



On Cauchy Numbers and Their Generalizations

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Highlights

- This paper focuses on a generalization of Cauchy numbers of both kinds.
- Some properties of this generalization are examined.
- Several new identities for classical Cauchy numbers of both kinds are obtained.

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Abstract

This paper is concerned with both kinds of the Cauchy numbers and their generalizations. Taking into account Mellin derivative, we relate p -Cauchy numbers of the second kind with shifted Cauchy numbers of the first kind, which yields new explicit formulas for the Cauchy numbers of the both kind. We introduce a generalization of the Cauchy numbers and investigate several properties, including recurrence relations, convolution identities and generating functions. In particular, these results give rise to new identities for Cauchy numbers.

1. INTRODUCTION

The Cauchy numbers of the first kind (CNFK) c_n and the Cauchy numbers of the second kind (CNSK) \hat{c}_n are given by the integral representations [1]

$$c_n = \int_0^1 (x)^{\underline{n}} dx \quad \text{and} \quad \hat{c}_n = \int_0^1 (x)^{\overline{n}} dx,$$

where $(x)^{\underline{n}} = x(x-1)\cdots(x-n+1)$ with $(x)^{\underline{0}} = 1$ and $(x)^{\overline{n}} = x(x+1)\cdots(x+n-1)$ with $(x)^{\overline{0}} = 1$. The numbers c_n , also known as Bernoulli numbers of the second kind b_n with the relation $c_n = n! b_n$ (see [2, 3]), have the generating function

$$\sum_{n=0}^{\infty} c_n \frac{z^n}{n!} = \frac{z}{\ln(1+z)} \quad (1)$$

and satisfy the combinatorial formula

$$c_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{n-k}}{k+1}. \quad (2)$$

Here $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ are the (unsigned) Stirling numbers of the first kind, defined by

$$\frac{\ln^k(1+z)}{k!} = \sum_{n=k}^{\infty} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] (-1)^{n-k} \frac{z^n}{n!}. \quad (3)$$

The numbers \hat{c}_n are generated by the function

$$\sum_{n=0}^{\infty} \hat{c}_n \frac{z^n}{n!} = \frac{z}{(1+z)\ln(1+z)}$$

and have an explicit formula

$$\hat{c}_n = (-1)^n \sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \frac{1}{k+1}. \quad (4)$$

It is good to note that the numbers \hat{c}_n are also called as Nörlund numbers and denoted by $B_n^{(n)}$ [4].

Recently, many studies have been carried out on generalizations of Cauchy numbers, according to the generating functions [5, 6], integral representations [7, 8, 9] and explicit formulas [10, 11, 12]. One of the generalizations of c_n (resp. \hat{c}_n) is $c_n^{(q)}(a)$, called shifted poly-CNFK, (resp. $\hat{c}_n^{(q)}(a)$ shifted poly-CNSK) and defined by [13]

$$c_n^{(q)}(a) = \int \underbrace{\int_0^1 \cdots \int_0^1}_{q \text{ times}} (x_1 \cdots x_q)^{a-1} (x_1 \cdots x_q)^{\bar{n}} dx_1 \cdots dx_q, \quad (5)$$

$$\hat{c}_n^{(q)}(a) = \int \underbrace{\int_0^1 \cdots \int_0^1}_{q \text{ times}} (x_1 \cdots x_q)^{a-1} (x_1 \cdots x_q)^{\bar{n}} dx_1 \cdots dx_q.$$

These numbers have the generating functions

$$\sum_{n=0}^{\infty} c_n^{(q)}(a) \frac{z^n}{n!} = e_q(\ln(1+z), a), \quad (6)$$

and

$$\sum_{n=0}^{\infty} \hat{c}_n^{(q)}(a) \frac{z^n}{n!} = e_q(-\ln(1+z), a), \quad (7)$$

where $e_q(z, a)$ is the poly-exponential function defined by [14]

$$e_q(z, a) = \sum_{k=0}^{\infty} \frac{z^k}{k! (k+a)^q}.$$

Moreover, these numbers can be written explicitly as [13]

$$c_n^{(q)}(a) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{n-k}}{(k+a)^q} \text{ and } \hat{c}_n^{(q)}(a) = (-1)^n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{1}{(k+a)^q}.$$

If $q = 1$ and $a = q = 1$, we have shifted Cauchy and Cauchy numbers of the both kinds (CNBK), respectively.

Another generalization of Cauchy numbers is the p -Cauchy numbers. The p -CNFK $c_{n,p}$ and the p -CNSK $\hat{c}_{n,p}$ are defined by [15]

$$\begin{aligned} \frac{c_{n,p}}{p+1} &= \int_0^1 (1-x)^p (x)^n dx, \\ \frac{\hat{c}_{n,p}}{p+1} &= \int_0^1 (1-x)^p (x)^{\bar{n}} dx \end{aligned} \quad (8)$$

and satisfy the explicit formulas

$$c_{n,p} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} \binom{k+p+1}{k}^{-1} \text{ and } \hat{c}_{n,p} = (-1)^n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \binom{k+p+1}{k}^{-1}.$$

In particular, we have $c_{n,0} = c_n$ and $\hat{c}_{n,0} = \hat{c}_n$.

This study is composed of two main parts. In the first part, we obtain a relation between p -CNSK and shifted CNFK. This relation gives rise to some new results for the p -CNBK. In the second part, we introduce a further generalization of CNBK, including both shifted poly-CNBK and p -CNBK. So, we call these numbers *shifted poly- p -Cauchy numbers of the both kinds*. We then examine several properties of the shifted poly- p -CNBK, such as recurrence relations, convolution identities and generating functions. In particular, we reach new explicit formulas and recurrence relations for shifted poly-Cauchy, p -Cauchy and Cauchy numbers of the both kinds.

2. p -CAUCHY NUMBERS

In this section, we express p -Cauchy numbers in terms of Cauchy numbers. Moreover, we relate p -CNSK and shifted CNFK. Then, we obtain new explicit formulas for CNBK.

Komatsu [13] showed that

$$c_n^{(q)}(a) = \sum_{\mu=0}^{a-1} \sum_{i=0}^{a-\mu-1} \binom{a-1}{i} \left\{ \begin{matrix} a-i-1 \\ \mu \end{matrix} \right\} n^i c_{n+\mu}^{(q)},$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the Stirling numbers of the second kind, a is a positive integer, n is a non-zero integer and $0 \leq \mu \leq a-1$. In the following theorem, we give a different relation for shifted poly-CNFK.

Theorem 1. For all integers $n, p \geq 0, q \geq 1$ and positive real number a , we have

$$c_n^{(q)}(a+p) = \sum_{k=0}^p \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \sum_{l=0}^k \binom{k}{l} \binom{n}{l} l! c_{n+k-l}^{(q)}. \quad (9)$$

Proof. Since

$$\sum_{n=0}^{\infty} (x)^n \frac{z^n}{n!} = (1+z)^x, \tag{10}$$

(5) can be rewritten as

$$\sum_{n=0}^{\infty} c_n^{(q)}(a) \frac{z^n}{n!} = \underbrace{\int_0^1 \cdots \int_0^1}_{q \text{ times}} (x_1 \cdots x_q)^{a-1} (1+z)^{x_1 \cdots x_q} dx_1 \cdots dx_q. \tag{11}$$

Applying the Mellin derivative, defined by $\left((1+z) \frac{d}{dz} \right)^p = \sum_{k=0}^p \left\{ \begin{matrix} p \\ k \end{matrix} \right\} (1+z)^k \frac{d^k}{dz^k}$ to the both sides of (11), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_n^{(q)}(a+p) \frac{z^n}{n!} &= \underbrace{\int_0^1 \cdots \int_0^1}_{q \text{ times}} (x_1 \cdots x_q)^{a+p-1} (1+z)^{x_1 \cdots x_q} dx_1 \cdots dx_q = \left((1+z) \frac{d}{dz} \right)^p \sum_{n=0}^{\infty} c_n^{(q)}(a) \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \left[\sum_{k=0}^p \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \sum_{l=0}^k \binom{k}{l} \binom{n}{l} l! c_{n+k-l}^{(q)}(a) \right] \frac{z^n}{n!}, \end{aligned}$$

which give the desired equation.

Setting $a = q = 1$ in (9) and comparing with the following formula for p -CNSK [15]

$$\frac{\hat{c}_{n-1,p+1}}{p+2} = \sum_{k=0}^p \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \sum_{l=0}^k \binom{k}{l} \binom{n}{l} l! c_{n+k-l}$$

give the following relation.

Corollary 2. For $n \geq 1$ and $p \geq 0$,

$$c_n(p+1) = \frac{1}{p+2} \hat{c}_{n-1,p+1}. \tag{12}$$

It is good to note that taking $p = 0$ in (12) and using the relation [15]

$$\hat{c}_{n+1,p} = \frac{p+1}{p+2} \hat{c}_{n,p+1} - (n+1) \hat{c}_{n,p},$$

we have the well-known identity for the Cauchy numbers

$$c_n = \hat{c}_n + n \hat{c}_{n-1}. \tag{13}$$

Moreover, from (12), we reach new explicit formulas as given in the following corollary.

Corollary 3. For $n, p \geq 0$,

$$\frac{1}{p+1} \hat{c}_{n,p} = \sum_{k=0}^n \frac{[n+1]}{[k+1]} \frac{(-1)^{n-k}}{k+p+1} \tag{14}$$

and

$$c_{n+1}(p+1) = \frac{(-1)^n}{p+2} \sum_{k=0}^n [n] \binom{k+p+2}{k}^{-1}. \tag{15}$$

When $p = 0$, we have

$$\hat{c}_n = \sum_{k=0}^n \frac{[n+1]}{[k+1]} \frac{(-1)^{n-k}}{k+1} \tag{16}$$

and

$$c_{n+1} = (-1)^n \sum_{k=0}^n [n] \frac{1}{(k+1)(k+2)}. \tag{17}$$

Now, we want to express p -CNFK in terms of CNFK. From (8), we have

$$\frac{1}{p+1} \sum_{n=0}^{\infty} c_{n,p} \frac{z^n}{n!} = \int_0^1 (1-x)^p (1+z)^x dx = \sum_{k=0}^p \binom{p}{k} (-1)^k \int_0^1 x^k (1+z)^x dx.$$

Setting $q = 1$ in (11) and using it in the above equation, we obtain

$$\frac{1}{p+1} \sum_{n=0}^{\infty} c_{n,p} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{k=0}^p \binom{p}{k} (-1)^k c_n (k+1) \right] \frac{z^n}{n!},$$

which can be written

$$\frac{1}{p+1} c_{n,p} = \sum_{k=0}^p \binom{p}{k} (-1)^k c_n (k+1). \tag{18}$$

Then, for $a = q = 1$, using (9), we have the following theorem.

Theorem 4. For $n, p \geq 0$,

$$\frac{1}{p+1} c_{n,p} = \sum_{i=0}^p \sum_{k=i}^p \sum_{l=0}^i \binom{p}{k} \{k\}_i \binom{i}{l} \binom{n}{l} l! (-1)^k c_{n+k-l}.$$

Now, we want to recall the binomial transform which will be useful in the next theorem:

$$a_n = \sum_{k=0}^n (-1)^k \binom{n}{k} b_k \quad (n \geq 0) \text{ if and only if } b_n = \sum_{k=0}^n (-1)^k \binom{n}{k} a_k.$$

Using this and (18) give

$$c_n(p + 1) = \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k c_{n,k}}{k + 1}.$$

From (12), the above equation can be written as

$$\frac{1}{p + 2} \hat{c}_{n,p+1} = \sum_{k=0}^p \binom{p}{k} \frac{(-1)^k c_{n+1,k}}{k + 1},$$

which is also a new recurrence relation between p -CNBK. For nonnegative integers n and p , utilizing [15, Theorem 4.5]

$$\frac{(-1)^p}{p + 1} c_{n,p} = \sum_{k=0}^p \left\{ \begin{matrix} p \\ k \end{matrix} \right\} \sum_{j=0}^{k+1} \binom{k + 1}{j} \binom{n}{j} j! \hat{c}_{n-j+k}$$

in the above equation, we have

$$\frac{1}{p + 2} \hat{c}_{n,p+1} = \sum_{k=0}^p \sum_{i=0}^k \sum_{l=0}^{i+1} \binom{p}{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} \binom{i + 1}{l} \binom{n + 1}{l} l! \hat{c}_{n+i+1-l}.$$

Since

$$\sum_{k=i}^n \binom{n}{k} \left\{ \begin{matrix} k \\ i \end{matrix} \right\} = \left\{ \begin{matrix} n + 1 \\ i + 1 \end{matrix} \right\}$$

we obtain a relation between p -CNSK and CNSK.

Theorem 5. For $n, p \geq 0$, we have

$$\frac{1}{p + 2} \hat{c}_{n,p+1} = \sum_{i=0}^p \left\{ \begin{matrix} p + 1 \\ i + 1 \end{matrix} \right\} \sum_{l=0}^{i+1} \binom{i + 1}{l} \binom{n + 1}{l} l! \hat{c}_{n+i+1-l}.$$

It is known that [13]

$$(-1)^n c_n^{(q)}(a) = \sum_{m=1}^n \left[\begin{matrix} n \\ m \end{matrix} \right] \hat{c}_m^{(q)}(a), \tag{19}$$

where $\left[\begin{matrix} n \\ m \end{matrix} \right]$ is the Lah numbers [16] defined by $\left[\begin{matrix} n \\ m \end{matrix} \right] = \binom{n - 1}{m - 1} \frac{n!}{m!}$. Setting $q = 1$ and $a = p + 1$ in (19) and using (12), we obtain a relation between p -CNSK and shifted CNSK.

Corollary 6. For $n, p \geq 0$, we have

$$\frac{(-1)^{n+1}}{p + 2} \hat{c}_{n,p+1} = \sum_{m=0}^n \left[\begin{matrix} n + 1 \\ m + 1 \end{matrix} \right] \hat{c}_{m+1}(p + 1).$$

3. SHIFTED POLY- p -CAUCHY NUMBERS OF THE FIRST KIND

Let $n \geq 0, q \geq 1, p \geq 0$ be integers. Let a be a positive real number. We define a sequence of rational numbers $c_{n,p}^{(q)}(a)$ by

$$\frac{1}{p+1} c_{n,p}^{(q)}(a) = \underbrace{\int_0^1 \cdots \int_0^1}_{q \text{ times}} (1-x_1)^p (x_2 \cdots x_q)^{a-1} (x_1 \cdots x_q)^n dx_1 \cdots dx_q. \tag{20}$$

Since $(x)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} x^k$, we obtain the following explicit formula for $c_{n,p}^{(q)}(a)$.

Theorem 7. For all integers $n \geq 0, q \geq 1, p \geq -1$ and a positive real number a , we have

$$c_{n,p}^{(q)}(a) = \sum_{k=0}^n \binom{n}{k} \binom{k+p+1}{k}^{-1} \frac{(-1)^{n-k}}{(k+a)^{q-1}}. \tag{21}$$

From (21), it is evident that

$$c_{n,p}^{(1)}(a) = c_{n,p}, c_{n,-1}^{(q)}(a) = c_n^{(q-1)}(a), c_{n,0}^{(q)}(1) = c_n^{(q)}, c_{n,0}^{(1)}(a) = c_{n,-1}^{(2)}(1) = c_n,$$

where $c_n^{(q)}$ is the n -th poly-CNFK [8]. Because of these relations we may call shifted poly- p -CNFK for $c_{n,p}^{(q)}(a)$.

Using Stirling transform, one can obtain that

$$\sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} c_{k,p}^{(q)}(a) = \binom{k+p+1}{k}^{-1} \frac{1}{(k+a)^{q-1}}.$$

We also note that such relations have been obtained for poly-Cauchy numbers in [8], for shifted poly-Cauchy numbers in [13] and for p -Cauchy numbers in [15]. Now, we want to generalize these relations with the use of generalized Stirling transform [17, Corollary 1]. We then obtain a relation between r -Stirling numbers of the second kind $\left\{ \begin{matrix} n+r \\ m+r \end{matrix} \right\}_r$ and shifted poly- p -CNFK.

Corollary 8. For $m \geq 0$, we have

$$\sum_{k=0}^n \left\{ \begin{matrix} n+m \\ k+m \end{matrix} \right\}_m c_{m+k,p}^{(q)}(a) = \sum_{k=0}^m \binom{m}{k} \binom{k+n+p+1}{k+n}^{-1} \frac{(-1)^{m-k}}{(k+n+a)^{q-1}}.$$

Using (10) and (20), the generating function of the number $c_{n,p}^{(q)}(a)$ satisfy the following iterated integrals.

Theorem 9. For all integers $q \geq 1, p \geq 0$ and a real number $a \geq 1$, we have

$$f_{p,q}(z, a) = \frac{1}{p+1} \sum_{n=0}^{\infty} c_{n,p}^{(q)}(a) \frac{z^n}{n!} = \underbrace{\int_0^1 \cdots \int_0^1}_{q \text{ times}} (1-x_1)^p (x_2 \cdots x_q)^{a-1} (1+z)^{x_1 \cdots x_q} dx_1 \cdots dx_q. \tag{22}$$

For integers $p \geq -1$ and q and a positive real number a , define the function $e_{p,q}(z, a)$ by

$$e_{p,q}(z, a) = \sum_{k=0}^{\infty} \frac{z^k}{(p)^{\bar{k}}(k+a)^{\bar{q}}}.$$

When $p = 1$, we have $e_{1,q}(z, a) = e_q(z, a)$. Using (21), we have

$$\sum_{n=0}^{\infty} c_{n,p}^{(q)}(a) \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{k!}{(p+2)^{\bar{k}}(k+a)^{q-1}} \sum_{n=k}^{\infty} [n] (-1)^{n-k} \frac{z^n}{n!} = \sum_{k=0}^{\infty} \frac{k!}{(p+2)^{\bar{k}}(k+a)^{q-1}} \frac{\ln^k(1+z)}{k!},$$

which can be stated as the following theorem.

Theorem 10. *The number $c_{n,p}^{(q)}(a)$ have the following generating function*

$$\sum_{n=0}^{\infty} c_{n,p}^{(q)}(a) \frac{z^n}{n!} = e_{p+2,q-1}(\ln(1+z), a). \tag{23}$$

The first few generating functions for $c_{n,p}^{(q)}(a)$ (for $p = 0,1,2$) are

$$\begin{aligned} \sum_{n=0}^{\infty} c_{n,0}^{(q)}(a) \frac{z^n}{n!} &= \frac{-1}{(a-1)^{q-1} \ln(1+z)} + \frac{e_{q-1}(\ln(1+z), a-1)}{\ln(1+z)}, \\ \sum_{n=0}^{\infty} c_{n,1}^{(q)}(a) \frac{z^n}{n!} &= \frac{-2}{(a-2)^{q-1} \ln^2(1+z)} - \frac{2}{(a-1)^{q-1} \ln(1+z)} + \frac{2e_{q-1}(\ln(1+z), a-2)}{\ln^2(1+z)}, \\ \sum_{n=0}^{\infty} c_{n,2}^{(q)}(a) \frac{z^n}{n!} &= \frac{-6}{(a-3)^{q-1} \ln^3(1+z)} - \frac{6}{(a-2)^{q-1} \ln^2(1+z)} - \frac{3}{(a-1)^{q-1} \ln(1+z)} \\ &\quad + \frac{6e_{q-1}(\ln(1+z), a-3)}{\ln^3(1+z)}. \end{aligned}$$

In general, we state the following closed formula for the generating function of shifted poly- p -CNFK.

Theorem 11. *For integers $q \geq 1, p \geq 0$ and a real number $a \geq p + 2$,*

$$\sum_{n=0}^{\infty} c_{n,p}^{(q)}(a) \frac{z^n}{n!} = \frac{(p+1)!}{\ln^{p+1}(1+z)} e_{q-1}(\ln(1+z), a-p-1) - \sum_{i=0}^p \frac{(p+1)^{i+1}}{(a-i-1)^{q-1} \ln^{i+1}(1+z)}. \tag{24}$$

Proof. After an integration by parts, (22) can be written as

$$f_{p,q}(z, a) = \frac{-1}{(a-1)^{q-1} \ln(1+z)} + \frac{p}{\ln(1+z)} f_{p-1,q}(z, a-1). \tag{25}$$

Applying inductively, we obtain

$$f_{p,q}(z, a) = \frac{p!}{\ln^p(1+z)} f_{0,q}(z, a-p) - \sum_{i=1}^p \frac{(p)^{i-1}}{(a-i)^{q-1} \ln^i(1+z)}. \tag{26}$$

Setting $p = 0$ and $a = a - p$ in (22), we have

$$\begin{aligned}
 f_{0,q}(z, a - p) &= \underbrace{\int_0^1 \cdots \int_0^1}_{q-1 \text{ times}} (x_2 \cdots x_q)^{a-p-1} \left[\int_0^1 (1+z)^{x_1 \cdots x_q} dx_1 \right] dx_2 \cdots dx_q \\
 &= \frac{1}{\ln(1+z)} \left[\sum_{n=0}^{\infty} c_n^{(q-1)} (a-p-1) \frac{z^n}{n!} - \frac{1}{(a-p-1)^{q-1}} \right].
 \end{aligned}$$

Using (6) in the above equation yields the formula.

It is good to note that setting $q = 1$ in (24) gives [15]

$$\sum_{n=0}^{\infty} c_{n,p} \frac{z^n}{n!} = \frac{(p+1)!(1+z)}{\ln^{p+1}(1+z)} - \sum_{i=0}^p \frac{(p+1)^{i+1}}{\ln^{i+1}(1+z)}. \tag{27}$$

Moreover, we have the following formula as an application of Theorem 11.

Theorem 12. For all integers $p \geq 0, n \geq p + 1, q \geq 2$ and real numbers $a \geq p + 2$, we have

$$\begin{aligned}
 \sum_{k=p+1}^n \binom{n}{k} \left[\begin{matrix} k \\ p+1 \end{matrix} \right] (-1)^{k+p+1} c_{n-k,p}^{(q)}(a) \\
 = c_n^{(q-1)}(a-p-1) + (-1)^{n+1} \sum_{i=0}^p \left[\begin{matrix} n \\ i \end{matrix} \right] \frac{(-1)^i}{(a+i-p-1)^{q-1}}.
 \end{aligned} \tag{28}$$

For $q = 1$, we have

$$\sum_{k=p+1}^n \binom{n}{k} \left[\begin{matrix} k \\ p+1 \end{matrix} \right] (-1)^{k+p+1} c_{n-k,p} = (-1)^{n+1} \sum_{i=0}^p \left[\begin{matrix} n \\ i \end{matrix} \right] (-1)^i, \quad n \geq 2. \tag{29}$$

We remark that for $p = 0$ in (29), using the relation $c_n = n! b_n$ and the well-known identity $\left[\begin{matrix} n \\ 1 \end{matrix} \right] = (n-1)!$, we have

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (k-1)! c_{n-k} = 0, \quad n \geq 2. \tag{30}$$

which was proved by Agoh and Dilcher [2] with a different method.

Proof of Theorem 12. Multiply both sides of (24) by $\frac{\ln^{p+1}(1+z)}{(p+1)!}$ and use (3) to obtain (28). Moreover, multiply both sides of (27) by $\frac{\ln^{p+1}(1+z)}{(p+1)!}$ and use (3) to have (29).

Now, we want to deal with some recurrence relations of shifted poly- p -CNFK. We first have the following:

Theorem 13. For all integers $n \geq 0, q \geq 1$ and real numbers $a \geq 1$, we have

$$c_{n+1,p}^{(q)}(a) + n c_{n,p}^{(q)}(a) = \frac{-(p+1)}{p+2} c_{n,p+1}^{(q)}(a+1) + c_{n,p}^{(q)}(a+1), \quad p \geq -1 \tag{31}$$

and

$$(a - p - 1)c_{n,p}^{(q+1)}(a) = c_{n,p}^{(q)}(a) - (p + 1)c_{n,p-1}^{(q+1)}(a), \quad p \geq 0. \quad (32)$$

Proof. Differentiate both sides of (22) with respect to z and multiply it by $(1 + z)$ to obtain

$$\begin{aligned} (1+z) \frac{d}{dz} f_{p,q}(z, a) &= \int_0^1 \cdots \int_0^1 \underbrace{(1-x_1)^p x_1 (x_2 \cdots x_q)^a}_{q \text{ times}} (1+z)^{x_1 \cdots x_q} dx_1 \cdots dx_q \\ &= - \int_0^1 \cdots \int_0^1 \underbrace{(1-x_1)^{p+1} (x_2 \cdots x_q)^a}_{q \text{ times}} (1+z)^{x_1 \cdots x_q} dx_1 \cdots dx_q \\ &\quad + \int_0^1 \cdots \int_0^1 \underbrace{(1-x_1)^p (x_2 \cdots x_q)^a}_{q \text{ times}} (1+z)^{x_1 \cdots x_q} dx_1 \cdots dx_q, \end{aligned}$$

or, equivalently,

$$\frac{(1+z)}{p+1} \sum_{n=0}^{\infty} c_{n+1,p}^{(q)}(a) \frac{z^n}{n!} = \frac{-1}{p+2} \sum_{n=0}^{\infty} c_{n,p+1}^{(q)}(a+1) \frac{z^n}{n!} + \frac{1}{p+1} \sum_{n=0}^{\infty} c_{n,p}^{(q)}(a+1) \frac{z^n}{n!}.$$

After some rearrangement, we obtain (31).

One can obtain that

$$e_{p,q}(x, a) = \frac{1}{a^q} {}_pF_{q+1} \left(\begin{matrix} 1, a, \dots, a \\ p, a+1, \dots, a+1 \end{matrix}; x \right), \quad (33)$$

where ${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x \right)$ is the generalized hypergeometric function defined by

$${}_pF_q \left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; x \right) = \sum_{k=0}^{\infty} \frac{(a_1)^{\bar{k}} \cdots (a_p)^{\bar{k}} x^k}{(b_1)^{\bar{k}} \cdots (b_q)^{\bar{k}} k!}.$$

For $p = p + 2$ and $x = \ln(1 + z)$, we have

$$\begin{aligned} &(a - p - 1)_{q+1} F_{q+1} \left(\begin{matrix} 1, a, \dots, a \\ p, a+1, \dots, a+1 \end{matrix}; \ln(1+z) \right) \\ &= a_{q+1} F_{q+1} \left(\begin{matrix} 1, a+1, a, \dots, a \\ p, a+1, \dots, a+1 \end{matrix}; \ln(1+z) \right) \\ &\quad - (p-1)_{q+1} F_{q+1} \left(\begin{matrix} 1, a, \dots, a \\ p-1, a+1, \dots, a+1 \end{matrix}; \ln(1+z) \right). \end{aligned}$$

From (23) and (33), we obtain

$$(a - p - 1) \sum_{n=0}^{\infty} c_{n,p}^{(q+1)}(a) \frac{z^n}{n!} = \sum_{n=0}^{\infty} c_{n,p}^{(q)}(a) \frac{z^n}{n!} - (p + 1) \sum_{n=0}^{\infty} c_{n,p-1}^{(q+1)}(a) \frac{z^n}{n!},$$

which gives (32).

Secondly, we obtain the following recurrence relations which are generalizations of (30).

Theorem 14. For all integers $n \geq 1, q \geq 1, p \geq 0$ and real number $a > 1$ we have

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (k - 1)! c_{n-k,p}^{(q)}(a) = (p + 1) c_{n,p-1}^{(q)}(a - 1). \tag{34}$$

In particular,

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (k - 1)! c_{n-k}^{(q)}(a) = c_n^{(q-1)}(a - 1) - (a - 1) c_n^{(q)}(a - 1), q \geq 2. \tag{35}$$

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (k - 1)! c_{n-k,p} = (p + 1) c_{n,p-1}, p \geq 1. \tag{36}$$

Proof. Considering (23) and (25) yield

$$\frac{\ln(1 + z)}{p + 1} \sum_{n=0}^{\infty} c_{n,p}^{(q)}(a) \frac{z^n}{n!} = \sum_{n=1}^{\infty} c_{n,p-1}^{(q)}(a - 1) \frac{z^n}{n!}.$$

Then, we have

$$\begin{aligned} (p + 1) \sum_{n=1}^{\infty} c_{n,p-1}^{(q)}(a - 1) \frac{z^n}{n!} &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^k c_{n,p}^{(q)}(a)}{(k + 1)n!} z^{n+k+1} \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (k - 1)! c_{n-k,p}^{(q)}(a) \frac{z^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ in the both sides of the above equation gives (34).

From (32), (34) can be written as

$$\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (k - 1)! c_{n-k,p}^{(q)}(a) = c_{n,p}^{(q-1)}(a - 1) - (a - p - 2) c_{n,p}^{(q)}(a - 1).$$

Setting $p = -1$ in the above equation yields (35).

Komatsu [18] showed that

$$\sum_{k=0}^n \binom{n}{k} c_k c_{n-k}^{(q)} = c_n + n c_{n-1} - n \sum_{j=1}^q (c_n^{(j)} + (n - 1) c_{n-1}^{(j)}), \tag{37}$$

as a generalization of [19]

$$\sum_{k=0}^n \binom{n}{k} c_k c_{n-k} = -(n-1)c_n - n(n-2)c_{n-1}.$$

We give analogues of those results in the following theorem.

Theorem 15. For all integers $n \geq 1, q \geq 1, p \geq 0$ and real number $a > 1$, we have

$$\sum_{k=0}^n \binom{n}{k} c_k c_{n-k,p-1}^{(q)}(a-1) = \frac{c_n}{(a-1)^{q-1}} + \frac{nc_{n-1,p}^{(q)}(a)}{p+1}. \tag{38}$$

When $q = 1$, it becomes

$$\sum_{k=0}^n \binom{n}{k} c_k c_{n-k,p-1} = c_n + \frac{n}{p+1} c_{n-1,p}.$$

It is good to note that for $a = 2$ and $p = 0$ in (38), we have

$$\sum_{k=0}^n \binom{n}{k} c_k c_{n-k}^{(q-1)} = nc_{n-1,0}^{(q)}(2) + c_n.$$

Comparing (37) with the above equation, we conclude that

$$\sum_{j=1}^{q-1} \left(c_n^{(j)} + (n-1)c_{n-1}^{(j)} \right) = c_{n-1} - c_{n-1,0}^{(q)}(2). \tag{39}$$

On the other hand, setting $p = -1$ and $a = 1$ in (31), we obtain $c_{n+1}^{(q-1)} + nc_n^{(q-1)} = c_n^{(q-1)}(2)$. Using this and (39), we have another conclusion as

$$\sum_{j=1}^{q-1} c_n^{(j)}(2) = c_n - c_{n,0}^{(q)}(2).$$

Proof of Theorem 15. With use of (1), (23) and (25), we have

$$\begin{aligned} \frac{1}{p+1} \sum_{n=0}^{\infty} c_{n,p}^{(q)}(a) \frac{z^{n+1}}{n!} + \frac{1}{(a-1)^{q-1}} \sum_{n=0}^{\infty} c_n \frac{z^n}{n!} &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{c_k c_{n,p-1}^{(q)}(a-1)}{k! n!} z^{n+k} \\ &= \sum_{n=0}^{\infty} \left[\sum_{k=0}^n \binom{n}{k} c_k c_{n-k,p-1}^{(q)}(a-1) \right] \frac{z^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{z^n}{n!}$ in the both sides of the above equation completes the proof.

4. SHIFTED POLY- p -CAUCHY NUMBERS OF THE SECOND KIND

Let a be a positive real number. We define shifted poly- p -CNSK $\hat{c}_{n,p}^{(q)}(a)$ ($n \geq 0, q \geq 1, p \geq 0$) by

$$\frac{1}{p+1} \hat{c}_{n,p}^{(q)}(a) = \underbrace{\int_0^1 \cdots \int_0^1}_{q \text{ times}} (1-x_1)^p (x_2 \cdots x_q)^{a-1} (-x_1 \cdots x_q)^n dx_1 \cdots dx_q. \tag{40}$$

Then, similar to Theorem 7, $\hat{c}_{n,p}^{(q)}(a)$ have the following explicit formula.

Theorem 16. For all integers $n \geq 0, q \geq 1, p \geq -1$ and positive real number a ,

$$\hat{c}_{n,p}^{(q)}(a) = (-1)^n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \binom{k+p+1}{k}^{-1} \frac{1}{(k+a)^{q-1}}.$$

In particular, we have

$$\hat{c}_{n,p}^{(1)}(a) = \hat{c}_{n,p}, \hat{c}_{n,-1}^{(q)}(a) = \hat{c}_n^{(q-1)}(a), \hat{c}_{n,0}^{(q)}(1) = \hat{c}_n^{(q)}, \hat{c}_{n,0}^{(1)}(a) = \hat{c}_{n,-1}^{(2)}(1) = \hat{c}_n,$$

where $\hat{c}_n^{(q)}$ is the n-th poly-CNSK.

The numbers $\hat{c}_{n,p}^{(q)}(a)$ have the following generating functions. The proof is similar to that of Theorem 9 and Theorem 10 and omitted.

Theorem 17. For all integers $q \geq 1, p \geq 0$ and positive real number a ,

$$\hat{f}_{p,q}(z, a) = \frac{1}{p+1} \sum_{n=0}^{\infty} \hat{c}_{n,p}^{(q)}(a) \frac{z^n}{n!} = \underbrace{\int_0^1 \cdots \int_0^1}_{q \text{ times}} (1-x_1)^p (x_2 \cdots x_q)^{a-1} (1+z)^{-x_1 \cdots x_q} dx_1 \cdots dx_q, \tag{41}$$

and

$$\sum_{n=0}^{\infty} \hat{c}_{n,p}^{(q)}(a) \frac{z^n}{n!} = e_{p+2,q-1}(-\ln(1+z), a) = \frac{1}{a^{q-1}} {}_qF_q \left(\begin{matrix} 1, a, \dots, a \\ p+2, a+1, \dots, a+1 \end{matrix}; -\ln(1+z) \right).$$

The first few generating function for $\hat{c}_{n,p}^{(q)}(a)$ (for $p = 0,1$) are

$$\sum_{n=0}^{\infty} \hat{c}_{n,0}^{(q)}(a) \frac{z^n}{n!} = \frac{1}{(a-1)^{q-1} \ln(1+z)} - \frac{e_{q-1}(-\ln(1+z), a-1)}{\ln(1+z)},$$

$$\sum_{n=0}^{\infty} \hat{c}_{n,1}^{(q)}(a) \frac{z^n}{n!} = \frac{-2}{(a-2)^{q-1} \ln^2(1+z)} + \frac{2}{(a-1)^{q-1} \ln(1+z)} + \frac{2e_{q-1}(-\ln(1+z), a-2)}{\ln^2(1+z)}.$$

In general, we have the following closed formula for the generating function of $\hat{c}_{n,p}^{(q)}(a)$.

Theorem 18. For integers $q \geq 2, p \geq 0$ and a real number $a \geq p+2$,

$$\sum_{n=0}^{\infty} \hat{c}_{n,p}^{(q)}(a) \frac{z^n}{n!} = \frac{(-1)^{p+1}(p+1)!}{\ln^{p+1}(1+z)} e_{q-1}(-\ln(1+z), a-p-1) + \sum_{i=0}^p \frac{(-1)^i (p+1)^{i+1}}{(a-i-1)^{q-1} \ln^{i+1}(1+z)}. \tag{42}$$

Proof. After an integration by parts, (41) can be written as

$$\hat{f}_{p,q}(z, a) = \frac{1}{(a-1)^{q-1} \ln(1+z)} - \frac{p}{\ln(1+z)} \hat{f}_{p-1,q}(z, a-1). \tag{43}$$

Applying inductively, we obtain

$$\hat{f}_{p,q}(z, a) = \frac{(-1)^p p!}{\ln^p(1+z)} \hat{f}_{0,q}(z, a-p) + \sum_{i=1}^p \frac{(-1)^{i+1} (p)^{i-1}}{(a-i)^{q-1} \ln^i(1+z)}. \tag{44}$$

Setting $p = 0, a = a - p$ in (41) and using (7), we have

$$f_{0,q}(z, a-p) = \frac{1}{\ln(1+z)} \left[-e_{q-1}(-\ln(1+z), a-p-1) + \frac{1}{(a-p-1)^{q-1}} \right].$$

Utilizing the above equation in (44) completes the proof.

We note that for $q = 1$ in (42), we have [15]

$$\sum_{n=0}^{\infty} \hat{c}_{n,p} \frac{z^n}{n!} = \frac{(-1)^{p+1}(p+1)!}{(1+z)\ln^{p+1}(1+z)} + \sum_{i=0}^p \frac{(-1)^i (p+1)^{i+1}}{\ln^{i+1}(1+z)}. \tag{45}$$

Multiplying both sides of (42) with $\frac{\ln^{p+1}(1+z)}{(p+1)!}$ and then applying the method used in the proof of Theorem 12, we arrive at the following:

Theorem 19. For integers $q \geq 2, n \geq p + 1, p \geq 0$ and a real number $a \geq p + 2,$

$$\sum_{k=p+1}^n \binom{n}{k} \left[\begin{matrix} k \\ p+1 \end{matrix} \right] (-1)^k \hat{c}_{n-k,p}^{(q)}(a) = \hat{c}_n^{(q-1)}(a-p-1) + (-1)^{n+1} \sum_{i=0}^p \left[\begin{matrix} n \\ i \end{matrix} \right] \frac{1}{(a+i-p-1)^{q-1}} \tag{46}$$

For $q = 1,$ we have

$$\sum_{k=p+1}^n \binom{n}{k} \left[\begin{matrix} k \\ p+1 \end{matrix} \right] (-1)^k \hat{c}_{n-k,p} = (-1)^n n! + (-1)^{n+1} \sum_{i=0}^p \left[\begin{matrix} n \\ i \end{matrix} \right], \quad n \geq 1. \tag{47}$$

Remark that for $p = 0,$ (47) becomes

$$\sum_{j=0}^{n-1} \frac{(-1)^j B_j^{(j)}}{n-j} = 1, \quad n \geq 1, \tag{48}$$

proved by Nörlund [4]. Moreover, using (12) in (47) yield $(n \geq p + 1, p \geq 1)$

$$\sum_{k=p+1}^n \binom{n}{k} \left[\begin{matrix} k \\ p+1 \end{matrix} \right] (-1)^k c_{n-k+1}(p) = \frac{(-1)^n n!}{p+1} + \frac{(-1)^{n+1}}{p+1} \sum_{i=0}^p \left[\begin{matrix} n \\ i \end{matrix} \right]. \tag{49}$$

Setting $p = 1$ in (49) and using the well-known identity

$$\left[\begin{matrix} n \\ 2 \end{matrix} \right] = (n-1)! H_{n-1},$$

where $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ is the harmonic number, we deduce an analogue of

$$\sum_{k=0}^{n-2} \frac{(-1)^k}{k!(n-k)} H_{n-k-1} c_k = \frac{1}{2(n-1)},$$

recorded by Agoh and Dilcher [2].

Corollary 20. For $n \geq 2$,

$$\sum_{k=0}^{n-2} \frac{(-1)^k}{k!(n-k)} H_{n-k-1} c_{k+1} = \frac{(n-1)}{2n}. \tag{50}$$

The following theorem relates both kinds of the shifted poly- p -Cauchy numbers.

Theorem 21. Let q be an integer and a be a positive real number. For $n \geq 1$ and $p \geq -1$, we have

$$(-1)^n \hat{c}_{n,p}^{(q)}(a) = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] c_{k,p}^{(q)}(a) \tag{51}$$

and

$$(-1)^n c_{n,p}^{(q)}(a) = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] \hat{c}_{k,p}^{(q)}(a). \tag{52}$$

Proof. With the use of $(-x)^{\bar{n}} = (-1)^n (x)^{\bar{n}}$ and $(x)^{\bar{n}} = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] (x)^{\underline{k}}$ in (40), we have

$$\frac{(-1)^n}{p+1} \hat{c}_{n,p}^{(q)}(a) = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] \underbrace{\int_0^1 \dots \int_0^1}_{q \text{ times}} (1-x_1)^p (x_2 \dots x_q)^{a-1} (x_1 \dots x_q)^k dx_1 \dots dx_q.$$

Then, from (20), we have the first identity. Moreover, let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of complex numbers. Then $b_n = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] a_k$ if and only if $a_n = \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] (-1)^{n-k} b_k$. Using this relation in the first identity yields the second formula.

These relations enable us to examine some properties of $\hat{c}_{n,p}^{(q)}(a)$ from $c_{n,p}^{(q)}(a)$. For instance, the following recurrence relations holds for shifted poly- p -CNSK.

Corollary 22. For all integers $n \geq 0$, $q \geq 1$ and real numbers $a \geq 1$, we have

$$\hat{c}_{n+1,p}^{(q)}(a) + n\hat{c}_{n,p}^{(q)}(a) = \frac{(p+1)}{p+2}\hat{c}_{n,p+1}^{(q)}(a+1) + \hat{c}_{n,p}^{(q)}(a+1), \quad p \geq -1 \tag{53}$$

and

$$(a-p-1)\hat{c}_{n,p}^{(q+1)}(a) = \hat{c}_{n,p}^{(q)}(a) - (p+1)\hat{c}_{n,p-1}^{(q+1)}(a), \quad p \geq 0. \tag{54}$$

Proof. Multiply both sides of (31) by $\begin{bmatrix} m \\ n \end{bmatrix}$ and sum over n from 1 to m to obtain

$$\sum_{n=1}^m \begin{bmatrix} m \\ n \end{bmatrix} c_{n+1,p}^{(q)}(a) + \sum_{n=1}^m \begin{bmatrix} m \\ n \end{bmatrix} n c_{n,p}^{(q)}(a) = \frac{-(p+1)}{p+2} \sum_{n=1}^m \begin{bmatrix} m \\ n \end{bmatrix} c_{n,p+1}^{(q)}(a+1) + \sum_{n=1}^m \begin{bmatrix} m \\ n \end{bmatrix} c_{n,p}^{(q)}(a+1).$$

Using (51) and the recurrence relation of Lah numbers

$$\begin{bmatrix} m+1 \\ n \end{bmatrix} = \begin{bmatrix} m \\ n-1 \end{bmatrix} + (m+n) \begin{bmatrix} m \\ n \end{bmatrix},$$

we obtain (53). Similarly, using (32) and (51) gives (54).

Using (41) and (43) and applying the same method used for the proof of Theorem 14, we obtain the following recurrence relations which are the analogues of (48).

Theorem 23. For all integers $n \geq 1$, and real number $a > 1$, we have

$$\sum_{k=1}^n (-1)^k \binom{n}{k} (k-1)! \hat{c}_{n-k,p}^{(q)}(a) = (p+1)\hat{c}_{n,p-1}^{(q)}(a-1), \quad q \geq 1, p \geq 0. \tag{55}$$

In particular,

$$\sum_{k=1}^n (-1)^k \binom{n}{k} (k-1)! \hat{c}_{n-k}^{(q)}(a) = \hat{c}_n^{(q-1)}(a-1) - (a-1)\hat{c}_n^{(q)}(a-1), \quad q \geq 2, \tag{56}$$

$$\sum_{k=1}^n (-1)^k \binom{n}{k} (k-1)! \hat{c}_{n-k,p} = (p+1)\hat{c}_{n,p-1}, \quad p \geq 1. \tag{57}$$

On the other hand, using (12) and (57), we have

$$\sum_{k=1}^n (-1)^k \binom{n}{k} (k-1)! c_{n+1-k}(p) = \hat{c}_{n,p-1}.$$

Setting $p = 1$ in the above equation gives a new relation between CNBK.

Corollary 24. For $n \geq 1$, we have

$$\sum_{k=1}^n (-1)^k \binom{n}{k} (k-1)! c_{n+1-k} = \hat{c}_n. \tag{58}$$

Note that Howard [20] gave the relation

$$B_n^{(n)} = n! \sum_{k=0}^n (-1)^{n-k} b_k.$$

Using the relation $c_n = n! b_n$, one can easily see that (58) can be written in the following form

$$B_n^{(n)} = n! \sum_{k=1}^n \frac{(-1)^{n-k-1} k}{n+1-k} b_k.$$

Komatsu [18] gave

$$\sum_{k=0}^n \binom{n}{k} c_k \hat{c}_{n-k}^{(q-1)} = \hat{c}_n - n \sum_{j=1}^{q-1} \left(\hat{c}_n^{(j)} + (n-1) \hat{c}_{n-1}^{(j)} \right) \tag{59}$$

to generalize [19]

$$\sum_{k=0}^n \binom{n}{k} c_k \hat{c}_{n-k} = -(n-1)(\hat{c}_n + n\hat{c}_{n-1}).$$

Now, we want to generalize these formulas. The proof is similar to that of Theorem 15 with the use of (41) and (43), so it is omitted.

Theorem 25. For all integers $n \geq 1, q \geq 1, p \geq 0$ and real number $a > 1$, we have

$$\sum_{k=0}^n \binom{n}{k} c_k \hat{c}_{n-k,p-1}^{(q)}(a-1) = \frac{c_n}{(a-1)^{q-1}} - \frac{n \hat{c}_{n-1,p}^{(q)}(a)}{p+1}. \tag{60}$$

When $q = 1$, we obtain

$$\sum_{k=0}^n \binom{n}{k} c_k \hat{c}_{n-k,p-1} = c_n - \frac{n}{p+1} \hat{c}_{n-1,p}. \tag{61}$$

It is good to note that for $a = 2$ and $p = 0$ in (60), we have

$$\sum_{k=0}^n \binom{n}{k} c_k \hat{c}_{n-k}^{(q-1)} = c_n - n \hat{c}_{n-1,0}^{(q)}(2).$$

Comparing (59) with the above equation and using (13), we conclude that

$$\sum_{j=1}^{q-1} \left(\hat{c}_n^{(j)} + (n-1) \hat{c}_{n-1}^{(j)} \right) = \hat{c}_{n-1,0}^{(q)}(2) - \hat{c}_{n-1}. \tag{62}$$

On the other hand, setting $p = -1$ and $a = 1$ in (53) and using it in the above equation, we arrive at

$$\sum_{j=1}^{q-1} \hat{c}_n^{(j)}(2) = \hat{c}_{n,0}^{(q)}(2) - \hat{c}_n.$$

Moreover, utilizing (12), (61) can be written as

$$\sum_{k=0}^n \binom{n}{k} c_k c_{n+1-k} (p-1) = \frac{c_n}{p} - \frac{nc_n(p)}{p}.$$

From (9), we have $c_n(2) = c_{n+1} + nc_n$. Thus, we obtain a new convolution identity for CNFK.

Corollary 26. For $n \geq 0$

$$\sum_{k=0}^n \binom{n}{k} c_k c_{n+1-k} = \frac{(1-n^2)c_n - nc_{n+1}}{2}.$$

5. CONCLUSIONS

In this paper, a new generalization of Cauchy numbers is introduced, and some arithmetical properties are discussed. In particular, new identities for Cauchy numbers are achieved. In the recent paper [21], Hurwitz-Lerch type multi-poly-Cauchy numbers are studied. For the further research, considering this paper with our results, one can introduce the concept of Hurwitz-Lerch type multi-poly- p -Cauchy numbers.

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CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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