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Combinatorial Sums and Identities associated with Functional Equations of Generating Functions for Fubini Type Polynomials

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Highlights

- This paper contains some combinatorial sums associated with Fubini type polynomials and numbers.
- By using generating function for Fubini type polynomials, some identities and formulas are given.
- Applying Riemann integral to this generating function, some integral formulas are also given.

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Abstract

Using generating functions with their functional equations method, a great number of novel combinatorial sums, formulas, and recurrence relation including Fubini type polynomials and numbers, Stirling type numbers, and Apostol type polynomials are given. Applying Riemann integral to this generating function with their functional equations, some identities involving Cauchy and Stirling numbers are obtained. Moreover, some interpretations about the results are given.

1. INTRODUCTION

Recently, generating functions of certain polynomials and numbers have been applied not only in mathematics, but also in statistics and in other sciences. Using these functions, one finds a great number of characteristics of these certain polynomials.

The motivation of this paper is to give a great number of relations for Fubini type polynomials and numbers with aid of these functions. These relations are comprised combinatorial sums, Apostol type polynomials and numbers, the Cauchy and (λ-) Stirling numbers, and λ-array polynomials.

The following specific notations can be used: let $\mathbb{N} = \{1,2,3, \dots\}$ and $\mathbb{N}_0 = \{0,1,2,3, \dots\}$. Let $\mathbb{R}$, set of real numbers, $\mathbb{C}$, set of complex numbers, and

$$\binom{s}{0} = 1 \text{ and } \binom{s}{u} = \frac{(s)_u}{u!}, \quad (u \in \mathbb{N}, s \in \mathbb{C})$$

where

$$(s)_0 = 1 \text{ and } (s)_u = s(s-1)\dots(s-u+1)$$

(*cf.* [1-24]).

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The Stirling numbers of the second kind are given by

$$H_{S2}(z,p) = \frac{(e^z - 1)^p}{p!} = \sum_{d=0}^{\infty} \frac{S_2(d,p)}{d!} z^d \tag{1}$$

and

$$y^d = \sum_{m=0}^{d} S_2(d,m)(y)_m \tag{2}$$

(*cf.* [1-24]).

The Fubini numbers are given by

$$H_w(z) = \frac{1}{2 - e^z} = \sum_{d=0}^{\infty} \frac{w_g(d)}{d!} z^d \tag{3}$$

(*cf.* [3,4,8,10,12]).

Using Equation (3), we have

$$w_g(d) = \sum_{m=0}^{d} m!\, S_2(d,m) \tag{4}$$

(*cf.* [3,4,8,10,12]).

By using (3) and (4), Good [5] gave many combinatorial applications for these numbers.

The Fubini type polynomials of order p are given by

$$H_a(z,p,y) = \frac{2^p}{(2 - e^z)^{2p}} e^{yz} = \sum_{s=0}^{\infty} \frac{a_s^{(p)}(y)}{s!} z^s, \tag{5}$$

where $p \in \mathbb{N}_0$ and $|z| < \ln 2$ (*cf.* [10]).

When $y = 0$, using (5), one gets

$$a_s^{(p)}(0) = a_s^{(p)}.$$

If $p = 1$, one gets

$$a_s^{(1)}(0) = a_s.$$

By using (3), we have

$$a_s = \sum_{j=0}^{s} \binom{s}{j} w_g(j) w_g(s-j) \tag{6}$$

(*cf.* [10, Theorem 4.7]).

For $n = 1, 2, 3, ...$, few values of a_s are given by

$$a_0 = 2, \qquad a_1 = 4, \qquad a_2 = 16, \qquad a_3 = 88, \quad a_4 = 616,$$

and so on.

By using (5), we have

$$a_s^{(m+l)} = \sum_{j=0}^{s} \binom{s}{j} a_j^{(m)} a_{s-j}^{(l)}$$

and

$$a_s^{(p)}(y) = \sum_{j=0}^{s} \binom{s}{j} a_j^{(p)} y^{s-j} \tag{7}$$

(*cf*. [10, Equations (20) and (21)]).

The Apostol-Bernoulli numbers of order p are given by

$$H_{AB}(z, p; \lambda) = \left(\frac{z}{\lambda e^z - 1}\right)^P = \sum_{s=0}^{\infty} \frac{\mathcal{B}_s^{(p)}(\lambda)}{s!} z^s, \tag{8}$$

where $|z| < 2\pi$ when $\lambda = 1$; $|z| < |\log(\lambda)|$ when $\lambda \neq 1$, $\lambda \in \mathbb{C}$, $p \in \mathbb{N}$ (*cf*. [9,16,17,21,23]).

The Apostol-Bernoulli polynomials of order p are given by

$$F_{AB}(z, p, y; \lambda) = H_{AB}(z, p; \lambda) e^{yz} = \sum_{s=0}^{\infty} \frac{\mathcal{B}_s^{(p)}(y; \lambda)}{s!} z^s \tag{9}$$

(*cf*. [9,16,17,21,23]).

When $y = 0$, using (9), one gets

$$\mathcal{B}_s^{(p)}(0; \lambda) = \mathcal{B}_s^{(p)}(\lambda)$$

(*cf*. [9,16,17,21,23]).

Substituting $p = 0$ into (9), we also have

$$\mathcal{B}_s^{(0)}(y; \lambda) = y^s.$$

Let $p \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. The λ-Stirling numbers of the second kind are given by

$$H_S(z, p; \lambda) = \frac{(\lambda e^z - 1)^P}{p!} = \sum_{d=0}^{\infty} \frac{S_2(d, p; \lambda)}{d!} z^d \tag{10}$$

(*cf*. [19,21,22]).

When $\lambda = 1$ in (10), we obtain

$$S_2(d,p;1) = S_2(d,p).$$

Let $p \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. The λ-array polynomials are given by

$$G(z,p,y;\lambda) = \frac{(\lambda e^z - 1)^P}{p!} e^{yz} = \sum_{d=0}^{\infty} \frac{S_p^d(y;\lambda)}{d!} z^d \tag{11}$$

(*cf.* [2,19,21]).

When $y = 0$, using (11), one gets

$$S_p^d(0;\lambda) = S_2(d,p;\lambda).$$

The Cauchy numbers are given by

$$\frac{z}{\log(1+z)} = \sum_{s=0}^{\infty} \frac{b_s(0)}{s!} z^s$$

(*cf.* [15,18,21]).

The numbers $b_s(0)$ are also given by

$$b_s(0) = \int_0^1 (y)_s dy \tag{10}$$

(*cf.* [15,18,21]).

Let us briefly summarize the results of this paper:
By generating functions with their functional equations, a great number of novel combinatorial sums including Fubini type numbers and polynomials, (λ-) Stirling numbers, and λ-array polynomials are given by section 2. A great number of novel identities for the Fubini type polynomials and numbers, the Apostol-Bernoulli polynomials are obtained. By applying the Riemann integral to these formulas, some integral formulas involving the Cauchy numbers are given by section 3. In section 4, conclusion section is given.

2. COMBINATORIAL SUMS INVOLVING FUBINI TYPE POLYNOMIALS

With the aid of generating functions with their functional equations, a great number of novel combinatorial sums involving Fubini type numbers and polynomials, (λ-) Stirling numbers, and λ-array polynomials are given below.

Theorem 1. For $d \in \mathbb{N}_0$ and $s \in \mathbb{N}$, we have

$$\sum_{c=0}^{s} \sum_{m=0}^{2d} (-1)^m \binom{2d}{m} \binom{s}{c} (m-y)^c 2^{d-m} a_{s-c}^{(d)}(\mathrm{y}) = 0. \tag{13}$$

Proof. Using (5), we get

$$2^d = \sum_{m=0}^{2d} (-1)^m \binom{2d}{m} e^{u(m-y)} 2^{2d-m} H_a(u,d,y).$$

Thus

$$1=\sum_{m=0}^{2d}(-1)^m\binom{2d}{m}2^{d-m}\sum_{s=0}^{\infty}(m-y)^s\frac{u^s}{s!}\sum_{s=0}^{\infty}a_s^{(d)}(y)\frac{u^s}{s!}.$$

Therefore

$$1=\sum_{s=0}^{\infty}\sum_{c=0}^{s}\sum_{m=0}^{2d}(-1)^m\binom{2d}{m}\binom{s}{c}(m-y)^c 2^{d-m}a_{s-c}^{(d)}(\mathrm{y})\frac{u^s}{s!}. \tag{14}$$

The coefficients of $\frac{u^s}{s!}$ the previous equation are equalized, we get Equation (13).

When $y=0$, using (13), we have the following result:

Corollary 2 (*cf*. [8]). For $d\in\mathbb{N}_0$ and $s\in\mathbb{N}$, we have

$$\sum_{c=0}^{s}\sum_{m=0}^{2d}(-1)^m\binom{2d}{m}\binom{s}{c}m^c 2^{d-m}a_{s-c}^{(d)}=0.$$

Substituting $s=0$ and $y=0$ into (14), we get

$$1=\sum_{m=0}^{2d}(-1)^m\binom{2d}{m}2^{d-m}a_0^{(d)}.$$

By the aid of the previous equation and the following sum:

$$a_0^{(d)}=\sum_{j=0}^{d}\binom{d}{j},$$

we obtain

$$\sum_{m=0}^{2d}(-1)^m\binom{2d}{m}2^{2d-m}=1.$$

Thus, we get the following result:

Corollary 3. For $d\in\mathbb{N}_0$, we have

$$\sum_{m=0}^{2d}\frac{(-1)^m}{2^m}\binom{2d}{m}=\frac{1}{\sum_{v=0}^{2d}\binom{2d}{v}}.$$

Theorem 4. For $d\in\mathbb{N}$, we have

$$\sum_{m=0}^{d}\binom{d}{m}S_{2p}^{m}\left(-y;\frac{1}{2}\right)a_{d-m}^{(p)}(\mathrm{y})=0.$$

Proof. By (5), we have

$$2^p = (2-e^u)^{2p} e^{-yu} H_a(u,p,y). \tag{15}$$

Combining (15) with (11), we get

$$1 = (2p)!\, 2^p \sum_{d=0}^{\infty} \sum_{m=0}^{d} \binom{d}{m} S_{2p}^{m}\left(-y; \frac{1}{2}\right) a_{d-m}^{(p)}(\mathrm{y}) \frac{u^d}{d!}.$$

Therefore

$$\sum_{m=0}^{d} \binom{d}{m} S_{2p}^{m}\left(-y; \frac{1}{2}\right) a_{d-m}^{(p)}(\mathrm{y}) = 0.$$

Hence, proof of theorem is complemented.

Theorem 5 (*cf*. [8]). For $s \in \mathbb{N}_0$, we have

$$y^s = (2p)!\, 2^p \sum_{c=0}^{s} \binom{s}{c} S_2\left(c, 2p; \frac{1}{2}\right) a_{s-c}^{(p)}(\mathrm{y}). \tag{16}$$

Proof. Combining (15) with (10), we get

$$\sum_{s=0}^{\infty} y^s \frac{u^s}{s!} = (2p)!\, 2^p \sum_{s=0}^{\infty} S_2\left(s, 2p; \frac{1}{2}\right) \frac{u^s}{s!} \sum_{s=0}^{\infty} a_s^{(p)}(\mathrm{y}) \frac{u^s}{s!}.$$

Therefore

$$\sum_{s=0}^{\infty} y^s \frac{u^s}{s!} = (2p)!\, 2^p \sum_{s=0}^{\infty} \sum_{c=0}^{s} \binom{s}{c} S_2\left(c, 2p; \frac{1}{2}\right) a_{s-c}^{(p)}(\mathrm{y}) \frac{u^s}{s!}.$$

The coefficients of $\frac{u^s}{s!}$ the previous equation are equalized, we get Equation (16).

When $y = 0$, using (16), we get the following result:

Corollary 6 (*cf*. [8]). For $s \in \mathbb{N}$, we have

$$\sum_{c=0}^{s} \binom{s}{c} S_2\left(c, 2p; \frac{1}{2}\right) a_{s-c}^{(p)} = 0.$$

3. RELATIONS AND INTEGRAL REPRESENTATIONS FOR FUBINI TYPE POLYNOMIALS

A great number of formulae and recurrence relation involving Fubini type polynomials, and the Apostol-Bernoulli polynomials and numbers are obtained. By applying Riemann integral to these formulas, some integral formulae for these polynomials and the Cauchy numbers are given below.

Theorem 7. For $s \in \mathbb{N}_0$, we have

$$a_s^{(p)}(\mathrm{y})=\frac{1}{2}a_s^{(p-1)}(\mathrm{y})-\frac{1}{4}a_s^{(p)}(\mathrm{y}+2)+a_s^{(p)}(\mathrm{y}+1).$$

Proof. By (5), we get

$$4H_a(u,p,y)-4H_a(u,p,y+1)+H_a(u,p,y+2)=2H_a(u,p-1,y),$$

where for $p\in\mathbb{N}$. From the above equation, we have

$$4\sum_{s=0}^{\infty}a_s^{(p)}(\mathrm{y})\frac{u^s}{s!}-2\sum_{s=0}^{\infty}a_s^{(p-1)}(\mathrm{y})\frac{u^s}{s!}=4\sum_{s=0}^{\infty}a_s^{(p)}(\mathrm{y}+1)\frac{u^s}{s!}-\sum_{s=0}^{\infty}a_s^{(p)}(\mathrm{y}+2)\frac{u^s}{s!}.$$

The coefficients of $\frac{u^s}{s!}$ the previous equation are equalized, we obtain

$$a_s^{(p)}(\mathrm{y})=\frac{1}{2}a_s^{(p-1)}(\mathrm{y})-\frac{1}{4}a_s^{(p)}(\mathrm{y}+2)+a_s^{(p)}(\mathrm{y}+1).$$

Hence, proof of theorem is complemented.

Theorem 8. For $s\in\mathbb{N}_0$, we have

$$a_s^{(m+l)}(\mathrm{w}+\mathrm{y})=\sum_{d=0}^{s}\binom{s}{d}a_d^{(m)}(\mathrm{w})a_{s-d}^{(l)}(\mathrm{y}).\tag{17}$$

Proof. Using (5), we get

$$\sum_{s=0}^{\infty}a_s^{(m+l)}(\mathrm{w}+\mathrm{y})\frac{u^s}{s!}=\sum_{s=0}^{\infty}a_s^{(m)}(\mathrm{w})\frac{u^s}{s!}\sum_{s=0}^{\infty}a_s^{(l)}(\mathrm{y})\frac{u^s}{s!}.$$

Therefore

$$\sum_{s=0}^{\infty}a_s^{(m+l)}(\mathrm{w}+\mathrm{y})\frac{u^s}{s!}=\sum_{s=0}^{\infty}\sum_{d=0}^{s}\binom{s}{d}a_d^{(m)}(\mathrm{w})a_{s-d}^{(l)}(\mathrm{y})\frac{u^s}{s!}.$$

The coefficients of $\frac{u^s}{s!}$ the previous equation are equalized, we get Equation (17).

When $l=0$, using (17) and the following identity

$$a_n^{(0)}=y^n,$$

we have the following result:

Corollary 9 (*cf.* [8]). For $s\in\mathbb{N}_0$, we have

$$a_s^{(m)}(\mathrm{w}+\mathrm{y})=\sum_{d=0}^{s}\binom{s}{d}a_d^{(m)}(\mathrm{w})y^{s-d}.\tag{18}$$

Remark 10. Substituting $y=1$ into (18), we have

$$a_s^{(m)}(\mathrm{w}+1)=\sum_{d=0}^{s}\binom{s}{d}a_d^{(m)}(\mathrm{w})$$

(*cf.* [10, Equation (29)]).

Theorem 11 (*cf.* [8]). For $s\in\mathbb{N}_0$, we have

$$a_s^{(p)}(\mathrm{y})=\frac{\mathcal{B}_{s+2p}^{(2p)}\left(y;\frac{1}{2}\right)}{2^p(s+2p)_{2p}}. \tag{19}$$

Proof. By using (5) and (9), we get

$$\sum_{s=0}^{\infty}a_s^{(p)}(\mathrm{y})\frac{u^s}{s!}=\frac{1}{2^pu^{2p}}\sum_{s=0}^{\infty}\mathcal{B}_s^{(2p)}\left(y;\frac{1}{2}\right)\frac{u^s}{s!}.$$

Thus

$$\sum_{s=0}^{\infty}a_s^{(p)}(\mathrm{y})\frac{u^s}{s!}=\frac{1}{2^p}\sum_{s=0}^{\infty}\frac{\mathcal{B}_{s+2p}^{(2p)}\left(y;\frac{1}{2}\right)}{(s+2p)_{2p}}\frac{u^s}{s!}.$$

The coefficients of $\frac{u^s}{s!}$ the previous equation are equalized, we get Equation (19).

Remark 12. When $p=1$ and $y=0$ in (19), we obtain

$$a_s=\frac{\mathcal{B}_{s+2}^{(2)}\left(\frac{1}{2}\right)}{2(s+1)(s+2)}$$

(*cf.* [10, Equation (25)]).

By using Equations (7) and (19), we get the Theorem 13:

Theorem 13. For $s\in\mathbb{N}_0$, we have

$$a_s^{(p)}(\mathrm{y})=\sum_{m=0}^{s}\binom{s}{m}\frac{y^{s-m}}{2^p(m+2p)_{2p}}\mathcal{B}_{m+2p}^{(2p)}\left(\frac{1}{2}\right).$$

3.1. Applications of Integral Representations for Fubini Type Polynomials

Applying the Riemann integral to the Fubini type polynomials of order p, some identities and integral representations for these polynomials are derived.

Theorem 14. For $s\in\mathbb{N}_0$, we have

$$\int_c^d a_s^{(p)}(y)dy=\frac{1}{s+1}\left(a_{s+1}^{(p)}(d)-a_{s+1}^{(p)}(c)\right). \tag{20}$$

Proof. Integrating the Equation (5) by parts with respect to y from c to d, we get

$$\sum_{s=0}^{\infty}\int_{c}^{d} a_s^{(p)}(y)\frac{u^s}{s!}dy = \frac{2^p}{(2-e^u)^{2p}}\int_c^d e^{yu}dy\,.$$

Therefore

$$\sum_{s=0}^{\infty}\int_{c}^{d} a_s^{(p)}(y)\frac{u^s}{s!}dy = \left(\sum_{s=0}^{\infty}\frac{a_{s+1}^{(p)}(d)}{s+1}\frac{u^s}{s!} - \sum_{s=0}^{\infty}\frac{a_{s+1}^{(p)}(c)}{s+1}\frac{u^s}{s!}\right).$$

The coefficients of $\frac{u^s}{s!}$ the previous equation are equalized, we get Equation (20).

Combining (7) with (20), we get the following result:

Corollary 15. For $s \in \mathbb{N}_0$, we have

$$\int_c^d a_s^{(p)}(y)dy = \sum_{m=0}^{s}\binom{s}{m}\frac{a_m^{(p)}}{s-m+1}\left(d^{s-m+1} - c^{s-m+1}\right). \tag{21}$$

Remark 16. Substituting $c = 0$ and $d = 1$ into (21), we have

$$\int_0^1 a_s^{(p)}(y)dy = \sum_{v=0}^{s}\frac{1}{s-v+1}\binom{s}{v}a_v^{(p)}$$

(*cf.* [10]).

Theorem 17. For $s \in \mathbb{N}_0$, we have

$$\int_0^1 a_s^{(p)}(w+y)dy = \sum_{d=0}^{s}\sum_{v=0}^{s-d}\binom{s}{d}S_2(s-d,v)a_d^{(p)}(w)b_v(0)\,.$$

Proof. From the Equations (2) and (18), we get

$$a_s^{(p)}(w+y) = \sum_{d=0}^{s}\binom{s}{d}a_d^{(p)}(w)\sum_{v=0}^{s-d}S_2(s-d,v)\,(y)_v.$$

Integrating the above equation by parts with respect to y from 0 to 1, we get

$$\int_0^1 a_s^{(p)}(w+y)dy = \sum_{d=0}^{s}\binom{s}{d}a_d^{(p)}(w)\sum_{v=0}^{s-d}S_2(s-d,v)\int_0^1 (y)_v dy\,.$$

Combining the above equation with (12), we have

$$\int_0^1 a_s^{(p)}(w+y)dy = \sum_{d=0}^{s}\binom{s}{d}\sum_{v=0}^{s-d}S_2(s-d,v)a_d^{(p)}(w)b_v(0)\,.$$

Hence, proof of theorem is complemented.

4. CONCLUSION

In this present paper, by the aid of generating functions with their functional equations, we obtained a great number of novel combinatorial sums, identities, and recurrence relation involving Fubini type polynomials and numbers, the Cauchy numbers, the Apostol-Bernoulli polynomials and Stirling type polynomials. Moreover, applications of integral representations for Fubini type polynomials, Cauchy and Stirling numbers are presented. The results of this paper may contribute to diverse fields of science such as mathematics, engineering and physics.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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