



On some Diagonalized and Regularized Hotelling's T^2 Tests of Location for High Dimensional Data

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Highlights

- A revisit to the diagonalized and regularized Hotelling's T^2 tests.
- Notions of data depth for trimming of mean vectors and covariance matrices.
- Comparison of diagonalized and regularized Hotelling's T^2 tests with the usual Hotelling's T^2 test.
- Limitations of diagonalized and regularized Hotelling's T^2 tests in high dimension.
- Proposal of robust versions of diagonalized and regularized Hotelling's T^2 tests in high dimension.

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Abstract

A widely used statistical test of hypothesis for location parameter in \mathbb{R}^p is the Hotelling's T^2 test. This test is efficient if data is normally distributed, ratio of sample size to dimension diverges and there are no outliers in the data. However, it is practically impossible to implement when dimension is greater than sample size. As a remedial measure, diagonalized and regularized Hotelling's T^2 tests were proposed. In this paper, powers of regularized and diagonalized Hotelling's T^2 tests are compared with the usual Hotelling's T^2 test in low dimension and the usual Hotelling's T^2 perform much better. It is observed that diagonalized Hotelling's T^2 test may have low power for mixture distributions. Due to a comparative performance of regularized and diagonalized Hotelling's T^2 tests, robust versions of diagonalized and regularized Hotelling's T^2 tests are proposed in high dimension in the presence of outliers. The powers of these tests were compared using simulated as well as real datasets.

1. INTRODUCTION

Let $\mathbf{X} = (X_1, X_2, \dots, X_p)^T$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_p)^T$ be two p -dimensional random vectors from distributions F and G with mean vectors $\boldsymbol{\mu}_X$ and $\boldsymbol{\mu}_Y$ and sample covariance matrices \mathbf{S}_X and \mathbf{S}_Y respectively. Suppose F and G have multivariate normal distributions. Under the assumption that $\mathbf{S}_X = \mathbf{S}_Y$, we are interested in testing hypotheses

$$H_0: \boldsymbol{\mu}_X = \boldsymbol{\mu}_Y \text{ against } H_1: \boldsymbol{\mu}_X \neq \boldsymbol{\mu}_Y.$$

The traditional Hotelling's T^2 test, with test statistic defined as

$$T^2 = (\bar{\mathbf{X}} - \bar{\mathbf{Y}})^T \left[\mathbf{S} \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \right]^{-1} (\bar{\mathbf{X}} - \bar{\mathbf{Y}}),$$

where n_1 and n_2 are the sample sizes of observations from F and G respectively, the pooled sample covariance matrix \mathbf{S} and the sample mean vectors $\bar{\mathbf{X}}$ and $\bar{\mathbf{Y}}$ are defined, respectively by

$$\mathbf{S} = \frac{(n_1 - 1)\mathbf{S}_X + (n_2 - 1)\mathbf{S}_Y}{(n_1 + n_2 - 2)}, \quad \bar{\mathbf{X}} = \frac{1}{n_1} \sum_{i=1}^{n_1} \mathbf{X}_i, \quad \bar{\mathbf{Y}} = \frac{1}{n_2} \sum_{i=1}^{n_2} \mathbf{Y}_i,$$

can be used. It has been established that Hotelling's T^2 test is uniformly the most powerful test in a class of affine invariant test [1]. Capilla [2] considered the use of Hotelling's T^2 test statistic in constructing a statistical control chart for wastewater treatment process. When the dimension is higher than the sample size, the Hotelling's T^2 test cannot be practically implemented [3].

Hu and Bai [1] considered the defect of Hotelling's T^2 test when the dimension (p) is greater than the sample size (n) or degree of freedom. Dempster [4,5] developed non-exact test (NET) for testing whether mean vectors of two populations are equal when $p > n$. Bai and Saranadasa [3] argued that Dempster's non-exact test serve as a replacement for Hotelling T^2 test. Bai and Saranadasa [3] suggested asymptotic normal test (ANT) to test whether \bar{X} and \bar{Y} are the same and discussed asymptotic power of Hotelling's T^2 test, NET and ANT using three different norms. In comparing the power of the Hotelling T^2 test and ANT, the authors found out that ANT has higher power than that of Hotelling T^2 test. Chen and Qin [6] also argued that ANT is not robust against outlier and thus suggested a robust version of ANT based on the square of norms difference of the means. However, Srivastava and Du [7] observed that both NET and ANT are not invariant under scale transformation of the data. Srivastava and Du [7] modified ANT statistic by replacing covariance matrix by either the diagonal of the covariance matrix or correlation matrix.

Chattinnawat and Bilen [8] analysed the effect of multivariate normal inspection errors on the performance of the Hotelling's T^2 test for individual observation and derive explicitly how the multivariate inspection errors are related to the Hotelling T^2 test statistic. Chen *et al.* [9] suggested a regularized Hotelling's T^2 test for cases where $p < n$ and $p > n$. Regularization is employed to stabilize the inverse of the sample covariance matrix S_n in $T^2 = n(\bar{X}_n - \mu_o)^T S_n^{-1} (\bar{X}_n - \mu_o)$ in one sample case. The regularized Hotelling's T^2 test statistic is $n(\bar{X}_n - \mu_o)^T (S_n + \lambda I)^{-1} (\bar{X}_n - \mu_o)$, where λ is a regularization parameter. It is expected that regularized Hotelling's T^2 test enjoy superior power compared to Hotelling's T^2 when dimension is close to sample size. However, this requires further investigation. Chen *et al.* [9] also derived an asymptotic null distribution of the regularized Hotelling's T^2 test statistic as both sample size and dimension increase to infinity at a comparable rate. The authors derived mathematical expression for power of regularized Hotelling's T^2 test and provide sufficient conditions for consistency of the regularized Hotelling's T^2 test.

A related study to test of location is a test of dispersion matrix. Chen *et al.* [10] considered tests for covariance matrix of multivariate distribution when dimension is much larger than the sample size. The authors considered critically two structures of covariance: $H_0: \Sigma = \sigma^2 I_p$ versus $H_1: \Sigma \neq \sigma^2 I_p$ and $H_0: \Sigma = I_p$ versus $H_1: \Sigma \neq I_p$ where I_p is the p -dimensional identity matrix and σ^2 is an unknown but finite positive constant. For inference on high-dimensional covariance matrices, there has been an array of works on the convergence of the sample covariance matrices based on the spectral analysis of large-dimensional random matrices. Chen *et al.* [10] also proposed a new tests for the hypotheses without the normality assumptions and without specifying explicit relationship between dimension and sample size as long as both dimension and sample size tend to infinity. In another development, Lu *et al.* [11] presented a modified version of Hotelling's T^2 test using a multiple forward search algorithm.

In this paper, robust versions of diagonalized and regularized Hotelling's T^2 test are proposed for testing equality of location vectors of two samples with very high dimension and small sample sizes. The performance of the robust versions of diagonalized and regularized Hotelling's T^2 test will be compared with the usual diagonalized and regularized Hotelling's T^2 test in terms of their powers using simulated as well as real data examples.

2. METHOD

Considering a two-sample location problem. Let $\mathbf{X} = (X_1, X_2, \dots, X_p)^T$ and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_p)^T$ be any random vectors from two independent populations, having multivariate normal distributions F and G respectively. Suppose μ_X and μ_Y denote the mean vectors and Σ_X and Σ_Y denote the covariance matrices of the distributions F and G respectively. Assuming that $\Sigma_X = \Sigma_Y$, the Hotelling's T^2 test can be employed

to investigate equality of the mean vectors $\boldsymbol{\mu}_X$ and $\boldsymbol{\mu}_Y$ under the hypotheses: $H_0: \boldsymbol{\mu}_X = \boldsymbol{\mu}_Y$ against $H_1: \boldsymbol{\mu}_X \neq \boldsymbol{\mu}_Y$. The distribution of Hotelling's T^2 test statistic is F distribution with p and $n_1 + n_2 - p - 1$ degrees of freedom while the critical value is given as

$$\frac{n_1 + n_2 - p - 1}{p(n_1 + n_2 - 2)} F_{p, n_1 + n_2 - p - 1}(\alpha).$$

The decision to reject null hypothesis is based on the comparison of the value of test statistic and critical value. That is, reject the null hypothesis at α level of significance if the value of test statistic is greater than critical value or probability that the value of test statistic greater than critical value is less than α .

Mathematically, H_0 is rejected if

$$T^2 > \frac{n_1 + n_2 - p - 1}{p(n_1 + n_2 - 2)} F_{p, n_1 + n_2 - p - 1}(\alpha).$$

Jureckova and Kalina [12] argued that the Hotelling's T^2 test is invariant with respect to affine transformation and is optimal unbiased against two-sample normal alternatives with $\boldsymbol{\mu}_X \neq \boldsymbol{\mu}_Y$ whenever $\boldsymbol{\Sigma}_X = \boldsymbol{\Sigma}_Y$. The asymptotical null distribution does not depend on normality whenever $n_1, n_2 \rightarrow \infty$ and $\frac{n_1}{n_2} \rightarrow 1$.

2.1. Diagonalized Hotelling's T^2 test

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_1}$ be a random sample in \mathbb{R}^p having distribution F with mean vector $\boldsymbol{\mu}_X$ and covariance matrix $\boldsymbol{\Sigma}_X$. Also, let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n_2}$ be a random sample in \mathbb{R}^p having distribution G with mean vector $\boldsymbol{\mu}_Y$ and covariance matrix $\boldsymbol{\Sigma}_Y$. Hotelling's T^2 test can not be implemented when the dimension is greater than or equal to the sample size ($p \geq n_1 + n_2$). For $\boldsymbol{\Sigma}_X = \boldsymbol{\Sigma}_Y = \boldsymbol{\Sigma}$, Srivastava and Du [7] proposed a diagonalized Hotelling's T^2 test. This involves replacing the pooled covariance matrix in the usual Hotelling's T^2 statistic with a diagonal matrix of the pooled covariance matrix. Mathematically, defining the diagonal matrix of sample covariance matrix as

$$\mathbf{D}_s = \text{diag}(s_{11}, \dots, s_{pp}),$$

where s_{11}, \dots, s_{pp} are the diagonal elements of \mathbf{S} , the pooled covariance matrix. The diagonalized Hotelling's T^2 test statistic is given by:

$$T_D^2 = \frac{n_1 n_2}{n_1 + n_2} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \mathbf{D}_s^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2).$$

[7] has shown that:

$$\frac{\frac{n_1 n_2}{n_1 + n_2} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T \mathbf{D}_s^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2) - \frac{np}{n-2}}{\left[2 \left(\text{tr}(\mathbf{R}^2) - \frac{p^2}{n_1 + n_2} \right) c_{p, n_1 + n_2} \right]^{\frac{1}{2}}} \rightarrow N(0, 1)$$

in distribution, where $c_{p, n_1 + n_2} = 1 + \frac{\text{tr} \mathbf{R}^2}{p^{3/2}}$ is the adjustment coefficient, which converges to 1 in probability

as $n_1 + n_2$ and p tend to infinity, \mathbf{R} is sample correlation matrix defined as $\mathbf{R} = \mathbf{D}_s^{-\frac{1}{2}} \mathbf{S} \mathbf{D}_s^{-\frac{1}{2}} = (r_{ii})$, $n = n_1 + n_2$, \mathbf{D}_s is the diagonal of pooled covariance matrix \mathbf{S} and $\text{tr}(\mathbf{R}^2)$ is the trace of \mathbf{R}^2 . Srivastava and Du [7] also gave an expression for the asymptotic power function of the diagonalized Hotelling's T^2 test as:

$$\Phi \left(-z_{1-\alpha} + \frac{n_1 n_2}{n^2} \frac{(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)^T \mathbf{D}_s^{-1} (\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1)}{\sqrt{2 \text{tr}(\mathbf{R}^2)}} \right),$$

where Φ is the standard normal distribution function. When $\Sigma_X = \Sigma_Y$, the computation of T_D^2 is straightforward using the fact that $D_S = \frac{1}{n_1}D_{S_1} + \frac{1}{n_2}D_{S_2}$, where D_{S_1} and D_{S_2} are diagonal matrices of sample covariance matrix from distributions F and G respectively.

2.2. Regularized Hotelling’s T^2 test

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n_1}$ be a random sample in \mathbb{R}^p having distribution F with mean vector μ_X and covariance matrix Σ . Also, let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n_2}$ be a random sample in \mathbb{R}^p having distribution G with mean vector μ_Y and covariance matrix Σ . Chen *et al.* [9] proposed a regularized Hotelling’s T^2 test in one sample problem. The regularized Hotelling’s T^2 test statistic is $n(\bar{\mathbf{X}}_n - \mu_o)^T(\mathbf{S}_n + \lambda I)^{-1}(\bar{\mathbf{X}}_n - \mu_o)$, where λ is a regularization parameter. In regularized Hotelling’s T^2 statistic, product of regularization parameter and identity matrix is added to covariance matrix in the usual Hotelling’s T^2 statistic to stabilize the inverse of the sample covariance matrix. The idea was first applied in ridge regression and regularized discriminant analysis to regularize \mathbf{S}_n or stabilize the inverse of \mathbf{S}_n . Hu and Bai [1] derived the asymptotic distribution for the regularized Hotelling’s T^2 statistic. That is, suppose \mathbf{X} is independent and identically distributed as $N(\mu, \Sigma)$, then

$$\sqrt{p} \left(nT_R^2 - \frac{1 - \lambda m(\lambda)}{1 - \frac{p(1 - \lambda m(\lambda))}{n}} \right) \xrightarrow{D} N(0,1),$$

$$\frac{1 - \lambda m(\lambda)}{\left(1 - \frac{p}{n} + \frac{p\lambda m(\lambda)}{n}\right)^3} - \lambda \frac{m(\lambda) - \lambda m^1(\lambda)}{\left(1 - \frac{p}{n} + \frac{p\lambda m(\lambda)}{n}\right)^4}$$

where $m(\lambda) = \frac{1}{p} \text{tr}(\mathbf{S} + \lambda I)^{-1}$ and $m^1(\lambda) = \frac{1}{p} \text{tr}(\mathbf{S} + \lambda I)^{-2}$.

For two sample problem, the regularized Hotelling’s T^2 test statistic is given by

$$T_R^2 = \frac{n_1 n_2}{n_1 + n_2} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)^T (\mathbf{S}_n + \lambda I)^{-1} (\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2),$$

where $n = n_1 + n_2$, $\lambda \in R^+$ is the regularization parameter.

2.3. Proposed robust versions of regularized and diagonalized Hotelling’s T^2 test

An α -trimmed mean is an average of the $n - [n\omega]$ deepest observations from the sample, where $[n\omega]$ is the integer part of $n\omega$. Let $\mathbf{x}_{(1)}, \dots, \mathbf{x}_{(n)}$ be the center-observed ordered sample, based on Λ , where $\mathbf{x}_{(1)}$ is the deepest (or most central) observation and $\mathbf{x}_{(n)}$ is the most outlying observation and Λ is any depth function [13]. Due to presence of extreme values and outliers in the data, a robust diagonalized and robust regularized Hotelling T^2 test is proposed.

To compute the trimmed mean, observations are ranked based on their depth values. Data depth is a measure of how central an observation is with respect to a data cloud or probability distribution [14]. Depth function provides ordering of observations in R^p with respect to some probability measure F defined on R^p . The depth notion creates possible basis of non-parametric multivariate analysis. Some depth functions commonly used in statistical inference include half-space depth, projection depth, spatial depth, among others. For details on data depth for multivariate data, we refer readers to Zuo and Serfling [15], Makinde and Adewumi [16].

The spatial depth of an observation \mathbf{x} in R^p with F is defined as:

$$D(\mathbf{x}, F) = 1 - \left\| E_F \left[\frac{(\mathbf{x} - \mathbf{X})}{\|(\mathbf{x} - \mathbf{X})\|} \right] \right\|.$$

Using spatial depth, all observations are ranked and $n_1 - [n_1\omega]$ most central observations are obtained and their mean vector and covariance matrix are obtained for the population 1, where $[n_1\omega]$ is the integer part

of $n_1\omega$. Similarly, $n_2 - [n_2\omega]$ most central observations are obtained and their mean vector and covariance matrix are obtained for the population 2, where $[n_2\omega]$ is the integer part of $n_2\omega$.

Mathematically, the ω -trimmed mean vectors for populations 1 and 2 are defined as

$$\mathbf{m}_1^\omega = \frac{\sum_{i=1}^{n_1-[n_1\omega]} \mathbf{x}_{(i)}}{n_1 - [n_1\omega]} \text{ and } \mathbf{m}_2^\omega = \frac{\sum_{i=1}^{n_2-[n_2\omega]} \mathbf{x}_{(i)}}{n_2 - [n_2\omega]},$$

respectively. The ω -trimmed sample covariance matrices \mathbf{S}_1^ω and \mathbf{S}_2^ω are defined as

$$\mathbf{S}_1^\omega = \frac{1}{n_1 - [n_1\omega] - 1} \sum_{i=1}^{n_1-[n_1\omega]} (\mathbf{x}_{1i} - \mathbf{m}_1^\omega)(\mathbf{x}_{1i} - \mathbf{m}_1^\omega)^T$$

and

$$\mathbf{S}_2^\omega = \frac{1}{n_2 - [n_2\omega] - 1} \sum_{i=1}^{n_2-[n_2\omega]} (\mathbf{x}_{2i} - \mathbf{m}_2^\omega)(\mathbf{x}_{2i} - \mathbf{m}_2^\omega)^T$$

respectively. The ω -trimmed pooled sample covariance matrix is defined as:

$$\mathbf{S}^\omega = \frac{(n_1 - [n_1\omega] - 1)\mathbf{S}_1^\omega + (n_2 - [n_2\omega] - 1)\mathbf{S}_2^\omega}{n_1 + n_2 - [n_1\omega] - [n_2\omega] - 2}.$$

The ω -trimmed diagonal matrix of pooled sample covariance matrix is defined as:

$$\mathbf{D}^\omega = \text{diag}(\mathbf{S}^\omega).$$

Therefore, robust diagonalized Hotelling T^2 statistic is given as

$$T^2_{D,\omega} = (\mathbf{m}_1^\omega - \mathbf{m}_2^\omega)^T \left[\mathbf{D}^\omega \left(\frac{1}{n_1 - [n_1\omega]} + \frac{1}{n_2 - [n_2\omega]} \right) \right]^{-1} (\mathbf{m}_1^\omega - \mathbf{m}_2^\omega).$$

The robust regularized Hotelling T^2 is given as:

$$T^2_{R,\omega} = (\mathbf{m}_1^\omega - \mathbf{m}_2^\omega)^T [\mathbf{S}^\omega + \lambda \mathbf{I}]^{-1} (\mathbf{m}_1^\omega - \mathbf{m}_2^\omega).$$

The test statistics $T^2_{D,\omega}$ and $T^2_{R,\omega}$ follow normal distributions [1,7,9]. The choice of value for λ in this study is based on the value that maximizes the power of the test.

3. NUMERICAL RESULTS

3.1. Simulation study

We present two simulation studies in low dimension to illustrate the performance of Hotelling’s T^2 test, diagonalized Hotelling’s T^2 test and regularized Hotelling’s T^2 test. This is because Hotelling’s T^2 test is a most powerful test in low dimension and comparison of its power with diagonalized Hotelling’s T^2 test and regularized Hotelling’s T^2 test will guide us whether to use the diagonalized Hotelling’s T^2 test and regularized Hotelling’s T^2 test in high dimension. Based on the power of diagonalized Hotelling’s T^2 test and regularized Hotelling’s T^2 test, we want to carry out simulation studies on diagonalized Hotelling’s T^2 test and regularized Hotelling’s T^2 test in high dimension. Simulation studies will also be carried out on the proposed robust versions of the diagonalized and regularized Hotelling’s T^2 tests. Each of the experiment consists of 100,000 replications. In each replication, p-value is estimated and power is determined as the average proportion of p-values less than level of significance. We do not compare type 1 error rates of these tests because it has been established that Hotelling’s T^2 test generates higher type 1 error rates compared to regularized Hotelling’s T^2 test irrespective of sample sizes.

Simulation 1 (Elliptical case). Suppose there are two populations π_k , $k = 1, 2$ from $N(\boldsymbol{\mu}_k, \boldsymbol{\Sigma})$ such that $\boldsymbol{\mu}_k = (\mu_{k1}, \mu_{k2}, \dots, \mu_{kp})'$ with $\mu_{1j} = 0$ for $1 \leq j \leq p$, $\mu_{2j} = 0.6$ for $1 \leq j \leq 100$ and $\mu_{2j} = 0$ if otherwise. The covariance structure $\boldsymbol{\Sigma}$ consists of 3×3 blocks, each block of dimension 3×3 with (j, j') element $0.6^{|j-j'|}$. The value of p is taken to be 9. This is a modified version of simulation example in Guo *et al.* [17] and Makinde [18]. Sample of sizes n_1 and n_2 are generated from each of populations π_k . The Hotelling’s T^2 test, diagonalized Hotelling’s T^2 test and regularized Hotelling’s T^2 test are performed on the samples to determine if the difference in mean vectors of the samples is statistically significant. Power of the three test are compared with varying sample sizes.

Simulation 2 (Spherical case). Suppose there are two populations $\pi_k, k = 1, 2$ from $N(\mu_k, \Sigma)$ such that mean vectors $\mu_1 = (0, 0, 0, 0, 0, 0)'$ and $\mu_2 = (0.7, 0.7, 0.7, 0, 0, 0)'$ and covariance matrix

$$\Sigma = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

Sample of sizes n_1 and n_2 are generated from each of the populations π_k . The Hotelling's T^2 test, diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test are performed on the samples to determine if the difference in mean vectors of the samples is statistically significant. Power of the three test are compared with varying sample sizes. The regularization parameter (λ) is chosen to be 0.1 for simulations 1 and 2.

The Table 1 gives the powers of Hotelling T^2 test, diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test at low dimension. The Hotelling's T^2 test has the highest power for small sample sizes in simulation 1 at all levels of significance. At $\alpha = 0.05$, for large sample, regularized Hotelling T^2 test performs optimal while diagonalized Hotelling's T^2 test performs better than regularized Hotelling T^2 test for large sample sizes when $\alpha = 0.01$. The Hotelling T^2 test has the highest power at low level of significance for various sample sizes. The regularized Hotelling's T^2 test performs best for large sample sizes at $\alpha = 0.05$. The usual Hotelling T^2 test has the highest power for varying sample sizes at $\alpha = 0.025$ and $\alpha = 0.1$ respectively.

In general, the usual Hotelling's T^2 test is more efficient when the sample size is small while the regularized Hotelling's T^2 test performs best at large sample size for $\alpha = 0.05$. Also, at the other level of significance, the regularized Hotelling's T^2 test has low power compared to the Hotelling's T^2 test and diagonalized Hotelling's T^2 test.

Table 1. Comparison of powers of Hotelling T^2 test, diagonalized Hotelling's T^2 and regularized Hotelling's T^2 test based on Simulation 4.1 for various levels of significance

| Sample sizes | $\alpha = 0.01$ | | | $\alpha = 0.025$ | | |
|-----------------------|----------------------|-----------------------------------|----------------------------------|----------------------|-----------------------------------|----------------------------------|
| | Hotelling T^2 test | Diagonalized Hotelling T^2 test | Regularized Hotelling T^2 test | Hotelling T^2 test | Diagonalized Hotelling T^2 test | Regularized Hotelling T^2 test |
| $n_1 = n_2 = 20$ | 0.9381 | 0.8276 | 0.8802 | 0.9626 | 0.8794 | 0.7187 |
| $n_1 = n_2 = 30$ | 0.9632 | 0.8946 | 0.9069 | 0.9772 | 0.9261 | 0.7131 |
| $n_1 = n_2 = 50$ | 0.9743 | 0.9303 | 0.9075 | 0.9834 | 0.9517 | 0.6333 |
| $n_1 = n_2 = 100$ | 0.9845 | 0.9583 | 0.8821 | 0.9901 | 0.9714 | 0.4696 |
| $n_1 = n_2 = 200$ | 0.9916 | 0.9786 | 0.8384 | 0.9947 | 0.9851 | 0.3215 |
| $n_1 = n_2 = 500$ | 0.9985 | 0.9962 | 0.8200 | 0.999 | 0.9973 | 0.1822 |
| $n_1 = 20, n_2 = 50$ | 0.9652 | 0.9046 | 0.9182 | 0.9778 | 0.9327 | 0.7465 |
| $n_1 = 100, n_2 = 50$ | 0.9803 | 0.9472 | 0.9059 | 0.9877 | 0.9629 | 0.5794 |
| Sample sizes | $\alpha = 0.05$ | | | $\alpha = 0.1$ | | |
| | Hotelling T^2 test | Diagonalized Hotelling T^2 test | Regularized Hotelling T^2 test | Hotelling T^2 test | Diagonalized Hotelling T^2 test | Regularized Hotelling T^2 test |
| $n_1 = n_2 = 20$ | 0.9760 | 0.9125 | 0.9419 | 0.9855 | 0.9414 | 0.8555 |
| $n_1 = n_2 = 30$ | 0.9851 | 0.9676 | 0.9691 | 0.9909 | 0.9640 | 0.8416 |
| $n_1 = n_2 = 50$ | 0.9887 | 0.9652 | 0.9872 | 0.9931 | 0.9763 | 0.7740 |

| | | | | | | |
|--------------------|--------|--------|--------|--------|--------|--------|
| $n_1 = n_2 = 100$ | 0.9931 | 0.9796 | 0.9984 | 0.9958 | 0.9865 | 0.6207 |
| $n_1 = n_2 = 200$ | 0.9965 | 0.9892 | 0.9997 | 0.9979 | 0.9928 | 0.4595 |
| $n_1 = n_2 = 500$ | 0.9993 | 0.9982 | 1.0000 | 0.9996 | 0.9988 | 0.3089 |
| $n_1= 20 n_2 = 50$ | 0.9854 | 0.951 | 0.9711 | 0.9909 | 0.9671 | 0.8626 |
| $n_1= 100 n_2= 50$ | 0.9917 | 0.9733 | 0.9944 | 0.9948 | 0.9820 | 0.7243 |

The Table 2 gives the powers of Hotelling T^2 test, diagonalized Hotelling’s T^2 test and regularized Hotelling’s T^2 test at different levels of significance. The Hotelling’s T^2 test has the highest power for low sample sizes in simulation 2 at the four levels of significance. At $\alpha = 0.05$, for large sample, regularized Hotelling T^2 test performs best while Hotelling’s T^2 test performs better than regularized Hotelling T^2 test for large sample sizes when $\alpha = 0.01$. The regularized Hotelling’s T^2 test performs best for sample sizes at $\alpha = 0.05$ while at $\alpha = 0.01$ the Hotelling’s T^2 test performs better than the diagonalized Hotelling’s T^2 test and regularized Hotelling’s T^2 test. The Hotelling’s T^2 test has the highest power for low sample sizes in simulation 2 at $\alpha = 0.025$ and $\alpha = 0.1$ respectively. At $\alpha = 0.025$ and $\alpha = 0.1$ for large sample, the power of regularized Hotelling T^2 test is small compared to both Hotelling’s T^2 test and diagonalized Hotelling’s T^2 test. In general, Hotelling’s T^2 test is more efficient when the sample size is small compare to regularized Hotelling’s T^2 test and diagonalized Hotelling’s T^2 test. The regularized Hotelling’s T^2 test performs best for large sample sizes at 5% level of significance. Also, at the other levels of significance the power of regularized Hotelling’s T^2 test is less compared to the Hotelling’s T^2 test and diagonalized Hotelling’s T^2 test.

Table 2. Comparison of powers of Hotelling T^2 test, diagonalized Hotelling’s T^2 test and regularized Hotelling’s T^2 test based on Simulation 2 for various levels of significance

| Sample sizes | $\alpha = 0.05$ | | | $\alpha = 0.1$ | | |
|--------------------|----------------------|-----------------------------------|----------------------------------|----------------------|-----------------------------------|----------------------------------|
| | Hotelling T^2 test | Diagonalized Hotelling T^2 test | Regularized Hotelling T^2 test | Hotelling T^2 test | Diagonalized Hotelling T^2 test | Regularized Hotelling T^2 test |
| $n_1 = n_2 = 20$ | 0.933 | 0.811 | 0.8012 | 0.9591 | 0.8666 | 0.8177 |
| $n_1 = n_2 = 30$ | 0.9572 | 0.877 | 0.8769 | 0.973 | 0.913 | 0.7850 |
| $n_1 = n_2 = 50$ | 0.9657 | 0.9107 | 0.9075 | 0.9781 | 0.9365 | 0.7183 |
| $n_1 = n_2 = 100$ | 0.9741 | 0.9298 | 0.9117 | 0.9828 | 0.9496 | 0.6669 |
| $n_1 = n_2 = 200$ | 0.9749 | 0.9354 | 0.8814 | 0.9836 | 0.9538 | 0.4285 |
| $n_1 = n_2 = 500$ | 0.9769 | 0.9684 | 0.8320 | 0.9845 | 0.9575 | 0.2921 |
| $n_1= 20 n_2 = 50$ | 0.9594 | 0.8897 | 0.9102 | 0.974 | 0.9211 | 0.8015 |
| $n_1= 100 n_2= 50$ | 0.9709 | 0.9241 | 0.9134 | 0.9816 | 0.9461 | 0.6994 |
| Sample sizes | $\alpha = 0.05$ | | | $\alpha = 0.1$ | | |
| | Hotelling T^2 test | Diagonalized Hotelling T^2 test | Regularized Hotelling T^2 test | Hotelling T^2 test | Diagonalized Hotelling T^2 test | Regularized Hotelling T^2 test |
| $n_1 = n_2 = 20$ | 0.9736 | 0.903 | 0.9314 | 0.9841 | 0.9344 | 0.8555 |
| $n_1 = n_2 = 30$ | 0.982 | 0.9371 | 0.9541 | 0.9892 | 0.9573 | 0.8416 |
| $n_1 = n_2 = 50$ | 0.9851 | 0.9534 | 0.9652 | 0.9906 | 0.9683 | 0.7740 |
| $n_1 = n_2 = 100$ | 0.9882 | 0.9632 | 0.9894 | 0.9923 | 0.9754 | 0.6207 |
| $n_1 = n_2 = 200$ | 0.9889 | 0.9661 | 0.9976 | 0.9929 | 0.9770 | 0.4595 |
| $n_1 = n_2 = 500$ | 0.9893 | 0.9684 | 0.9990 | 0.9931 | 0.9575 | 0.3089 |
| $n_1= 20 n_2 = 50$ | 0.9825 | 0.9426 | 0.9681 | 0.9889 | 0.9618 | 0.8626 |
| $n_1= 100 n_2= 50$ | 0.9874 | 0.9602 | 0.9914 | 0.9919 | 0.9723 | 0.7243 |

The Table 3 gives the results obtain on regularized Hotelling's T^2 test at different α level of significance. The power of regularized Hotelling's T^2 test was found at varying regularization parameter (λ). At regularization parameter of 0.05, it was observed that the values of the power of the regularized Hotelling's T^2 test keep increasing as the sample size increases. It was also revealed that at varying sample sizes, the power keeps increasing. When the regularization parameter is 0.10, it shows that the power of regularized Hotelling's T^2 test keeps increasing as the sample size increases. At other values of regularization parameter, the power of the regularized Hotelling's T^2 test also increases at various sample sizes. The results show that at equal number of sample size of 20 across the regularization parameter, the performance was found to be highest at low level of regularization parameter 0.05. Also, across sample size 30 and above, the regularized Hotelling's T^2 test was found to be effective at regularization parameter 0.05. At varying sample size, it reveals that regularized Hotelling's T^2 test was found efficient and effective at regularization parameter of 0.05. Conclusively, the regularized Hotelling's T^2 test perform better at low level of regularization parameter. Hence, as the regularization parameter increases the power of the regularized Hotelling's T^2 test decreases.

Table 3. Effect of regularization parameter on power of the regularized Hotelling T^2 test based on Simulation 1 at 5% level of significance

| Sample sizes | Various values of regularization parameter (λ) | | | | | | | |
|-------------------|--|--------|--------|--------|--------|--------|--------|--------|
| | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 | 0.40 | 0.50 |
| $n_1 = n_2 = 20$ | 0.9419 | 0.8554 | 0.7570 | 0.6621 | 0.5753 | 0.4976 | 0.3709 | 0.2730 |
| $n_1 = n_2 = 30$ | 0.9691 | 0.9054 | 0.8242 | 0.7408 | 0.6592 | 0.5792 | 0.4393 | 0.3242 |
| $n_1 = n_2 = 50$ | 0.9872 | 0.9485 | 0.8898 | 0.8170 | 0.7366 | 0.6520 | 0.4884 | 0.3487 |
| $n_1 = n_2 = 100$ | 0.9984 | 0.9879 | 0.9586 | 0.9081 | 0.8354 | 0.7458 | 0.5464 | 0.3603 |
| $n_1 = n_2 = 200$ | 0.9997 | 0.9992 | 0.9940 | 0.9737 | 0.9280 | 0.8469 | 0.6045 | 0.3468 |
| $n_1 = n_2 = 500$ | 1.0000 | 1.0000 | 0.9999 | 0.9989 | 0.9905 | 0.9538 | 0.6933 | 0.2980 |
| $n_1=20 n_2=50$ | 0.9711 | 0.9129 | 0.8390 | 0.7619 | 0.6826 | 0.6057 | 0.4697 | 0.3578 |
| $n_1=50 n_2=20$ | 0.9722 | 0.9147 | 0.8399 | 0.7620 | 0.6835 | 0.6077 | 0.4700 | 0.3563 |
| $n_1=50 n_2=100$ | 0.9940 | 0.9698 | 0.9258 | 0.8619 | 0.7856 | 0.6996 | 0.5262 | 0.3703 |
| $n_1=100 n_2=50$ | 0.9944 | 0.9705 | 0.9248 | 0.8608 | 0.7834 | 0.7002 | 0.5249 | 0.3713 |

Simulation 3. Suppose i th observation is in k th class, $X_{ki} \sim N(\mu_k, \Sigma)$, where $\mu_k = (\mu_{k1}, \mu_{k2}, \dots, \mu_{kp})'$ with $\mu_{1j} = 0$ for $1 \leq j \leq p$, $\mu_{2j} = 0.7$ for $1 \leq j \leq 100$ and $\mu_{2j} = 0$ otherwise and $k = 1, 2$. The covariance structure Σ consists of 5×5 blocks, each block of dimension 100×100 with (j, j') element $0.6^{|j-j'|}$. This simulation example is considered in Guo et al. [17] and Makinde [18].

Simulation 4. Suppose F and G are normal mixture distributions defined as

$$F = \begin{cases} N(\mu_1^1, \Sigma) & \text{with probability } p \\ N(\mu_1^2, \Sigma) & \text{with probability } 1 - p \end{cases} \quad \text{and} \quad G = \begin{cases} N(\mu_2^1, \Sigma) & \text{with probability } p \\ N(\mu_2^2, \Sigma) & \text{with probability } 1 - p \end{cases}$$

where $p \in (0,1)$ is the mixing proportion, $\mu_{1,j}^1 = 0$ for $1 \leq j \leq 500$, $\mu_{1,j}^2 = 0.5$ if $1 \leq j \leq 100$, $\mu_{1,j}^2 = 0$ if $101 \leq j \leq 500$, $\mu_{2,j}^1 = 0.5$ if $1 \leq j \leq 100$, $\mu_{2,j}^1 = 0$ if $101 \leq j \leq 500$, $\mu_{2,j}^2 = 0.7$ if $1 \leq j \leq 100$, $\mu_{2,j}^2 = 0$ if $101 \leq j \leq 500$ and Σ is as defined in Simulation 4.3.

Simulation 5. Suppose each experiment consists of measurements on independent features such that $X_{1j} \sim \text{cauchy}(0,1)$ if $1 \leq j \leq 160$ and $X_{1j} \sim \text{cauchy}(1,1)$ if $161 \leq j \leq 200$. Similarly, $X_{2j} \sim \text{cauchy}(0,1)$ if $1 \leq j \leq 200$.

Table 4 presents three simulation studies on four modified versions of Hotelling's T^2 test for high dimensional data using sample sizes (20, 30, 50 and 100). In simulation 3, both the diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test perform equally with high power of 1.000 for various sample

sizes. The robust versions of the diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test perform equivalently.

In simulation 4, the power of the diagonalized Hotelling's T^2 test and that of the regularized Hotelling's T^2 test is the same as that of the robust versions of the diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test for sample size of 20. When the sample size is 30, the regularized Hotelling's T^2 test performs better than the diagonalized Hotelling's T^2 test while the robust version of the diagonalized Hotelling's T^2 test performs better than the usual diagonalized Hotelling's T^2 test. When the sample size increases to 50, the power of the diagonalized Hotelling's T^2 test dropped compared to that of the regularized Hotelling's T^2 test and the robust version for the diagonalized Hotelling's T^2 test perform much better than the usual diagonalized Hotelling's T^2 test. When the sample size increases to 100, the diagonalized Hotelling's T^2 test has a very low power. However, the robust versions of the diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test perform equivalently as the usual regularized Hotelling's T^2 test in terms of their powers.

Table 4. Comparison of powers of diagonalized, regularized Hotelling's T^2 test and their robust versions in high dimension at 5% level of significance

| Simulation | Dimension | Sample sizes | Diagonalized Hotelling T^2 Test | Regularized Hotelling T^2 Test | Robust Diagonalized Hotelling's T^2 test | Robust Regularized Hotelling's T^2 test |
|------------|-----------|---------------|-----------------------------------|----------------------------------|--|---|
| 3 | 500 | $n = m = 20$ | 1.000 | 1.000 | 1.000 | 1.000 |
| | | $n = m = 30$ | 1.000 | 1.000 | 1.000 | 1.000 |
| | | $n = m = 50$ | 1.000 | 1.000 | 1.000 | 1.000 |
| | | $n = m = 100$ | 1.000 | 1.000 | 1.000 | 1.000 |
| 4 | 500 | $n = m = 20$ | 1.000 | 1.000 | 1.000 | 1.000 |
| | | $n = m = 30$ | 0.988 | 1.000 | 1.000 | 1.000 |
| | | $n = m = 50$ | 0.124 | 1.000 | 1.000 | 1.000 |
| | | $n = m = 100$ | 0.000 | 1.000 | 1.000 | 1.000 |
| 5 | 200 | $n = m = 20$ | 1.000 | 1.000 | 1.000 | 1.000 |
| | | $n = m = 30$ | 1.000 | 1.000 | 1.000 | 1.000 |
| | | $n = m = 50$ | 0.208 | 1.000 | 1.000 | 1.000 |
| | | $n = m = 100$ | 0.000 | 0.000 | 1.000 | 1.000 |

In simulation 5, both the diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test have equal power performance with the robust versions for small sample sizes (20 and 30). As the sample sizes increase to 50, the power of diagonalized Hotelling's T^2 test reduce while the robust version power is very high. Both the regularized Hotelling's T^2 test and the robust version are the same with high performance. At same size 100, the power of diagonalized Hotelling's T^2 test and the regularized Hotelling's T^2 test goes to zero while their robust versions maintain high power. In general, the robust versions of the diagonalized Hotelling's T^2 test and the regularized Hotelling's T^2 test performs better than the diagonalized Hotelling's T^2 test and the regularized Hotelling's T^2 test which can therefore be implemented for high dimensional data.

3.2. Analysis of Real Data

We analyse five real data sets to illustrate the performance of the proposed robust diagonalized Hotelling's T^2 test and robust regularized Hotelling's T^2 test in high dimensional data. The real data sets are Arcene data, Madelon data, Small round blue cell tumor data set (SRBCT), Gisette data and colon data. Arcene data consists of 10,000 variables and two groups, positive and negative examples with sizes 56 and 44 respectively. A random subset of 500 variables are selected for each of the two groups. The experiment was

repeated 500 times. In each replication, p-value is estimated and power is determined as the proportion of p-values less than level of significance.

Madelon data is an artificial dataset, which consists of two groups with 500 continuous input variables. The groups are positive and negative, a random sample of size 200 is selected from each of the groups. The experiment was repeated 500 times. In each replication, p-value is estimated and power is determined as the proportion of p-values less than level of significance.

Small round blue cell tumor data set, denoted by SRBCT, consists of gene expression level on 2308 genes for 83 patients. This dataset arose from the study of Khan *et al.* [19] on childhood cancer and is available on R package *rda*. The data set contains four classes; Ewing sarcoma (ES) of size 29, Burkitt lymphoma (BL) of size 11, neuroblastoma (NB) of size 18 and rhabdomyosarcoma (RMS) of size 25. The four classes are categorized in two groups. Group 1 consists of ES and BL while group 2 consists of NB and RMS. A random sample of size 35 is taken from each group.

We analyse Gisette data. The data consist of two groups, each with size 300 and 50,001 features. Following Makinde and Fasoranbaku [20], zero and near-zero-valued variables are excluded in the data. A random sample of size 100 is selected from each of the groups with 2189 variables. Makinde and Fasoranbaku [20] used gisette data to validate maximum depth classifiers based on depth distribution approach. Colon tissue data set [21], denoted by colon, and contains 62 samples with 2000 genes from two classes. The classes are tumor tissues of size 40 and normal tissues of size 22. Random training samples of sizes 30 and 22 were selected from the two classes. Colon data has been used for gene expression data classification based on some distance-based methods [18].

Table 5 presents the powers of the diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test with the robust versions. The diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test and their robust versions achieve the statistical power of unity (power = 1) for the five. This shows that all these statistical tests perform well.

Table 5. Comparison of powers of diagonalized, regularized Hotelling's T^2 test and their robust versions for arcene and madelon data

| Real data | Diagonalized Hotelling's T^2 test | Regularized Hotelling's T^2 test | Robust Diagonalized Hotelling's T^2 test | Robust Regularized Hotelling's T^2 test |
|-----------|-------------------------------------|------------------------------------|--|---|
| Arcene | 1.000 | 1.000 | 1.000 | 1.000 |
| Madelon | 1.000 | 1.000 | 1.000 | 1.000 |
| SRBCT | 1.000 | 1.000 | 1.000 | 1.000 |
| Gisette | 1.000 | 1.000 | 1.000 | 1.000 |
| Colon | 1.000 | 1.000 | 1.000 | 1.000 |

4. CONCLUSION

This paper demonstrates equivalence in the performance of Hotelling's T^2 test, diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test in low dimension at different levels of significance. The performance of the regularized Hotelling's T^2 test depends on the values of the regularization parameter chosen. For high-dimensional data, the regularized Hotelling's T^2 test has higher power compared to the diagonalized Hotelling's T^2 test when the two competing samples have mixture distribution and their sample sizes are large. This study proposed robust diagonalized and regularized Hotelling's T^2 test and demonstrated better performance of the robust versions of the diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test over the existing diagonalized Hotelling's T^2 test and regularized Hotelling's T^2 test for simulated data. These tests were also applied on five deal datasets and all performed well in terms of their power.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

- [1] Hu, J., Bai, Z., “A review of 20 years of naive tests of significance for high-dimensional mean vectors and covariance matrices”, *Science China Mathematics*, 59(12): 2281-2300, (2016).
- [2] Capilla, C., “Application and Simulation on Study of the Hotelling’s T^2 Control Chart to monitor a wastewater treatment Process”, *Journal of Environmental Engineering Science*, 26(2): 333–42, (2009).
- [3] Bai, Z., Saranadasa, H., “Effect of high dimension: by an example of a two sample problem”, *Statistica Sinica*, 6: 311–29, (1996).
- [4] Dempster, A.P., “A high dimensional two sample significance test”, *The Annals of Mathematical Statistics*, 29(4): 995-1010, (1958).
- [5] Dempster, A.P., “A significance test for the separation of two highly multivariate small sample”, *Biometrics*, 16(1): 41-50, (1960).
- [6] Chen, S.X., Qin, Y.L., “A two-sample test for high-dimensional data with applications to gene set testing”, *The Annals of Statistics*, 38(2): 808–35, (2010).
- [7] Srivastava, M.S., Du, M., “A test for the mean vector with fewer observations than the dimension”, *Journal of Multivariate Analysis*, 99(3): 386–402, (2008).
- [8] Chattinnawat, W., Bilen, C., “Performance analysis of Hotelling T^2 under multivariate inspection errors”, *Quality Technology and Quantitative Management*, 14(3): 249 – 68, (2017).
- [9] Chen, L.S., Paul, D., Prentice, R.L., Wang, P., “A regularized Hotelling’s T^2 test for pathway analysis in proteomic studies”, *Journal of American Statistical Association*, (496): 1345 – 60, (2011).
- [10] Chen, S.X., Zhang, L.X., Zhong, P.S., “Tests for High-Dimensional Covariance Matrices”, *Journal of the American Statistical Association*, 105(490): 810-19, (2010).
- [11] Lu, Y., Liu, P.Y., Xiao, P., Deng, H.W., “Hotelling's T^2 multivariate profiling for detecting differential expression in microarrays”, *Bioinformatics*, 21(14): 3105-13, (2005).
- [12] Jureckova, J., Kalina, J., “Nonparametric multivariate rank tests and their unbiasedness”, *Bernoulli*, 18(1): 229-51, (2012).
- [13] Lopez-Pintado, S., Romo, J., “Depth based classification of functional data”, *DIMACS Series in Discrete Mathematics and Theoretical Computer Science. Data Depth: Robust Multivariate Analysis, Computational Geometry and Applications*. American Mathematical Society, 72: 103-20, (2006).
- [14] Liu, R., Singh, K., “A quality index based on data depth and multivariate rank test”, *Journal of the American Statistical Association*, 88(421): 252-59, (1993).
- [15] Zuo, Y., Serfling, R., “General notions of Statistical depth function”. *The Annals of Statistics*, 28(2): 461 – 82, (2000).

- [16] Makinde, O.S., Adewumi, A.D., “A comparison of depth functions in maximal depth classification rules”, *Journal of Modern Applied Statistics and Methods*, 16(1): 388 – 405, (2017).
- [17] Guo, Y., Hastie, T., Tibshirani, R., “Regularized linear discriminant analysis and its application in micro-arrays”, *Biostatistics*, 8: 86-100, (2007).
- [18] Makinde, O.S., “Gene expression data classification: some distance-based methods”, *Kuwait Journal of Science*, 46(3): 31-39, (2019).
- [19] Khan, J., Wei, J.S., Ringner, M., Saal, L.H., Ladanyi, M., Westermann, F., Berthold, F., Schwab, M., Antonescu, C.R., Peterson, C., Meltzer, P.S., "Classification and diagnostic prediction of cancers using gene expression profiling and artificial neural networks", *Nature Medicine*, 7: 673-79, (2001).
- [20] Makinde, O.S., Fasoranbaku. O.A., "On maximum depth classifiers: depth distribution approach", *Journal of Applied Statistics*, 45(6): 1106-17, (2018).
- [21] Alon, U., Barkai, N., Notterman, D.A., Gish, K., Ybarra, S., Mack, D., Levine, A.J., "Broad patterns of gene expression revealed by clustering analysis of tumor and normal colon tissues probed by oligonucleotide arrays", *Proceedings of the National Academy of Sciences of the United States of America*, 96(12): 6745-50, (1999).