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The Analytical Solutions of the Schrödinger Equation with Generalized Hulthen Plus A New Ring Shaped Like Potential

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ABSTRACT

In this paper, we studied the Schrödinger equation with the generalized Hulthen potential plus a new ring shaped potential. We obtained approximately the bound state energy spectra and the corresponding wave function using the functional analysis method. We used the well-known Hellmann-Feynmann theorem to calculate the expectation values for $\left\langle \left(e^{\delta r}-1\right)^{-1}\right\rangle, \left\langle \left(e^{\delta r}-1\right)^{-2}\right\rangle$ $\left\langle r^{-1}\right\rangle, \left\langle r^{-2}\right\rangle, \left\langle \cot^2\theta\right\rangle,$ $\left\langle \tan^2\theta\right\rangle$ and $\left\langle \cos ec^2\theta\right\rangle \left\langle \sec^2\theta\right\rangle$.We also discussed the special case of the potential which was consistent with the results found in the literature.

Keywords: *Schrödinger equation, generalized Hulthen potential, new ring shaped potential, Hellmann-Feynmann Theorem*

1. INTRODUCTION

The analytical solutions of the Schrödinger equation have been a subject of interest in recent years in different fields of physics and quantum chemistry because of the information arising from its solutions. It is well-known that the exact solutions of the Schrödinger equation can only be found for a few special simple potentials [1-3]. Recently, we proposed a new ring shaped like potential which may have useful applications in quantum chemistry and nuclear physics to study ro-vibrational energy level of molecules, atoms and deformed nucleus [4-5]. The investigation of the occurrence of accidental degeneracy and hidden symmetry for the non-central potentials have also been reported [6-7]. These accidental degeneracy occurring in ring shaped potential was explain by constructing an SU(2) dynamical algebra[8].The ring shaped potential has many possible applications to ring

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shaped organic molecules like cyclic polyenes and benzene [9-10]. Many authors have investigated the quantum properties of the ring shaped potentials with Schrödinger, Klein-Gordon and Dirac equation such as ring shaped harmonic potentials [11],Hartmann potentials [12],ring shaped non-oscillator potentials [13], noncentral electric dipole ring shaped potential [14] , Poschl-Teller double ring shaped Coulomb potential [15],Makarov potential [16],double ring shaped oscillator potential [17], harmonic oscillatory ring shaped potential [18],novel angle dependent(NAD) potential[19], new harmonic oscillatory ring shaped noncentral potential [20],ring shaped non-spherical harmonic potential [21],new Coulomb ring shaped potential [22], noncentral electric dipole plus a novel angle dependent component [23]. Berdemir[24] had shown that the concept of the Coulomb potential or the harmonic oscillator gives a good approximation for understanding the spectroscopy and the structure of diatomic molecules in the ground electronic states. Just recently, Chen et al [25-27] studied the ring shaped potentials using the universal associated Legendre polynomials and they when further to discussed the super-universal associated Legendre polynomials.

Motivated by the study of the ring- shaped-like potential [4-5,25-27] we proposed the novel generalized Hulthen plus a new ring shaped potential of the form,

$$V_{GHNR}(r,\theta)=-\frac{ze^2\delta e^{-\delta r}}{\left(1-qe^{-\delta r}\right)}+\frac{D\delta^2 e^{-2\delta r}}{\left(1-qe^{-\delta r}\right)^2}+\frac{\hbar^2}{2\mu r^2}\left(\frac{A\sin^2\theta+B\cos^2\theta+C}{\sin\theta\cos\theta}\right)^2 \tag{1}$$

where A, B and C are the three dimensionless ring shaped parameters, z is the atomic number, e is the electronic charge, δ is the screening parameter and D is the potential depth. The purpose of the present paper is to attempt study the non-relativistic Schrödinger equation with generalized Hulthen plus new ring shaped(GHNR) potential using factorization method and show our result reduces to the well known potential when the potential parameters are varied.

2. SEPARATION OF VARIABLES FOR SCHRÖDINGER EQUATION

The Schrödinger equation of a particle with reduced mass μ and potential $V(r,\theta)$ is given by

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2-\frac{ze^2\delta e^{-\delta r}}{\left(1-qe^{-\delta r}\right)}+\frac{D\delta^2 e^{-2\delta r}}{\left(1-qe^{-\delta r}\right)^2}+\frac{\hbar^2}{2\mu r^2}\left(\frac{A\sin^2\theta+B\cos^2\theta+C}{\sin\theta\cos\theta}\right)^2\right]\psi(r,\theta,\varphi)=E\psi(r,\theta,\varphi) \tag{2}$$

Now using an ansatz for the wave function as $\psi(r,\theta,\varphi)=r^{-1}R(r)H(\theta)\Phi(\varphi)$ and substituting into Eq.(2), we get

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}\left(E+\frac{ze^2\delta e^{-\delta r}}{\left(1-qe^{-\delta r}\right)}-\frac{D\delta^2 e^{-2\delta r}}{\left(1-qe^{-\delta r}\right)^2}\right)-\frac{\lambda}{r^2}\right]R(r)=0, \tag{3}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH(\theta)}{d\theta}\right)+\left[\lambda-\frac{m^2}{\sin^2\theta}-\left(\frac{A\sin^2\theta+B\cos^2\theta+C}{\sin\theta\cos\theta}\right)^2\right]H(\theta)=0, \tag{4}$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m^2\Phi(\varphi)=0 \tag{5}$$

where m^2 and $\lambda=l'(l'+1)$ are separation constants, which are real and dimensionless. The solution of E.(29) is periodic and for bound state $\Phi(\varphi)$ satisfy the periodic boundary condition $\Phi(\varphi+2\pi)$ and its solutions become,

$$\Phi(\varphi)=\frac{1}{\sqrt{2\pi}}e^{im\varphi}, m=0,\pm1,\pm2... \tag{6}$$

Similarly, the boundary conditions for Eq.(4) which are finite values are given as $H(0), H\left(\frac{\pi}{2}\right)$ and $H(\pi)$.

3. SOLUTION OF $H(\theta)$ ANGULAR EQUATION FOR SCHRÖDINGER EQUATION

Setting

$$m' = \sqrt{m^2 + (B+C)^2} \tag{7}$$

$$\beta = \sqrt{l'(l'+1) + (B-A)^2 + \frac{1}{4}} - \frac{1}{2} \tag{8}$$

$$\Lambda = \sqrt{(A+C)^2 + \frac{1}{4}} + \frac{1}{2}$$

and letting $x = \cos\theta \in [-1,1]$ then equation (4) becomes

$$\left(1-x^2\right)\frac{d^2H(x)}{dx^2} - 2x\frac{dH(x)}{dx} + \left[\beta(\beta+1) - \frac{\Lambda(\Lambda-1)}{x^2} - \frac{m'^2}{1-x^2}\right]H(x) = 0 \tag{9}$$

As a result, the asymptotic solutions of the wave function of equation (9) becomes,

$$H(x) = x^{\Lambda}\left(1-x^2\right)^{\frac{m'}{2}}\varphi(x) \tag{10}$$

and the wave function satisfies the boundary conditions $H(x) \to 0$ as $x \to 0$ and $H(x) \to 0$ as $x \to \pm 1$.
Substituting equation (10) into Eq.(9), we obtain

$$\left(1-x^2\right)\varphi'' + 2\left[\frac{\Lambda}{x} - \left(1+m'+\Lambda\right)x\right]\varphi'(x) + \left(\beta(\beta+1) - \left(\Lambda+m'+1\right)\left(\Lambda+m'\right)\right)\varphi(x) = 0 \tag{11}$$

Introducing the new variables, $z = x^2$, we can rearrange equation (11) as,

$$z(1-z)\varphi''(z) + \left[\frac{1}{2} + \Lambda - \left(m' + \Lambda + \frac{3}{2}\right)z\right]\varphi'(z) + \left(\frac{\beta(\beta+1)}{4} - \frac{\left(\Lambda+m'+1\right)\left(\Lambda+m'\right)}{4}\right)\varphi(z) = 0 \tag{12}$$

The hypergeometric function is defined as [28],

$$z(1-z)\varphi'' + \left[c - \left(a+b+1\right)z\right]\varphi' - ab\varphi(z) = 0 \tag{13}$$

and one of the solutions which satisfies the physical boundary condition is ,

$$\varphi_n(z) = {}_2F_1\left(a,b,c;z\right) \tag{14}$$

Now comparing Eqs.(12) and (13), we obtain the a, b and c parameters as follows,

$$
\begin{aligned}
a &= \frac{m' + \Lambda + \beta + 1}{2}, \\
b &= \frac{m' + \Lambda - \beta}{2}, \\
c &= \frac{1}{2} + \Lambda
\end{aligned}
\tag{15}
$$

Using the quantization conditions [29], $\frac{m' + \Lambda - \beta}{2} = -n_r$ and incorporating Eqs.((7) and (8), we obtain

$$
\lambda = \left(2n_r + 1 + \sqrt{m^2 + \left(B + C\right)^2} + \sqrt{\left(A + C\right)^2 + \frac{1}{4}} \right)^2 - \left(B - A\right)^2 - \frac{1}{4}
\tag{16}
$$

Equation (16) is the contribution of the angle-dependent part of generalized Hulthen plus ring shaped like potential. . However, when the ring-shaped term potential vanishes, that is $A = B = C = 0$, then the constant of separation becomes $\lambda = l'(l' + 1)$, where $l' = 2n_r + 1 + |m|, m = 0,1,2...$ It can be observed that the angular part of the generalized Hulthen plus ring shaped potential has singularities at $\theta = t\pi (t = 0,1,2...)$ and also at very small and very large values of r. The complete angular wave function can be written as,

$$
H(x) = N_{l'm'} x^{\Lambda} \left(1 - x^2\right)^{\frac{m'}{2}} {}_2F_1\left(a, b, c; x^2\right)
\tag{17}
$$

where $N_{l'm}$ is the normalization constant of the angular wave function $H_{l'm'}(x)$.In order to determine the normalization constant, we first set $z = x^2$ and used the following normalization conditions [28,30],

$$
\int_0^1 z^{\gamma - 1} \left(1 - z\right)^{s - \gamma} \left[{}_2F_1\left(-n, n + s; \gamma; z\right) \right]^2 dz = \frac{n!}{\left(s + 2n\right)} \frac{\Gamma\left(\gamma\right)^2 \Gamma\left(n + s - \gamma + 1\right)}{\Gamma\left(n + s\right) \Gamma\left(s + \gamma\right)}
\tag{18}
$$

$$
\int_{-1}^1 \left| H_{l'm}(x) \right|^2 dx = 1
\tag{19}
$$

and we obtain the normalization constant as,

$$
N_{lm} = \left(\frac{\left(1 + \beta\right)}{\Gamma^2\left(\Lambda + 1\right)} \frac{\Gamma\left(\frac{\beta + m' + \Lambda + 2}{2}\right)}{\left(\frac{\beta - m' - \Lambda}{2}\right)!} \frac{\Gamma\left(m' + 2\Lambda + 1\right)}{\Gamma\left(\frac{\beta + m' + \Lambda + 2}{2}\right)} \right)^{\frac{1}{2}}
\tag{20}
$$

4. SOLUTIONS OF THE RADIAL EQUATION

By considering the appropriate approximation for the centrifugal term of Eq.(3) for the generalized Hulthen potential given in Eq.(1), the radial equation (3) becomes,

$$
\frac{d^2R(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2} \left(E + \frac{ze^2 \delta e^{-\delta r}}{\left(1 - qe^{-\delta r}\right)} - \frac{D\delta^2 e^{-2\delta r}}{\left(1 - qe^{-\delta r}\right)^2} \right) - \delta^2 \lambda \left(\frac{1}{12} + \frac{\omega e^{-\delta r}}{1 - qe^{-\delta r}} + \frac{qe^{-2\delta r}}{\left(1 - qe^{-\delta r}\right)^2} \right) \right] R(r) = 0,
\tag{22}
$$

where the approximation for $\frac{1}{r^2}$ is taken as [31],

$$\frac{1}{r^2} \approx \left(\frac{1}{12} + \frac{\omega e^{-\delta r}}{1 - q e^{-\delta r}} + \frac{q e^{-2\delta r}}{\left(1 - q e^{-\delta r}\right)^2} \right) \tag{23}$$

since Eq.(3) cannot be solve exactly because of the centrifugal term. In order to solve Eq.(22) by algebraic method, we first let

$$\begin{gathered} -\varepsilon_{nl}^2 = \frac{1}{\delta^2}\left(\frac{2\mu E_{nl}}{\hbar^2} - \frac{\delta^2 \lambda}{12} \right), \alpha^2 = \frac{1}{q\delta^2}\left(\frac{2\mu z e^2 \delta}{\hbar^2} - \omega \delta^2 \lambda \right) \\ \gamma(\gamma+1) = \frac{1}{q^2 \delta^2}\left(\frac{2\mu D}{\hbar^2} + \delta^2 \lambda q \right) \end{gathered} \tag{24}$$

and defining a new variable $x = q e^{-\delta r}$, which transform it to the form,

$$x^2 \frac{d^2 R(x)}{dx^2} + x \frac{dR(x)}{dx} + \left[-\varepsilon_{nl}^2 + \frac{\alpha^2 x}{1-x} - \frac{\gamma(\gamma+1) x^2}{(1-x)^2} \right] R(x) = 0 \tag{25}$$

We proposed an ansatz for the wave function of the form,

$$R(x) = x^{\varepsilon_{nl}} (1-x)^{\gamma+1} \varphi(x) \tag{26}$$

Substituting Eq.(26) into Eq.(25) gives,

$$x(1-x)\varphi''(x) + \left[1 + 2\varepsilon_{nl} - \left(2\varepsilon_{nl} + 2\gamma + 3\right)x\right]\varphi'(x) - \left(\gamma + \varepsilon_{nl} + 1 + \sqrt{\alpha^2 + \varepsilon_{nl}^2}\right)\left(\gamma + \varepsilon_{nl} + 1 - \sqrt{\alpha^2 + \varepsilon_{nl}^2}\right)\varphi(x) = 0, \tag{27}$$

The solutions of Eq.(27) is nothing but hypergeometric function $\varphi(x) = {}_2F_1\left(a, b, c; x\right)$, where,

$$\begin{gathered} a = \left(\gamma + 1 + \varepsilon_{nl} + \sqrt{\alpha^2 + \varepsilon_{nl}^2}\right), \\ b = \left(\gamma + 1 + \varepsilon_{nl} - \sqrt{\alpha^2 + \varepsilon_{nl}^2}\right), c = 1 + 2\varepsilon_{nl} \end{gathered} \tag{28}$$

For bound states, the solution of Eq.(27) can be expressed in Gauss hypergeometric form as,

$$R(r) = C_{nl}\left(q e^{-\delta r}\right)^{\varepsilon_{nl}}\left(1 - q e^{-\delta r}\right)^{\gamma+1} {}_2F_1\left(\gamma + 1 + \varepsilon_{nl} + \sqrt{\alpha^2 + \varepsilon_{nl}^2}, \gamma + \varepsilon_{nl} + 1 - \sqrt{\alpha^2 + \varepsilon_{nl}^2}; 1 + 2\varepsilon_{nl}; q e^{-\delta r}\right) \tag{29}$$

when $\gamma + \varepsilon_{nl} + 1 + \sqrt{\alpha^2 + \varepsilon_{nl}^2} = -n_r$ or $\gamma + \varepsilon_{nl} + 1 - \sqrt{\alpha^2 + \varepsilon_{nl}^2} = -n_r$, for $n_r = 0, 1, 2, 3...$ and the hypergeometric function above reduces to a polynomial of degree n_r. Based on the quantization or quantum conditions, $\gamma + \varepsilon_{nl} + 1 - \sqrt{\alpha^2 + \varepsilon_{nl}^2} = -n_r$, we obtain the energy eigenvalues as,

$$E_{nl} = -\frac{\hbar^2 \delta^2}{2\mu}\left[\frac{\alpha^2}{2\left(1 + \gamma + n_r\right)} - \frac{\left(1 + \gamma + n_r\right)}{2} \right]^2 + \frac{\hbar^2}{2\mu}\frac{\delta^2 \lambda}{12} \tag{30}$$

Incorporating Eq.(24) into Eq.(30), we obtain the complete energy eigenvalues for the GHNR potential as,

$$E_{nl} = -\frac{\hbar^2\delta^2}{2\mu}\left[\frac{\frac{1}{q\delta^2}\left(\frac{2\mu ze^2\delta}{\hbar^2} - \omega\delta^2 l'(l'+1)\right)}{2\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)} - \frac{\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)}{2}\right]^2 \text{w}$$

$$+\frac{\hbar^2}{2\mu}\frac{\delta^2 l'(l'+1)}{12} \qquad (31)$$

here l' is given in Eq.(16). In the limiting case when the screening parameter $\delta \to 0, q=1$, i.e low screening regime then the GHNR potential turns to pseudo-Coulomb plus new ring shaped (PCNR) potential

$$V_{PCNR}(r,\theta) = \lim_{\delta\to 0} V_{GHNR}(r,\theta) = -\frac{V_0}{r} + \frac{D}{r^2} + \frac{1}{r^2}\left(\frac{A\sin^2\theta + B\cos^2\theta + C}{\sin\theta\cos\theta}\right)^2 \qquad (32)$$

where $V_0 = ze^2$ .In the special case when the ring-shaped term potential vanishes, that is $A=B=C=0$, then the energy equation for the pseudo-Coulomb potential from Eq.(31) becomes,

$$E_{nl} = -\frac{\hbar^2}{4\mu}\left[\frac{\frac{2\mu V_0}{\hbar^2}}{(n_r+\sigma)} - \sqrt{\left(\frac{2\mu D}{\hbar^2}\right)}\right]^2 \qquad (33)$$

where $\sigma = \frac{1}{2}\left(1+\sqrt{1+4l'(l'+1)}\right)$ and $l' = 2n_r + 1 + |m|$. This result is consistent with the one reported in Ref.[32] when $\alpha \to 1$ in their result. When $D=0$, we obtain the energy spectrum for Coulomb-like potential as,

$$E_{nl} = -\frac{\mu V_0^2}{\hbar^2(n_r+\sigma)^2} \qquad (34)$$

This result is consistent with those found in the literature [33].

5. EXPECTATIONS VALUES AND HFT

In this section, we obtain the expectation values $\left\langle (e^{\delta r}-1)^{-1}\right\rangle, \left\langle (e^{\delta r}-1)^{-2}\right\rangle$ $\langle r^{-1}\rangle, \langle r^{-2}\rangle$, $\langle\cot^2\theta\rangle$, $\langle\tan^2\theta\rangle$ and $\langle\cos ec^2\theta\rangle\langle\sec^2\theta\rangle$ using the Hellmann-Feynman theorem (HFT) [34-35].According to the HFT, if the Hamiltonian H of a quantum mechanical system is a function of some parameters q and letting $E_n(q)$ and $\psi_n(q)$ be the energy eigenvalues and the eigenfunctions of the Hamiltonian $H(q)$ respectively satisfy the relation,

$$\frac{\partial E_n(q)}{\partial q} = \left\langle \psi_{n,l}\left|\frac{\partial H(q)}{\partial q}\right|\psi_{n,l}\right\rangle. \qquad (35)$$

The effective Hamiltonian is given as,

$$H=-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{\hbar^2}{2\mu}\frac{l'(l'+1)}{r^2}-\frac{V_0\delta e^{-\delta r}}{\left(1-qe^{-\delta r}\right)}+\frac{D\delta^2 e^{-2\delta r}}{\left(1-qe^{-\delta r}\right)^2}+\frac{1}{r^2}\left(\frac{A\sin^2\theta+B\cos^2\theta+C}{\sin\theta\cos\theta}\right)^2. \tag{36}$$

n order to calculate $\left\langle\left(e^{\delta r}-1\right)^{-1}\right\rangle$, we set $q=V_0$ such that,

$$\frac{\partial E_{nl}(V_0)}{\partial V_0}=\left\langle\Psi_{nl}(V_0)\right|\frac{\partial H}{\partial V_0}\left|\Psi_{nl}(V_0)\right\rangle=-\delta\left\langle\left(e^{\delta r}-q\right)^{-1}\right\rangle \tag{37}$$

and By HFT, we obtained,

$$\left\langle\left(e^{\delta r}-q\right)^{-1}\right\rangle=\frac{1}{\delta}\left[\frac{\frac{2}{q^2}\left(\frac{2\mu V_0}{\hbar^2}-\frac{\delta\omega l'(l'+1)}{q}\right)}{\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)^2}-\frac{\delta}{q}\right] \tag{37}$$

For the calculation of $\left\langle r^{-1}\right\rangle$, we obtain from equation (37) in the limiting case $\delta\to 0, q=1$ as,

$$\left\langle r^{-1}\right\rangle=\lim_{\delta\to 0,q=1}\delta\left\langle\left(e^{\delta r}-1\right)^{-1}\right\rangle=\frac{4\mu V_0^2}{\hbar^2\left(n_r+\sigma\right)^2} \tag{38}$$

Similarly, by setting $q=B$ we obtain $\left\langle\left(e^{\delta r}-q\right)^{-2}\right\rangle$ as,

$$\delta^2\left\langle\left(e^{\delta r}-q\right)^{-2}\right\rangle=\frac{\hbar^2\delta^2}{4\mu}\left[\frac{\frac{1}{q\delta^2}\left(\frac{2\mu ze^2\delta}{\hbar^2}-\omega\delta^2 l'(l'+1)\right)}{\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)^2}\frac{\frac{\mu}{\hbar^2 q^2\delta^2}}{\sqrt{\frac{1}{4}+\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)\right)}}+\frac{\frac{\mu}{\hbar^2 q^2\delta^2}}{2\sqrt{\frac{1}{4}+\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)\right)}}\right]$$
$$\times\left(\frac{\frac{1}{q\delta^2}\left(\frac{2\mu ze^2\delta}{\hbar^2}-\omega\delta^2 l'(l'+1)\right)}{\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)}-\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)\right) \tag{39}$$

Also taking the limit, $\left\langle r^{-2}\right\rangle=\lim_{\delta\to 0,q=1}\delta^2\left\langle\left(e^{\delta r}-1\right)^{-2}\right\rangle$, we get

$$\left\langle r^{-2}\right\rangle = -\frac{\hbar}{16\mu}\sqrt{\frac{2\mu D}{\left(1+4l'(l'+1)\right)}} \tag{40}$$

Also by letting $q = A$, we obtain $\left\langle \tan^2\theta\right\rangle$ as

$$\left\langle \tan^2\theta\right\rangle = \frac{1}{2A\left\langle r^{-2}\right\rangle}\frac{\partial E_{nl}}{\partial A} \tag{41}$$

Where,

$$\begin{aligned}
\frac{\partial E_{nl}(A)}{\partial A} = &-\frac{\hbar^2\delta^2}{4\mu}\left[\frac{-\frac{2\omega}{q}\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)\left(\frac{\partial\Lambda}{\partial A}\left(2n_r+m'+\Lambda+\frac{1}{2}\right)+B-A\right)}{\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)^2}\right] \\
&\times\left(\frac{\frac{1}{q\delta^2}\left(\frac{2\mu z e^2\delta}{\hbar^2}-\omega\delta^2 l'(l'+1)\right)}{\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)}-\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)\right) \\
&+\frac{\hbar^2\delta^2}{4\mu}\left[\frac{\frac{2}{q\delta^2}\left(\frac{2\mu z e^2\delta}{\hbar^2}-\omega\delta^2 l'(l'+1)\right)\left(\frac{\partial\Lambda}{\partial A}\left(2n_r+m'+\Lambda+\frac{1}{2}\right)+B-A\right)}{\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)^2\sqrt{\frac{4}{\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2\lambda q+\frac{1}{4}\right)}}\right] \\
&\times\left(\frac{\frac{1}{q\delta^2}\left(\frac{2\mu z e^2\delta}{\hbar^2}-\omega\delta^2 l'(l'+1)\right)}{\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)}-\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)\right) \\
&+\frac{\hbar^2\delta^2}{4\mu}\left[\frac{2\left(\frac{\partial\Lambda}{\partial A}\left(2n_r+m'+\Lambda+\frac{1}{2}\right)+B-A\right)}{\sqrt{\frac{4}{\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2\lambda q+\frac{1}{4}\right)}}\right] \\
&\times\left(\frac{\frac{1}{q\delta^2}\left(\frac{2\mu z e^2\delta}{\hbar^2}-\omega\delta^2 l'(l'+1)\right)}{\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)}-\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)\right) \\
&+\frac{\hbar^2\delta^2}{12\mu}\left(\frac{\partial\Lambda}{\partial A}\left(2n_r+m'+\Lambda+\frac{1}{2}\right)+A-B\right)
\end{aligned} \tag{42}$$

With

$$\frac{\partial\Lambda}{\partial A}=\frac{(A+C)}{\sqrt{\left(A+C\right)^2+\frac{1}{4}}} \tag{43}$$

Similarly, we obtain $\left\langle\cot^2\theta\right\rangle,\left\langle\sec^2\theta\right\rangle\left\langle\cos ec^2\theta\right\rangle$ by setting $q=B$ and $q=C$ as,

$$\left\langle\cot^2\theta\right\rangle=\frac{1}{2B\left\langle r^{-2}\right\rangle}\frac{\partial E_{nl}}{\partial B} \tag{44}$$

$$\left\langle\sec^2\theta\right\rangle\left\langle\cos ec^2\theta\right\rangle=\frac{1}{2C\left\langle r^{-2}\right\rangle}\frac{\partial E_{nl}}{\partial C} \tag{45}$$

respectively, where

$$\frac{\partial E}{\partial B}=-\frac{\hbar^2\delta^2}{4\mu}\left(\frac{V\frac{\partial U}{\partial\lambda}\frac{\partial\lambda}{\partial B}-U\frac{\partial V}{\partial\lambda}\frac{\partial\lambda}{\partial B}}{V^2}-\frac{\partial V}{\partial\lambda}\frac{\partial\lambda}{\partial B}\right)\left(\frac{U}{V}-V\right)+\frac{\hbar^2\delta^2}{24\mu}\frac{\partial\lambda}{\partial B} \tag{46}$$

$$\frac{\partial E}{\partial C}=-\frac{\hbar^2\delta^2}{4\mu}\left(\frac{V\frac{\partial U}{\partial\lambda}\frac{\partial\lambda}{\partial C}-U\frac{\partial V}{\partial\lambda}\frac{\partial\lambda}{\partial C}}{V^2}-\frac{\partial V}{\partial\lambda}\frac{\partial\lambda}{\partial C}\right)\left(\frac{U}{V}-V\right)+\frac{\hbar^2\delta^2}{24\mu}\frac{\partial\lambda}{\partial C} \tag{47}$$

With

$$U=\frac{1}{q\delta^2}\left(\frac{2\mu ze^2\delta}{\hbar^2}-\omega\delta^2 l'(l'+1)\right),V=\left(\frac{1}{2}+\sqrt{\frac{1}{q^2\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2 l'(l'+1)q\right)+\frac{1}{4}}+n_r\right)$$

$$\frac{\partial V}{\partial\lambda}=\frac{1}{\sqrt{\frac{4}{\delta^2}\left(\frac{2\mu D}{\hbar^2}+\delta^2\lambda q+\frac{1}{4}\right)}},\frac{\partial U}{\partial\lambda}=-\frac{2\omega}{q},\frac{\partial\lambda}{\partial B}=2\frac{\partial m'}{\partial B}\left(2n_r+m'+\Lambda+\frac{1}{2}\right)-2\left(B-A\right),$$

$$\frac{\partial\lambda}{\partial C}=2\left(\frac{\partial m'}{\partial C}+\frac{\partial\Lambda}{\partial C}\right)\left(2n_r+m'+\Lambda+\frac{1}{2}\right),\frac{\partial m'}{\partial B}=\frac{(B+C)}{\sqrt{m^2+\left(B+C\right)^2}},\frac{\partial\Lambda}{\partial C}=\frac{(A+C)}{\sqrt{\frac{1}{4}+\left(A+C\right)^2}} \tag{48}$$

6. CONCLUSIONS

In this paper, we investigate analytically the Schrödinger equation with the GHNR potential. By virtue of the suitable approximation for the centrifugal term we obtain approximately the energy eigenvalues and the corresponding eigenfunction for the radial part of the Schrödinger equation for the generalized Hulthen potential. We also obtain the eigenvalues for the angular part of the Schrödinger equation. These results are obtained by the use of the functional analysis method. The approximate analytical energy spectra and the corresponding wave function are obtained in a closed form, In addition, we use the HFT to obtain the expectation values for $\left\langle\left(e^{\delta r}-1\right)^{-1}\right\rangle,\left\langle\left(e^{\delta r}-1\right)^{-2}\right\rangle$ $\left\langle r^{-1}\right\rangle,\left\langle r^{-2}\right\rangle,\left\langle\cot^2\theta\right\rangle$, $\left\langle\tan^2\theta\right\rangle$ and $\left\langle\cos ec^2\theta\right\rangle\left\langle\sec^2\theta\right\rangle$. As a special case, when $\delta\to 0$ and $A=C=0,$ we have

$V(r,\theta)=-\frac{V_0}{r}+\frac{D}{r^2}+\frac{\hbar^2}{2\mu}\frac{B\cos^2\theta}{r^2\sin^2\theta}$ which is the Kratzer plus a new ring shaped potential [36]. Also when $D=A=B=0$, we obtain the double ring shaped Hulthen potential

$$V_{GHNR}(r,\theta)=-\frac{ze^2\delta e^{-\delta r}}{\left(1-qe^{-\delta r}\right)}+\frac{\hbar^2}{2\mu}\left(\frac{C^2}{r^2\sin^2\theta}+\frac{C^2}{r^2\cos^2\theta}\right)$$

[37] which are consistent with results under these conditions. Finally, because of the applications of this potential model in quantum chemistry and nuclear physics, then our results can be used in these fields of study direct or with some modifications.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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