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A class of Hartley-Ross type unbiased estimators for population mean using ranked set sampling

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Abstract

In this paper, we propose a class of Hartley-Ross type unbiased estimators for estimating the finite population mean of the study variable under ranked set sampling (RSS), when population mean of the auxiliary variable is known. The variances of the proposed class of unbiased estimators are obtained to first degree of approximation. Both theoretically and numerically, the proposed estimators are compared with some competitor estimators, using three different data sets. It is identified through numerical study that the proposed estimators are more efficient as compared to all other competitor estimators.

Keywords: Ranked set sampling, auxiliary variable, variance.

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1. Introduction

In applications, there might be a situation when the variable of interest cannot be easily measured or is very expensive to do so, but it can be ranked easily at no cost or at very little cost. In view of this, Mclntyre [5] was the first who proposed the concept of ranked set sampling (RSS) in the context of obtaining reliable farm yield estimates based on sampling of pastures and crop yield. He provided a clear and insightful introduction to the basic framework of RSS and laid out the rationale for how it can be lead to improved estimation relative to simple random sampling (SRS). Takahasi and Wakimoto [11] have provided the necessary mathematical theory of RSS and showed that the sample mean under RSS is an unbiased estimator of the finite population mean and more precise than the sample mean estimator under SRS.

The auxiliary information plays an important role in increasing efficiency of the estimator. Samawi and Muttlak [6] have suggested an estimator for population ratio in RSS and showed that it has less variance as compared to usual ratio estimator in SRS.

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In RSS, perfect ranking of elements was considered by Mclntyre [5] and Takahasi and Wakimoto [11] for estimation of population mean. In some situations, ranking may not be perfect. Dell and Clutter [2] have studied the case in which there are errors in ranking. They pointed out that a loss in efficiency would be caused by the errors in ranking. The sample mean in RSS is an unbiased estimator of the population mean regardless of errors in ranking of the elements. To reduce the error in ranking, several modifications of the RSS method had been suggested. Stokes [10] has proposed use of the concomitant variable to aid in the ranking process to obtain ranked set data. She has also studied the ranked set sample approach for making inferences about the population variance and correlation coefficient. Here, the ranking of elements was done on basis of the auxiliary variable instead of judgment. Singh et al. [7] have proposed an estimator for population mean and ranking of the elements is observed on basis of the auxiliary variable. Singh et al.[9] have also proposed the ratio and the product type estimators for population mean under stratified ranked set sampling (SRSS).

Hartley and Ross [3] proposed an unbiased ratio estimators for finite population mean in SRS. Motivated by Singh et al. [8], we suggest a class of Hartley-Ross type unbiased estimators based on RSS for population mean, using some known population parameters of the auxiliary variable. It is shown that the proposed estimators outperform as compared to some existing estimators in RSS.

2. RSS procedure

To create ranked sets, we must partition the selected first phase sample into sets of equal size. In order to plan RSS design, we must therefore choose a set of size m that is typically small, around three or four, to minimize ranking error. Here m is the number of sample units allocated to each set. The RSS procedure can be summarized as follows:

- **Step 1:**Randomly select m^2 bivariate sample units from the population.
- **Step 2:**Allocate m^2 selected units randomly as possible into m sets, each of size m.
- **Step 3:**Each sample is ranked with respect to one of the variables Y or X. Here, we assume that the perfect ranking is done on basis of the auxiliary variable X while the ranking of Y is with error.
- **Step 4:**An actual measurement from the first sample is then taken of the unit with the smallest rank of X, together with variable Y associated with smallest rank of X. From second sample of size m, the variable Y associated with the second smallest rank of X is measured. The process is continued until from the mth sample, the Y associated with the highest rank of X is measured.
- **Step 5:**Repeat Steps 1 through 4 for r cycles until the desired sample size $n = mr$, is obtained for analysis.

As an illustration, we select a sample of size 36 from a population by simple random sampling with replacement (SRSWR). These data are grouped into 3 sets each of size 3 and we repeat this process 4 times. According to RSS methodology, we order the X values from smaller to larger and assume that there is no judgment error in this ordering. Then, the smallest unit is selected from the first ordered set, the second smallest unit is selected from the second ordered set and so on. Similarly from the third ordered set, the third smallest unit is selected. By this way, we select $n = mr = 12$ observations. A ranked set sample design with set size $m = 3$ and number of sampling cycles $r = 4$ is illustrated in Figure 1. Although 36 sample units have been selected from the population, only the 12 circled units are actually included in the final sample for quantitative analysis.

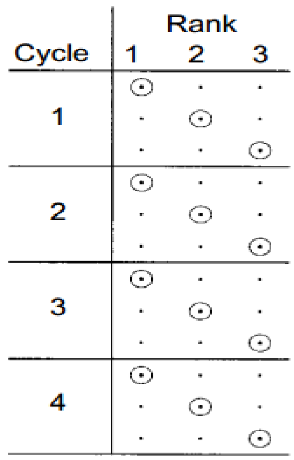

Figure 1. Illustration of ranked set sampling.

3. Symbols and Notations

We consider a situation when rank the elements on the auxiliary variable. Let $(y_{[i]j}, x_{(i)j})$ be the ith judgment ordering in the ith set for the study variable Y based on the ith order of the ith set of the auxiliary variable X in the jth cycle. To obtain bias and variance of the estimators, we define:
$\bar{y}_{[i]} = \bar{Y}(1+e_0)$, $\bar{x}_{(i)} = \bar{X}(1+e_1)$, $\bar{r}_{(i)} = \bar{R}(1+e_2)$, $\bar{x}^*_{(i)} = \bar{X}^*(1+e_1^*)$,
$\bar{r}^*_{(i)} = \bar{R}^*(1+e_2^*)$, such that
$E(e_i) = 0$, i=0,1,2. $E(e_i^*) = 0$, i=1,2.
and
$E(e_0^2) = \gamma C_y^2 - W_{y[i]}^2$, $E(e_1^2) = \gamma C_x^2 - W_{x(i)}^2$, $E(e_0 e_1) = \gamma C_{yx} - W_{yx(i)}$,
$E(e_1^{*2}) = \gamma C_{x^*}^2 - W_{x^*(i)}^2$, $E(e_0 e_1^*) = \gamma C_{yx^*} - W_{yx^*(i)}$, $E(e_1^* e_2^*) = \gamma C_{r^*x^*} - W_{r^*x^*(i)}$,
where

$$W_{yx(i)} = \frac{1}{m^2 r \bar{X}\bar{Y}} \sum_{i=1}^{m} \tau_{yx(i)},\ W_{x(i)}^2 = \frac{1}{m^2 r \bar{X}^2} \sum_{i=1}^{m} \tau_{x(i)}^2,\ W_{y[i]}^2 = \frac{1}{m^2 r \bar{Y}^2} \sum_{i=1}^{m} \tau_{y[i]}^2,$$

$W_{yx^*(i)} = \frac{1}{m^2 r \bar{X}^* \bar{Y}} \sum_{i=1}^{m} \tau_{yx^*(i)}$, $W_{x^*(i)}^2 = \frac{1}{m^2 r \bar{X}^{*2}} \sum_{i=1}^{m} \tau_{x^*(i)}^2$,
$W_{r^*x^*(i)} = \frac{1}{m^2 r \bar{X}^* \bar{R}^*} \sum_{i=1}^{m} \tau_{r^*x^*(i)}$,

$$\tau_{x(i)} = (\mu_{x(i)} - \bar{X}),\ \tau_{y[i]} = (\mu_{y([i]} - \bar{Y}),\ \tau_{yx(i)} = (\mu_{y[i]} - \bar{Y})(\mu_{x(i)} - \bar{X}),$$

$\tau_{x^*(i)} = (\mu_{x^*(i)} - \bar{X}^*)$, $\tau_{yx^*(i)} = (\mu_{y[i]} - \bar{Y})(\mu_{x^*(i)} - \bar{X}^*)$, $\tau_{r^*x^*(i)} = (\mu_{r^*(i)} - \bar{R}^*)(\mu_{x^*(i)} - \bar{X}^*)$.
Here $\gamma = (\frac{1}{mr})$ and $C_{yx} = \rho C_y C_x$, where C_y and C_x are the coefficients of variation of Y and X respectively. Also $\bar{Y}$ and $\bar{X}$ are the population means of Y and X respectively. The values of $\mu_{y[i]}$ and $\mu_{x(i)}$ depend on order statistics from some specific distributions (see Arnold et al.[1]).
The following notations will be used through out this paper.

$\bar{y}_{[i]} = (1/n)\sum_{j=1}^{n} y_{[i]j}$, $\bar{x}_{(i)} = (1/n)\sum_{j=1}^{n} x_{(i)j}$, $\bar{r}_{(i)} = \frac{\sum_{j=1}^{n} r_{(i)j}}{n}$, $r_{(i)j} = \frac{y_{[i]j}}{x_{(i)j}}$,
$\bar{R} = E(\bar{r}_{(i)})$, $\bar{r}^*_{(i)} = \frac{\sum_{j=1}^{n} r^*_{(i)j}}{n}$, $r^*_{(i)j} = \frac{y_{[i]j}}{x^*_{(i)j}}$, $x^*_{(i)j} = (ax_{(i)j} + b)$,
$\bar{x}^*_{(i)} = (a\bar{x}_{(i)} + b)$, $\bar{X}^* = (a\bar{X} + b)$, $\bar{R}^* = E(\bar{r}^*_{(i)})$, $\bar{r}'_{(i)} = \frac{\sum_{j=1}^{n} r'_{(i)j}}{n}$,
$r'_{(i)j} = \frac{y_{[i]j}}{x'_{(i)j}}$, $x'_{(i)j} = (x_{(i)j}C_x + \rho)$, $\bar{x}'_{(i)} = (\bar{x}_{(i)}C_x + \rho)$, $\bar{X}' = (\bar{X}C_x + \rho)$,
$\bar{R}' = E(\bar{r}'_{(i)})$, $\bar{r}''_{(i)} = \frac{\sum_{j=1}^{n} r''_{(i)j}}{n}$, $r''_{(i)j} = \frac{y_{[i]j}}{x''_{(i)j}}$, $x''_{(i)j} = (x_{(i)j}\beta_2(x) + C_x)$,
$\bar{x}''_{(i)} = (\bar{x}_{(i)}\beta_2(x) + C_x)$, $\bar{X}'' = (\bar{X}\beta_2(x) + C_x)$, $\bar{R}'' = E(\bar{r}''_{(i)})$, $\bar{r}'''_{(i)} = \frac{\sum_{j=1}^{n} r'''_{(i)j}}{n}$,
$r'''_{(i)j} = \frac{y_{[i]j}}{x'''_{(i)j}}$, $x'''_{(i)j} = (x_{(i)j}C_x + \beta_2(x))$, $\bar{x}'''_{(i)} = (\bar{x}_{(i)}C_x + \beta_2(x))$,
$\bar{X}''' = (\bar{X}C_x + \beta_2(x))$ and $\bar{R}''' = E(\bar{r}'''_{(i)})$,
where a and b are known population parameters, which can be coefficient of variation, coefficient of skewness and coefficient of kurtosis and the coefficient of correlation of the auxiliary variable.

Following Singh [8], the variance of the Hartley-Ross type unbiased estimator based on Upadhyaya and Singh [12] estimator in SRS, is given by

$$V(\bar{y}^{(u)}_{US2(SRS)}) \cong \gamma\left(\bar{Y}^2C_y^2 + \bar{X}'''^2\bar{R}'''^2C^2_{x'''} - 2\bar{R}'''\bar{Y}\bar{X}'''C_{yx'''}\right). \tag{3.1}$$

Under RSS scheme, the variance of $\bar{y}_{RSS} = \bar{y}_{[i]} = (1/n)\sum_{j=1}^{n} y_{[i]j}$, is given by

$$V(\bar{y}_{RSS}) = \bar{Y}^2\left(\gamma C_y^2 - W^2_{y[i]}\right). \tag{3.2}$$

4. Proposed Hartley-Ross unbiased estimator in RSS

Following Singh et al. [8], we consider the following ratio estimator:

$$\bar{y}_{H(RSS)} = \bar{r}_{(i)}\bar{X}. \tag{4.1}$$

The bias of $\bar{y}_{H(RSS)}$, is given by

$$B(\bar{y}_{H(RSS)}) = -\frac{(N-1)}{N}S_{r_{(i)}x_{(i)}},$$

where $S_{r_{(i)}x_{(i)}} = \frac{1}{N}\sum_{j=1}^{N}(r_{(i)j} - \bar{R})(x_{(i)j} - \bar{X})$ and an unbiased estimator of $S_{r_{(i)}x_{(i)}}$ is given by

$$\begin{aligned} s_{r_{(i)}x_{(i)}} &= \frac{1}{n-1}\sum_{j=1}^{n}(r_{(i)j} - \bar{r}_{(i)})(x_{(i)j} - \bar{x}_{(i)}) \\ &= \frac{n}{n-1}(\bar{y}_{[i]} - \bar{r}_{(i)}\bar{x}_{(i)}). \end{aligned}$$

So bias of $\bar{y}_{H(RSS)}$ becomes

$$B(\bar{y}_{H(RSS)}) = -\frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}_{(i)}\bar{x}_{(i)}). \tag{4.2}$$

Thus an unbiased Hartley-Ross type estimator of population mean based on RSS is given by

$$\bar{y}^{(u)}_{H(RSS)} = \bar{r}_{(i)}\bar{X} + \frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}_{(i)}\bar{x}_{(i)}). \tag{4.3}$$

In terms of $e's$, we have

$$\bar{y}^{(u)}_{H(RSS)} = \bar{X}\bar{R}(1+e_2) + \frac{n(N-1)}{N(n-1)}\left[\bar{Y}(1+e_0) - \bar{X}\bar{R}(1+e_1)(1+e_2)\right].$$

Under the assumption $\frac{n(N-1)}{N(n-1)} \cong 1$, we can write

$$(\bar{y}^{(u)}_{H(RSS)} - \bar{Y}) \cong (\bar{Y}e_0 - \bar{X}\bar{R}e_1).$$

Taking square and then expectation, the variance of $\bar{y}^{(u)}_{H(RSS)}$, is given by

$$V(\bar{y}^{(u)}_{H(RSS)}) \cong \bar{Y}^2(\gamma C_y^2 - W^2_{y[i]}) + \bar{X}^2\bar{R}^2(\gamma C_x^2 - W^2_{x(i)}) - 2\bar{R}\bar{Y}\bar{X}(\gamma C_{yx} - W_{yx(i)}). \tag{4.4}$$

5. Proposed class of Hartley-Ross type unbiased estimators in RSS

Consider the following ratio estimator:

$$\bar{y}_{P(RSS)} = \bar{r}^*_{(i)}\bar{X}^*. \tag{5.1}$$

The bias of $\bar{y}_{P(RSS)}$, is given by

$$B(\bar{y}_{P(RSS)}) = -\frac{(N-1)}{N}S_{r^*_{(i)}x^*_{(i)}},$$

where $S_{r^*_{(i)}x^*_{(i)}} = \frac{1}{N}\sum_{j=1}^{N}(r^*_{(i)j} - \bar{R}^*)(x^*_{(i)j} - \bar{X}^*)$ and an unbiased estimator of $S_{r^*_{(i)}x^*_{(i)}}$ is given by

$$\begin{aligned} s_{r^*_{(i)}x^*_{(i)}} =& \frac{1}{n-1}\sum_{j=1}^{n}(r^*_{(i)j} - \bar{r}^*_{(i)})(x^*_{(i)j} - \bar{x}^*_{(i)}) \\ =& \frac{n}{n-1}(\bar{y}_{[i]} - \bar{r}^*_{(i)}\bar{x}^*_{(i)}). \end{aligned}$$

We give the following theorem.

5.1. Theorem. *An unbiased estimator of* $S_{r^*_{(i)}x^*_{(i)}} = \frac{1}{N}\sum_{j=1}^{N}(r^*_{(i)j} - \bar{R}^*)(x^*_{(i)j} - \bar{X}^*)$ *is given by*
$s_{r^*_{(i)}x^*_{(i)}} = \frac{1}{n-1}\sum_{j=1}^{n}(r^*_{(i)j} - \bar{r}^*_{(i)})(x^*_{(i)j} - \bar{x}^*_{(i)})$.

Proof. We have to prove that $E(s_{r^*_{(i)}x^*_{(i)}}) = S_{r^*_{(i)}x^*_{(i)}}$. Here for fixed i, $j = 1, 2, ..., n$, $r^*_{(i)j}$ and $x^*_{(i)j}$ are simple random samples of size n.

$$\begin{aligned} E(s_{r^*_{(i)}x^*_{(i)}}) =& E\left[\frac{1}{n-1}\sum_{j=1}^{n}(r^*_{(i)j} - \bar{r}^*_{(i)})(x^*_{(i)j} - \bar{x}^*_{(i)})\right], \\ =& \frac{1}{n-1}E\left[\sum_{j=1}^{n} r^*_{(i)j}x^*_{(i)j} - n\bar{r}^*_{(i)}\bar{x}^*_{(i)}\right], \\ =& \frac{1}{n-1}\left[\sum_{j=1}^{n} E(r^*_{(i)j}x^*_{(i)j}) - nE(\bar{r}^*_{(i)}\bar{x}^*_{(i)})\right], \\ =& \frac{1}{n-1}\left[\frac{n}{N}\sum_{j=1}^{N} r^*_{(i)j}x^*_{(i)j} - n\left(Cov(\bar{r}^*_{(i)}, \bar{x}^*_{(i)}) + \bar{R}^*\bar{X}^*\right)\right], \\ =& \frac{n}{n-1}\left[\frac{1}{N}\sum_{j=1}^{N} r^*_{(i)j}x^*_{(i)j} - \bar{R}^*\bar{X}^* - \frac{S_{r^*_{(i)}x^*_{(i)}}}{n}\right], \\ =& \frac{n}{n-1}\left(S_{r^*_{(i)}x^*_{(i)}} - \frac{S_{r^*_{(i)}x^*_{(i)}}}{n}\right), \\ =& S_{r^*_{(i)}x^*_{(i)}}. \end{aligned}$$

□

So bias of $\bar{y}_{P(RSS)}$ becomes

$$B(\bar{y}_{KP(RSS)}) = -\frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}^*_{(i)}\bar{x}^*_{(i)}). \tag{5.2}$$

Thus an unbiased class of Hartley-Ross type estimators of population mean based on RSS is given by

$$\bar{y}^{(u)}_{P(RSS)} = \bar{r}^*_{(i)}\bar{X}^* + \frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}^*_{(i)}\bar{x}^*_{(i)}). \tag{5.3}$$

In terms of $e's$, we have

$$\bar{y}^{(u)}_{P(RSS)} = \bar{X}^*\bar{R}^*(1+e^*_2) + \frac{n(N-1)}{N(n-1)}\left(\bar{Y}(1+e_0) - \bar{X}^*\bar{R}^*(1+e^*_1)(1+e^*_2)\right).$$

Under the assumption $\frac{n(N-1)}{N(n-1)} \cong 1$, we have

$$(\bar{y}^{(u)}_{P(RSS)} - \bar{Y}) \cong (\bar{Y}e_0 - \bar{X}^*\bar{R}^*e^*_1).$$

Taking square and then expectation, the variance of $\bar{y}^{(u)}_{P(RSS)}$, is given by

$$\begin{aligned} V(\bar{y}^{(u)}_{P(RSS)}) \cong& \bar{Y}^2(\gamma C^2_y - W^2_{y[i]}) + \bar{X}^{*2}\bar{R}^{*2}(\gamma C^2_{x^*} - W^2_{x^*(i)}) \\ &- 2\bar{R}^*\bar{Y}\bar{X}^*(\gamma C_{yx^*} - W_{yx^*(i)}). \end{aligned} \tag{5.4}$$

Note: (i). If $a = C_x$ and $b = \rho$, then from Equation (5.3), we get the Hartley-Ross type unbiased estimator based on Kadilar and Cingi [4] estimator $\bar{y}^{(u)}_{KC(RSS)}$, as:

$$\bar{y}^{(u)}_{KC(RSS)} = \bar{r}^{'}_{(i)}\bar{X}^{'} + \frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}^{'}_{(i)}\bar{x}^{'}_{(i)}). \tag{5.5}$$

The variance of $\bar{y}_{KC(RSS)}$, is given by

$$\begin{aligned} V(\bar{y}^{(u)}_{KC(RSS)}) \cong& \bar{Y}^2(\gamma C^2_y - W^2_{y[i]}) + \bar{X}^{'2}\bar{R}^{'2}(\gamma C^2_{x^{'}} - W^2_{x^{'}(i)}) \\ &- 2\bar{R}^{'}\bar{Y}\bar{X}^{'}(\gamma C_{yx^{'}} - W_{yx^{'}(i)}). \end{aligned} \tag{5.6}$$

(ii). If $a = \beta_2(x)$ and $b = C_x$, then Equation (5.3) becomes the Hartley-Ross type unbiased estimator based on Upadhyaya and Singh [12] estimator $\bar{y}^{(u)}_{US1(RSS)}$ and is given by

$$\bar{y}^{(u)}_{US1(RSS)} = \bar{r}^{''}_{(i)}\bar{X}^{''} + \frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}^{''}_{(i)}\bar{x}^{''}_{(i)}). \tag{5.7}$$

The variance of $\bar{y}_{US1(RSS)}$, is given by

$$\begin{aligned} V(\bar{y}^{(u)}_{US1(RSS)}) \cong& \bar{Y}^2(\gamma C^2_y - W^2_{y[i]}) + \bar{X}^{''2}\bar{R}^{''2}(\gamma C^2_{x^{''}} - W^2_{x^{''}(i)}) \\ &- 2\bar{R}^{''}\bar{Y}\bar{X}^{''}(\gamma C_{yx^{''}} - W_{yx^{''}(i)}). \end{aligned} \tag{5.8}$$

(iii). If $a = C_x$ and $b = \beta_2(x)$, then Equation (5.3) becomes the Hartley-Ross type unbiased estimator based on Upadhyaya and Singh [12] estimator $\bar{y}^{(u)}_{US2(RSS)}$ and is given by

$$\bar{y}^{(u)}_{US2(RSS)} = \bar{r}^{'''}_{(i)}\bar{X}^{'''} + \frac{n(N-1)}{N(n-1)}(\bar{y}_{[i]} - \bar{r}^{'''}_{(i)}\bar{x}^{'''}_{(i)}). \tag{5.9}$$

The variance of $\bar{y}_{US2(RSS)}$, is given by

$$\begin{aligned} V(\bar{y}^{(u)}_{US2(RSS)}) \cong& \bar{Y}^2(\gamma C^2_y - W^2_{y[i]}) + \bar{X}^{'''2}\bar{R}^{'''2}(\gamma C^2_{x^{'''}} - W^2_{x^{'''}(i)}) \\ &- 2\bar{R}^{'''}\bar{Y}\bar{X}^{'''}(\gamma C_{yx^{'''}} - W_{yx^{'''}(i)}). \end{aligned} \tag{5.10}$$

6. Efficiency comparison

The proposed estimator $\bar{y}^{(u)}_{US2(RSS)}$ is more efficient than $\bar{y}^{(u)}_{US2(SRS)}$, $\bar{y}_{(RSS)}$, $\bar{y}^{(u)}_{H(RSS)}$, $\bar{y}^{(u)}_{KC(RSS)}$ and $\bar{y}^{(u)}_{US1(RSS)}$ respectively if the following conditions hold:

(i). $-(\bar{Y}W_{y[i]} - \bar{X}^{'''}\bar{R}^{'''}W_{x'''(i)})^2 < 0$

(ii). $\bar{X}^{'''}\bar{R}^{'''}(\gamma C^2_{x'''} - W^2_{x'''(i)}) - 2\bar{Y}(\gamma C_{yx'''} - W_{yx'''(i)}) < 0$

(iii). $\bar{X}^{'''2}\bar{R}^{'''2}(\gamma C^2_{x'''} - W^2_{x'''(i)}) - 2\bar{X}^{'''}\bar{R}^{'''}\bar{Y}(\gamma C_{yx'''} - W_{yx'''(i)}) - \bar{X}^2\bar{R}^2(\gamma C^2_x - W^2_{x(i)}) + 2\bar{R}\bar{X}\bar{Y}(\gamma C_{yx} - W_{yx(i)}) < 0$

(iv). $\bar{X}^{'''2}\bar{R}^{'''2}(\gamma C^2_{x'''} - W^2_{x'''(i)}) - 2\bar{X}^{'''}\bar{R}^{'''}\bar{Y}(\gamma C_{yx'''} - W_{yx'''(i)}) - \bar{X}^{'2}\bar{R}^{'2}(\gamma C^2_{x'} - W^2_{x'(i)}) + 2\bar{R}^{'}\bar{X}^{'}\bar{Y}(\gamma C_{yx'} - W_{yx'(i)}) < 0.$

(v). $\bar{X}^{'''2}\bar{R}^{'''2}(\gamma C^2_{x'''} - W^2_{x'''(i)}) - 2\bar{X}^{'''}\bar{R}^{'''}\bar{Y}(\gamma C_{yx'''} - W_{yx'''(i)}) - \bar{X}^{''2}\bar{R}^{''2}(\gamma C^2_{x''} - W^2_{x''(i)}) + 2\bar{R}^{''}\bar{X}^{''}\bar{Y}(\gamma C_{yx''} - W_{yx''(i)}) < 0.$

7. Numerical Illustration

To observe performances of the estimators, we use the following three data sets. The descriptions of these populations are given below.

Population I [source: Valliant et al.[13]]
The summary statistics are:
y : Breast cancer mortality in 1950-1969,
x : Adult female population in 1960.

$N = 301,$	$n = 12,$	$m = 3,$	$r = 4,$
$\bar{X} = 11288.1800,$	$\bar{Y} = 39.8500,$	$\rho = 0.9671,$	$\beta_2(x) = 10.79,$
$\bar{R} = 0.0039,$	$\bar{R}^{'} = 0.0032,$	$\bar{R}^{''} = 0.00036,$	$\bar{R}^{'''} = 0.0032,$
$\bar{X}^{'} = 13780.84,$	$\bar{X}^{''} = 121852.40,$	$\bar{X}^{'''} = 13290.67,$	$C_y = 1.2794,$
$C_x = 1.2207,$	$C_{x'} = 1.2206,$	$C_{x''} = 1.2207,$	$C_{x'''} = 1.2198,$
$C_{yx} = 1.5105,$	$C_{yx'} = 1.5104,$	$C_{yx''} = 1.5104,$	$C_{yx'''} = 1.5093,$
$W^2_{y[i]} = 0.014502,$	$W^2_{x(i)} = 0.002234,$	$W_{yx(i)} = 0.022478,$	$W^2_{x'(i)} = 0.002234,$
$W_{yx'(i)} = 0.022476,$	$W^2_{x''(i)} = 0.002234,$	$W_{yx''(i)} = 0.022478,$	$W^2_{x'''(i)} = 0.002231,$
$W_{yx'''(i)} = 0.022461.$			

Population II [source: Valliant et al. [13]]
The summary statistics are:
y : Number of patients discharged,
x : Number of beds.

Population III [source: Valliant et al. [13]]
The summary statistics are:
y : Population, excluding residents of group quarters in 1960,

$N = 393,$	$n = 15,$	$m = 3,$	$r = 5,$
$\bar{X} = 274.70,$	$\bar{Y} = 814.65,$	$\rho = 0.9105,$	$\beta_2(x) = 3.5670,$
$\bar{R} = 3.1548,$	$\bar{R}^{'} = 3.6842,$	$\bar{R}^{''} = 0.9286,$	$\bar{R}^{'''} = 3.5520,$
$\bar{X}^{'} = 214.13,$	$\bar{X}^{''} = 980.63,$	$\bar{X}^{'''} = 216.78,$	$C_y = 0.7239,$
$C_x = 0.7762,$	$C_{x'} = 0.7729,$	$C_{x''} = 0.7756,$	$C_{x'''} = 0.7634,$
$C_{yx} = 0.5116,$	$C_{yx'} = 0.5094,$	$C_{yx''} = 0.5112,$	$C_{yx'''} = 0.5031,$
$W^2_{y[i]} = .016234,$	$W^2_{x(i)} = 0.003354,$	$W_{yx(i)} = 0.041280,$	$W^2_{x'(i)} = 0.003353,$
$W_{yx'(i)} = 0.041277,$	$W^2_{x''(i)} = 0.003354,$	$W_{yx''(i)} = 0.041279,$	$W^2_{x'''(i)} = 0.003348,$
$W_{yx'''(i)} = 0.04148.$			

x : Number of households in 1960.

$N = 304,$	$n = 12,$	$m = 3,$	$r = 4,$
$\bar{X} = 8931.17,$	$\bar{Y} = 32916.19,$	$\rho = 0.9979,$	$\beta_2(x) = 14.6079,$
$\bar{R} = 3.7993,$	$\bar{R}^{'} = 2.9703,$	$\bar{R}^{''} = 0.2589,$	$\bar{R}^{'''} = 2.9580,$
$\bar{X}^{'} = 11627.52,$	$\bar{X}^{''} = 130466.90,$	$\bar{X}^{'''} = 11641.13,$	$C_y = 1.2390,$
$C_x = 1.3018,$	$C_{x'} = 1.3017,$	$C_{x''} = 1.3018,$	$C_{x'''} = 1.3002.98,$
$C_{yx} = 1.6096,$	$C_{yx'} = 1.6094,$	$C_{yx''} = 1.6095,$	$C_{yx'''} = 1.6075,$
$W^2_{y[i]} = .006744,$	$W^2_{x(i)} = 0.005193,$	$W_{yx(i)} = 0.023651,$	$W^2_{x'(i)} = 0.005192,$
$W_{yx'(i)} = 0.023649,$	$W^2_{x''(i)} = 0.005193,$	$W_{yx''(i)} = 0.023652,$	$W^2_{x'''(i)} = 0.005179,$
$W_{yx'''(i)} = 0.023622.$			

Table 1. Comparison values

Population	$V(\bar{y}^{(u)}_{US2(RSS)}) < V(\bar{y}^{(u)}_{US2(SRS)})$	$V(\bar{y}^{(u)}_{US2(RSS)}) < V(\bar{y}_{RSS})$	$V(\bar{y}^{(u)}_{US2(RSS)}) < V(\bar{y}^{(u)}_{H(RSS)})$	$V(\bar{y}^{(u)}_{US2(RSS)}) < V(\bar{y}^{(u)}_{KC(RSS)})$	$V(\bar{y}^{(u)}_{US2(RSS)}) < V(\bar{y}^{(u)}_{US1(RSS)})$
I	$-7.7800 < 0$	$-3.0556 < 0$	$-3.5006 < 0$	$-4.2868 < 0$	$-3.4670 < 0$
II	$-3509.73 < 0$	$-0.55052 < 0$	$-0.43858 < 0$	$-0.39893 < 0$	$-0.43292 < 0$
III	$-50649.03 < 0$	$-2773.45 < 0$	$-199146.50 < 0$	$-185356.01 < 0$	$-197541.90 < 0$

We investigate the percent relative efficiency (PRE) of Hartley-Ross unbiased estimator $\bar{y}^{(u)}_{H(RSS)} = \hat{\theta}_1$ (say), Hartley-Ross type unbiased estimator based on Kadilar and Cingi [4] estimator $\bar{y}^{(u)}_{KC(RSS)} = \hat{\theta}_2$, Hartley-Ross type unbiased estimator based on Upadhyaya and Singh [12] estimator $\bar{y}^{(u)}_{US1(RSS)} = \hat{\theta}_3$ and $\bar{y}^{(u)}_{US2(RSS)} = \hat{\theta}_4$ with respect to conventional estimator $\bar{y}_{RSS} = \hat{\theta}_0$ (say) .

The PRE of proposed estimators $\hat{\theta}_j, \quad j = 1, 2, 3, 4,$ with respect to conventional estimator $\bar{y}_{RSS} = \hat{\theta}_0$, is defined as:

$$PRE(\hat{\theta}_0, \hat{\theta}_j) = \frac{V(\hat{\theta}_0)}{V(\hat{\theta}_j)} \times 100, \quad j = 1, 2, 3, 4. \tag{7.1}$$

The $PRE's$ of our proposed estimators and other existing estimators for Populations I, II and III are given in Tables 2, 3 and 4 respectively.

Table 2. $PRE's$ of various estimators for Population I.

m	r	n	$\bar{y}_{RSS}$	$\bar{y}^{(u)}_{H(RSS)}$	$\bar{y}^{(u)}_{KC(RSS)}$	$\bar{y}^{(u)}_{US1(RSS)}$	$\bar{y}^{(u)}_{US2(RSS)}$
3	3	9	100	178.37	178.40	178.38	**178.74**
	4	12	100	354.74	354.77	354.75	**354.92**
	5	15	100	326.14	326.23	326.22	**326.96**
4	3	12	100	397.23	397.30	397.25	**397.80**
	4	16	100	119.37	119.40	119.38	**119.56**
	5	20	100	114.70	114.75	114.74	**114.86**
5	3	15	100	217.06	217.10	217.09	**217.30**
	4	20	100	108.68	108.71	108.70	**108.85**
	5	25	100	177.16	177.20	177.18	**177.50**
	10	50	100	355.90	355.98	355.94	**356.77**

Table 3. $PRE's$ of various estimators for Population II.

m	r	n	$\bar{y}_{RSS}$	$\bar{y}^{(u)}_{H(RSS)}$	$\bar{y}^{(u)}_{KC(RSS)}$	$\bar{y}^{(u)}_{US1(RSS)}$	$\bar{y}^{(u)}_{US2(RSS)}$
3	3	9	100	199.15	201.21	199.54	**207.09**
	4	12	100	147.75	149.47	148.08	**154.18**
	5	15	100	119.02	119.06	119.04	**119.45**
4	3	12	100	259.28	259.33	259.29	**259.86**
	4	16	100	177.56	177.60	177.57	**177.96**
	5	20	100	141.97	142.01	141.98	**142.78**
5	3	15	100	111.53	111.56	111.54	**112.72**
	4	20	100	138.47	138.50	138.48	**139.75**
	5	25	100	167.65	167.69	167.67	**168.08**
	10	50	100	260.15	260.20	260.17	**260.66**

Table 4. $PRE's$ of various estimators for Population III.

m	r	n	$\bar{y}_{RSS}$	$\bar{y}^{(u)}_{H(RSS)}$	$\bar{y}^{(u)}_{KC(RSS)}$	$\bar{y}^{(u)}_{US1(RSS)}$	$\bar{y}^{(u)}_{US2(RSS)}$
3	3	9	100	158.03	158.08	158.04	**158.66**
	4	12	100	330.60	330.71	330.61	**332.27**
	5	15	100	288.63	288.68	288.64	**289.37**
4	3	12	100	194.50	194.56	194.51	**195.34**
	4	16	100	116.76	116.82	116.78	**117.51**
	5	20	100	322.73	322.84	322.75	**324.23**
5	3	15	100	146.69	146.73	146.70	**147.21**
	4	20	100	122.19	122.23	122.20	**122.71**
	5	25	100	124.24	124.28	124.26	**124.76**
	10	50	100	215.39	215.46	215.40	**216.32**

From Tables 2, 3 and 4, we see that the proposed Hartley-Ross type unbiased estimators are more efficient than usual conventional estimator in RSS. Thus, if population coefficient of variation, population coefficient of kurtosis and population correlation coefficient are known in advance, then our proposed estimators can be used in practice.

8. Conclusion

Table 1 has established the conditions obtained in Section 6 numerically. It is shown that all conditions are satisfied for all considered populations. On the basis of results given in Tables 2, 3 and 4, we conclude that the proposed class of Hartley-Ross type unbiased estimators are preferable over its competitive estimators under RSS. It is also observed that the proposed unbiased estimator $\bar{y}^{(u)}_{US2(RSS)}$ has highest PRE in comparison to all other considered estimators in all three populations.

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