsilon}{2}\chi_F}(\lambda),$$

from this we have

$$\{\lambda > 0 : D_{\frac{\varepsilon}{2}\chi_F}(\lambda) \leq t\} \subseteq \{\lambda > 0 : D_{u\chi_F}(\lambda) \leq t\},$$

then

$$\inf\{\lambda > 0 : D_{u\chi_F}(\lambda) \leq t\} \leq \inf\{\lambda > 0 : D_{\frac{\varepsilon}{2}\chi_F}(\lambda) \leq t\}$$

that is,

$$(u\chi_F)^*(t) \leq \left(\frac{\varepsilon}{2}\chi_F\right)^*(t),$$

and so

$$(u\chi_F)^*(t) \leq \frac{\varepsilon}{2}(\chi_F)^*(t).$$

Therefore

$$\begin{aligned}\|M_u\chi_F\|_{\varphi,w} &= \inf\left\{\varepsilon>0:\int_0^\infty \varphi\left(\frac{(u\chi_F)^*(t)}{\varepsilon}\right)w(t)\,dt\le 1\right\}\\ &\le \inf\left\{\varepsilon>0:\int_0^\infty \varphi\left(\frac{\frac{\varepsilon}{2}(\chi_F)^*(t)}{\varepsilon}\right)w(t)\,dt\le 1\right\}\\ &= \frac{\varepsilon}{2}\|\chi_F\|_{\varphi,w},\end{aligned}$$

which is a contradiction. Therefore $\mu(E)=0$. This completes the proof. □

3.9. Corollary. If $\mu\neq 0$ μ-a.e. on X, and μ is a complete measure, then M_u has a closed range on $L_{\varphi,w}(X,\mathcal{A},\mu)$ if and only if there exists $\delta>0$ such that $|u(x)|\ge\delta$ μ-a.e. on X.

Proof. The result follows as a consequence of

$$L_{\varphi,w}(X,\mathcal{A},\mu)=L_{\varphi,w}(\operatorname{supp}u)$$ □

3.10. Theorem. The set of all multiplication operators on $L_{\varphi,w}$ is a maximal abelian subalgebra of the set $B(L_{\varphi,w})$, the algebra of all bounded linear operators on $L_{\varphi,w}$.

Proof. Let

$$\mathcal{H}=\{M_u : u\in L_\infty\}$$

and consider the operator product

$$M_u\cdot M_v=M_{uv},$$

where $M_u, M_v\in\mathcal{H}$. Let us check that it is a Banach algebra. Let $u,v\in L_\infty$, then $|u|\le\|u\|_\infty$ and $|v|\le\|v\|_\infty$, therefore

$$\|uv\|_\infty\le\|u\|_\infty\|v\|_\infty,$$

this implies that the product is an inner operation, moreover the usual function product is associative, commutative and distributive respect to the sum and the scalar product, thus we conclude that $\mathcal{H}$ is a subalgebra of $B(L_{\varphi,w})$. Now, we like to check that it is a maximal subalgebra, that is, given $N\in B(L_{\varphi,w})$, if N commute with $\mathcal{H}$, we have to prove that $N\in\mathcal{H}$. Consider the unit function $e:X\to\mathbb{C}$ defined by $e(x)=1$ for all $x\in X$. Let $N\in B(L_{\varphi,w})$ be an operator which commute with $\mathcal{H}$ and let χ_E the characteristic function of a measurable set E. Then

$$\begin{aligned}N(\chi_E)&=N[M_{\chi_E}(e)]\\ &=M_{\chi_E}[N(e)]\\ &=\chi_E\cdot N(e)\\ &=N(e)\cdot\chi_E\\ &=M_w\cdot\chi_E,\end{aligned}$$

where $w=N(e)$. Similarly

$$N(s)=M_w(s) \tag{3.3}$$

for any simple function.

Now, let us check that $w\in L_\infty$. By way of contradiction, assume that $w\notin L_\infty$, then the set

$$E_n=\{x\in X:|w(x)|>n\}$$

has a positive measure for each $n\in\mathbb{N}$. Note that

$$M_w(\chi_{E_n})(x)=w\chi_{E_n}(x)\ge n\chi_{E_n}(x)$$

for all $x \in X$. By the monotonicity property of the distribution function we have

$$D_{w\chi_{E_n}}(\lambda) \geq D_{\chi_{E_n}}\left(\frac{\lambda}{n}\right).$$

From this

$$\{\lambda > 0 : D_{w\chi_{E_n}}(\lambda) \leq t\} \subseteq \left\{\lambda > 0 : D_{\chi_{E_n}}\left(\frac{\lambda}{n}\right) \leq t\right\}.$$

Then

$$\inf\left\{\lambda > 0 : D_{\chi_{E_n}}\left(\frac{\lambda}{n}\right) \leq t\right\} \leq \inf\{\lambda > 0 : D_{w\chi_{E_n}}(\lambda) \leq t\}.$$

Putting $\alpha = \frac{\lambda}{n}$, we have

$$\|w\chi_{E_n}\|_{\varphi,w} \geq n\|\chi_{E_n}\|_{\varphi,w},$$

since χ_E is a simple function, then by (3.3) we have

$$M_w(\chi_{E_n}) = N(\chi_{E_n}).$$

Hence

$$\|N(\chi_{E_n})\|_{\varphi,w} \geq n\|\chi_{E_n}\|_{\varphi,w}.$$

Therefore N is an unbounded operator. This is a contradiction to the fact that N is bounded.

So then $w \in L_\infty$ and by theorem 3.4 M_w is bounded.

Next, given $f \in L_{\varphi,w}$, there exists a nondecreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of measurable simple functions such that $\lim_{n\to\infty} s_n = f$, then by (3.3) we have

$$\begin{aligned} N(f) &= N(\lim s_n) \\ &= \lim N(s_n) \\ &= \lim M_w(s_n) \\ &= M_w(\lim s_n) \\ &= M_w(f). \end{aligned}$$

Therefore $N(f) = M_w(f)$ for all $f \in L_{\varphi,w}$ and thus we conclude that $N \in \mathcal{H}$. □

3.11. Corollary. The multiplication operator is invertible on $B(L_{\varphi,w})$ if and only if is invertible on L_∞.

Proof. Let M_u be invertible. Then there exists $N \in B(L_{\varphi,w})$ such that

$$M_u \cdot N = N \cdot M_u = I \tag{3.4}$$

where I represent the identity operator. Let us check that N commute with $\mathcal{H}$.

Let $M_w \in \mathcal{H}$, then

$$M_w \cdot M_u = M_u \cdot M_w. \tag{3.5}$$

Applying N to (3.5) and by (3.4) we obtain

$$\begin{aligned} N \cdot M_w \cdot M_u \cdot N &= N \cdot M_u \cdot M_w \cdot N, \\ N \cdot M_w \cdot I &= I \cdot M_w \cdot N, \\ N \cdot M_w &= M_w \cdot N, \end{aligned}$$

and thus we conclude that N commute with $\mathcal{H}$. By theorem 3.10 $N \in \mathcal{H}$, then there exists $g \in L_\infty$ such that $N = M_g$, hence

$$M_u \cdot M_g = M_g \cdot M_u = I,$$

this implies that $ug = gu = 1$ μ-a.e., which means that u is invertible on L_∞.

On the other hand, assume u is invertible on L_∞, that is, $\frac{1}{u} \in L_\infty$, then

$$\begin{aligned} M_u \cdot M_{\frac{1}{u}} &= M_{\frac{1}{u}} \cdot M_u \\ &= M_{(\frac{1}{u})u} \\ &= M_1 = I, \end{aligned}$$

which means that M_u is invertible on $B(L_{\varphi,w})$. □

For the sake of completeness and the convenience of the reader, we gives here one definition and one lemma which will play an important role on the coming results.

3.12. Definition. Let T be an operator. A subspace V of X is said to be invariant under T (or simply T-invariant) whenever

$$T(V) \subseteq V.$$

3.13. Lemma. Let $T : X \to X$ be an operator. If T is compact and M is a closed T-invariant subspace of X, then $T|_M$ is compact.

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a subsequence in $M \subseteq X$. Then $\{x_n\}_{n\in\mathbb{N}} \subseteq X$, thus there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $T(x_{n_k})$ converges in X, but $T(x_{n_k}) \subseteq T(M)$ since $\{x_{n_k}\}_{k\in\mathbb{N}} \subseteq M$. Then $T(x_{n_k})$ converges on $\overline{T(M)} \subseteq \overline{M} = M$. Therefore $T(x_{n_k})$ converges on M, hence $T|_M$ is compact. □

3.14. Theorem. Let M_u be a compact operator. For $\varepsilon > 0$ define

$$A_\varepsilon(u) = \{x \in X : |u(x)| \geq \varepsilon\},$$

and

$$L_{\varphi,w}(A_\varepsilon(u)) = \{f\chi_{A_\varepsilon(u)} : f \in L_{\varphi,w}\}.$$

Then $L_{\varphi,w}(A_\varepsilon(u))$ is a closed invariant subspace of $L_{\varphi,w}$ under M_u. Moreover

$$M_u\big|_{L_{\varphi,w}(A_\varepsilon(u))}$$

is a compact operator.

Proof. Let $h, s \in L_{\varphi,w}(A_\varepsilon(u))$ and $\alpha, \beta \in \mathbb{R}$. Then $h = f\chi_{A_\varepsilon(u)}$ and $s = g\chi_{A_\varepsilon(u)}$ where $f, g \in L_{\varphi,w}$ thus

$$\begin{aligned} \alpha h + \beta s &= \alpha(f\chi_{A_\varepsilon(u)}) + \beta(g\chi_{A_\varepsilon(u)}) \\ &= (\alpha f + \beta g)\chi_{A_\varepsilon(u)} \in L_{\varphi,w}(A_\varepsilon(u)), \end{aligned}$$

which means that $L_{\varphi,w}(A_\varepsilon(u))$ is a subspace of $L_{\varphi,w}$.

Next, for all $h \in L_{\varphi,w}(A_\varepsilon(u))$ we have

$$\begin{aligned} M_u h &= uh \\ &= u(f\chi_{A_\varepsilon(u)}) \\ &= (uf)\chi_{A_\varepsilon(u)}, \end{aligned}$$

where $uf \in L_{\varphi,w}$. Therefore $M_u \in L_{\varphi,w}(A_\varepsilon(u))$, which means that $L_{\varphi,w}(A_\varepsilon(u))$ is an invariant subspace of $L_{\varphi,w}$ under M_u.

Now, let us show that $L_{\varphi,w}(A_\varepsilon(u))$ is a closed set. Indeed, let g a function belonging to the closure of $L_{\varphi,w}(A_\varepsilon(u))$ then there exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $L_{\varphi,w}(A_\varepsilon(u))$ such that

$$g_n \to g \text{ in } L_{\varphi,w}$$

Just remain to exhibit that g belongs to $L_{\varphi,w}(A_\varepsilon(u))$. Note that

$$g = g\chi_{A_\varepsilon(u)} + g\chi_{A^c_\varepsilon(u)}.$$

Next, we want to show that $g\chi_{A^c_\varepsilon(u)} = 0$. In fact, given $\varepsilon_1 > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\|g\chi_{A^c_\varepsilon(u)}\|_{\varphi,w} &= \|(g - g_{n_0} + g_{n_0})\chi_{A^c_\varepsilon(u)}\|_{\varphi,w}\\ &= \|(g - g_{n_0})\chi_{A^c_\varepsilon(u)}\|_{\varphi,w}\\ &\leq \|g - g_{n_0}\|_{\varphi,w} < \varepsilon_1.\end{aligned}$$

Thus, $g\chi_{A^c_\varepsilon(u)} = 0$ which means that $g = g\chi_{A_\varepsilon(u)}$, that is, $g \in L_{\varphi,w}(A_\varepsilon(u))$. Finally by lemma 3.13 we have

$$M_u\,\big|_{L_{\varphi,w}(A_\varepsilon(u))}$$

is a compact operator. And the proof is now complete. □

3.15. Theorem. Let $M_u \in B(L_{\varphi,w})$. Then M_u is compact if and only if $L_{\varphi,w}(A_\varepsilon(u))$ is finite dimensional for each $\varepsilon > 0$.

Proof. If $|u(x)| \geq \varepsilon$, we should note that

$$|uf\chi_{A_\varepsilon}(x)| \geq \varepsilon f\chi_{A_\varepsilon(u)}(x)$$

and so

$$\{x : \varepsilon f\chi_{A_\varepsilon(u)}(x) > \lambda\} \subseteq \{x : \big|uf\chi_{A_\varepsilon(u)}(x)\big| > \lambda\},$$

thus

$$D_{\varepsilon f\chi_{A_\varepsilon(u)}}(\lambda) \leq D_{uf\chi_{A_\varepsilon(u)}}(\lambda),$$

then

$$\{\lambda > 0 : D_{uf\chi_{A_\varepsilon(u)}}(\lambda) \leq t\} \subseteq \{\lambda > 0 : D_{\varepsilon f\chi_{A_\varepsilon(u)}}(\lambda) \leq t\}$$

from this we have

$$\inf\{\lambda > 0 : D_{\varepsilon f\chi_{A_\varepsilon(u)}}(\lambda) \leq t\} \leq \inf\{\lambda > 0 : D_{uf\chi_{A_\varepsilon(u)}}(\lambda) \leq t\}$$

that is

$$(\varepsilon f\chi_{A_\varepsilon(u)})^*(t) \leq (uf\chi_{A_\varepsilon(u)})^*(t).$$

Hence

$$\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(uf\chi_{A_\varepsilon(u)})^*(t)}{k}\right) w(t)\,dt \leq 1\right\} \subseteq \left\{k > 0 : \int_0^\infty \varphi\left(\frac{\varepsilon(f\chi_{A_\varepsilon(u)})^*(t)}{k}\right) w(t)\,dt \leq 1\right\}.$$

Therefore

$$\inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{\varepsilon(f\chi_{A_\varepsilon(u)})^*(t)}{k}\right) w(t)\,dt \leq 1\right\} \leq \inf\left\{k > 0 : \int_0^\infty \varphi\left(\frac{(uf\chi_{A_\varepsilon(u)})^*(t)}{k}\right) w(t)\,dt \leq 1\right\}.$$

And hence

$$\|M_u f\chi_{A_\varepsilon(u)}\|_{\varphi,w} \geq \varepsilon\|f\chi_{A_\varepsilon(u)}\|_{\varphi,w}. \tag{3.6}$$

Now, if M_u is a compact operator, then $L_{\varphi,w}(A_\varepsilon(u))$ is a closed invariant subspace of $L_{\varphi,w}$ under M_u and by lemma 3.13

$$M_u\,\big|_{L_{\varphi,w}(A_\varepsilon(u))}$$

is a compact operator. Then by (3.6) $M_u\big|_{L_{\varphi,w}(A_\varepsilon(u))}$ has a closed range in $L_{\varphi,w}(A_\varepsilon(u))$ and it is invertible, being compact $L_{\varphi,w}(A_\varepsilon(u))$ is finite dimensional.

Conversely, suppose that $L_{\varphi,w}(A_\varepsilon(u))$ is finite dimensional for each $\varepsilon > 0$. In particular, for each n, $L_{\varphi,w}(A_{\frac{1}{u}}(u))$ is finite dimensional, then for each n, define $u_n : X \to \mathbb{C}$ as

$$u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \geq \frac{1}{n} \\ 0 & \text{if } |u(x)| < \frac{1}{n}. \end{cases}$$

Then we find that

$$((u_n - u) \cdot f)^*(t) \leq \|u_n - u\|_\infty f^*(t) \ \forall \ t > 0.$$

Consequently

$$\begin{aligned} \|M_{u_n} f - M_u f\|_{\varphi,w} &\leq \|u_n - u\|_\infty \|f\|_{\varphi,w} \\ &\leq \frac{1}{n} \|f\|_{\varphi,w}, \end{aligned}$$

which implies that M_{u_n} converges to M_u uniformly. As $L_{\varphi,w}(A_\varepsilon(u))$ is finite dimensional so M_{u_n} is a finite rank operator. Therefore, M_{u_n} is a compact operator and hence M_u is a compact operator. □

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