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Characterization and Extendability of P_k Sets For $k \equiv 3(4)$

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ABSTRACT

In this paper the characterization of certain families of the P_k sets for $k \equiv 3(4)$ are given, and it is shown that some of them can not be extended.

Key Words: Diophantine Equation, Congruence, Legendre Symbol

1. INTRODUCTION

From its firs Let k be a non zero integer and X be a set of distinct positive integers. X is said to be a P_k set if any two distinct positive integers x_i and x_j of X, the integer $x_i.x_j + k$ is a perfect square. A P_k set X can be extended if there exists a positive integer $y \notin X$ such that $X \cup \{y\}$ is still a P_k set.

For simplicity, throughout this paper, $x \equiv y(n)$ will denote $x \equiv y(modn)$.

The problem of extending P_k sets is an old one dating from the time of Diophantus[1]. The most famous result in this area is due to Baker and Davenport[2], who proved that the P_1 set $\{1,3,8,120\}$ can not be extended. Recently the problem of extendibility of the P_k sets have been examined by Kanagasabapathy and Ponnudurai[3], Heichelheim[4], Thamotherampillai[5], Mohanty and Ramasamy[6],[7], Brown[8], Altindis[9], Dujella [10], Dujella and Luca[11] and Dujella and Ramasamy [12].

The purpose of this paper is to characterize certain families of the P_k sets for $k \equiv 3(4)$ and to show that some of them can not be extended.

2. CHARACTERIZATION OF P_k SETS FOR $k \equiv 3(4)$

THEOREM 1. If X is a P_k set for $k \equiv 3(4)$, then all of the elements of X either are odd and they are congruent to one another or at most one of them is even and it is congruent to 2 modulo 4.

PROOF. Let $X = \{x_1, x_2, x_3, x_4\}$ be a P_k set with $k \equiv 3(4)$. Then by the definition of P_k set we have

$$x_i.x_j + k = c^2$$

for some integers x_i, x_j and c with $i \neq j$. From the fact that perfect squares are congruent to 0 or 1 modulo 4, we have

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$x_i.x_j + k \equiv 0 \; or \; 1(4)$

so that

$x_i.x_j \equiv 1 \; or \; 2(4)$

this shows that the product of any two elements x_i, x_j of a P_k set is congruent to 1 or 2 modulo 4. Indeed;

a). Let $x_1 \equiv x_2 \equiv x_3 \equiv 1(4)$. If $x_4 \equiv 1 \; or \; 2(4)$, then $x_i.x_j \equiv 1 \; or \; 2(4), 1 \le i \ne j \le 4$

b). Let $x_1 \equiv x_2 \equiv x_3 \equiv 3(4)$. If $x_4 \equiv 3 \; or \; 2(4)$, then $x_i.x_j \equiv 1 \; or \; 2(4), 1 \le i \ne j \le 4$

For the remaining cases we have $x_i.x_j \equiv 0 \; or \; 3(4)$ which is impossible. This completes the proof.

3. NON EXTENDABILITY OF CERTAIN P_k SETS FOR $k \equiv 3(4)$

Let $X = \{a, b, c\}$ be a P_k set. By the definition of P_k set we have

$$ab + k = x^2 \quad (1)$$

$$ac + k = y^2 \quad (2)$$

$$bc + k = z^2 \quad (3)$$

where x, y, z are integers. Solving equations (1), (2) in terms of b and c, and plugging them into equation (3), we obtain

$$(x^2 - k)(y^2 - k) + a^2 k = (az)^2$$

Since the right hand side of this equation is a perfect square, the left hand side must be a perfect square, too. The left hand side can be written as

$$(xy - k)^2 - k[(x - y)^2 - a^2]$$

If we set $y - x = a$, then the equation becomes a perfect square.

The problem of choosing a is reduced to solving the congruence $x^2 \equiv k(a)$. If $(k, a) = 1$ and $\left(\frac{k}{a}\right) = 1$ then this congruence is solvable, where $\left(\frac{k}{a}\right)$ denotes the Legendre Symbol.

for $x = an + s, y = x + a$ we obtain

$$b = n(an + 2s) + \tfrac{s^2 - k}{a}$$

$$c = (n+1)[a(n+1) + 2s] + \tfrac{s^2 - k}{a}$$

where $n \in N, s^2 \equiv k(a)$. Hence adding k to the product if any two elements of

$$X = \{a, b, c\} = \{a, n(an + 2s) + \frac{s^2 - k}{a}, (n+1)[a(n+1) + 2s] + \frac{s^2 - k}{a}\}$$

is always a perfect square.

REMARK 1. If $k = -1, a = 1$ and $s = 0$ then we obtain the P_{-1} sets $\{1, n^2 + 1, (n+1)^2 + 1\}$ [6]. If $k = -1, a = 2$ and $s = 1, (k = -1, a = 17, s = 4$ and $n = 1)$ then we get the P_{-1} sets $\{2, 2n^2 + 2n + 1, 2n^2 + 6n + 5\}$, (respectively $\{17, 26, 85\}$)[8]. If $k = 3, a = 1$ and $s = 0, (k = 3, a = 2, s = 1)$ then we get the P_3 sets $\{1, n^2 - 3, n^2 + 2n - 2\}$,(respectively $\{2, 2n^2 + 2n - 1, 2n^2 + 6n + 3\}$)[9].

THEOREM 2. If $n \equiv 1(4)$ then the P_3 sets $\{2, 2n^2 + 2n - 1, 2n^2 + 6n + 3\}$ can not be extended.

PROOF. $\{2, 2n^2 + 2n - 1, 2n^2 + 6n + 3, d\}$ is a P_3 set. Then there exist x, y and z integers such that

$$2d + 3 = x^2 \quad (4)$$

$$(2n^2 + 2n - 1)d + 3 = y^2 \quad (5)$$

$$(2n^2 + 6n + 3)d + 3 = z^2 \quad (6)$$

Now the equations, (4), (5), and (6) lead to the equations

$$2y^2 - (2n^2 + 2n - 1)x^2 = 9 - 6n^2 - 6n \quad (7)$$

$$2z^2 - (2n^2 + 6n + 3)x^2 = -6n^2 - 18n - 3 \quad (8)$$

$$(2n^2 + 2n - 1)z^2 - (2n^2 + 6n + 3)y^2 = -12n - 12 \quad (9)$$

Write $n \equiv 1(4)$ and examining the equations (7) and (8) mod4 shows that

$2y^2 + x^2 \equiv 1(4)$

$2z^2 - 3x^2 \equiv -3(4)$

and consequently x is odd, y is even and z is even. Putting $y = 2u, z = 2v$ into (9) yields

$(2n^2+2n-1)v^2-(2n^2+6n43)u^2=-3(1+n)$

From the fact that $n \equiv 1(4)$ we have

$3u^2+v^2 \equiv 2(4)$

which is impossible. Indeed, if v is odd, this leads to the congruence

$3u^2 \equiv 1(4)$

which is impossible. if v is even then this leads to the congruence

$u^2 \equiv 2(4)$

which is impossible. Thus if $n \equiv 1(4)$, then the P_3 set $\{2, 2n^2+2n-1, 2n^2+6n+3\}$ can not be extended.

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