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On The Equivalence of Convergence and 2-Norm Convergence in Normed Spaces

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ABSTRACT

Let $(X, \|.\|)$ be a normed space and $\|.,\ldots,.\|_G$ be the n-norm given by Gähler. In this paper, we show that $\|.\|$-convergence and $\|.,\ldots,.\|_G$ 2-convergence are equivalent.

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1. INTRODUCTION

The theory of 2-normed spaces was first introduced by Gähler [1,2] as an interesting linear generalizations of a normed space which was subsequently studied by many others. For a fixed number $2 \leq n \in \mathbb{N}$, an n-norm on a real vector space X $(\dim(X) \geq n)$ is a mapping $\|.,\ldots,.\|: X^n \to \mathbb{R}$ which satisfies the following four conditions:

1. $\|x_1,\ldots,x_n\| = 0$ if and only if $x_1,\ldots,x_n$ are linearly independent,
2. $\|x_1,\ldots,x_n\|$ invariant under permutation,
3. $\|x_1,\ldots,x_{n-1},\alpha x_n\| = |\alpha|\|x_1,\ldots,x_{n-1},x_n\|$ for any $\alpha \in \mathbb{R}$,
4. $\|x_1,\ldots,x_{n-1},y+z\| \leq \|x_1,\ldots,x_{n-1},y\| + \|x_1,\ldots,x_{n-1},z\|$.

The pair $(X, \|.,\ldots,.\|)$ is called an n-normed space. An example of an n-normed space is $X = \mathbb{R}^n$ equipped with the following n-norm:

$$\|x_1,\ldots,x_n\| = \left|\det\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}\right|$$

where $x_i = (x_{i1},\ldots,x_{in}) \in \mathbb{R}^n$ for each $i = 1,2,\ldots,n$. For $n = 2$, this is the area of the parallelogram determined by the vectors x_1 and x_2. Another example of an n-normed space is the space l_p with $1 \leq p < \infty$ equipped with the following n-norm:

$$\|x_1,\ldots,x_n\|_p = \left[\frac{1}{n!}\sum_{j_1}\ldots\sum_{j_n}\left|\det\left(x_{ij_k}\right)\right|^p\right]^{\frac{1}{p}}, i = 1,2,\ldots,n.$$

The following definitions was first introduced by White [7]. A sequence (x_k) in an n-normed space $(X, \|.,\ldots,.\|)$ is said to cenverge to an $x \in X$ if

$$\lim_{k\to\infty}\|x_k - x, y_1,\ldots,y_{n-1}\| = 0$$

for all $y_1,\ldots,y_{n-1} \in X$.

Gähler showed that if $(X, \|.\|)$ is a normed space with dual X', the following Formula defines an n-norm on X

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$$\|x_1,\dots,x_n\|_G = \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,\dots,n}} \left| det\begin{pmatrix} f_1(x_1) & \cdots & f_n(x_1)\\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n)\end{pmatrix}\right|.$$

So every normed space is an *n*-normed space. Conversely, if $(X,\|.,\dots,.\|)$ is an *n*-normed space and $\{a_1,\dots,a_n\}$ is a linearly independent set in *X*, then the function

$$\|x\|_* = maks\|x,a_{i_2},\dots,a_{i_n}\|:\{i_2,\dots,i_n\}\subset\{1,\dots,n\}$$

[4]. Then, we say that every *n*-normed space is a normed space. Another norm obtained from *n*-norm is the following

$$\|x\|^* = \sum_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|x,a_{i_2},\dots,a_{i_n}\|$$

[3]. These norms are called as derived norm on *X* with respect to linearly independent set $\{a_1,\dots,a_n\}$. It is easily seen that the norms $\|.\|^*$ and $\|.\|_*$ are equivalent. Therefore, it is not importent which one taken in this work. There are two important question in this topic:

1. Let $(X,\|.,\dots,.\|)$ be an *n*-normed space. Is there any norm on *X* such that *n*-norm convergence and norm convergence are equivalent?
2. Let $(X,\|.\|)$ be a normed space. Is there any *n*-norm on *X* such that norm convergence and *n*-norm convergence are equivalent?

The answer of first question has been obtained partially. Gunawan and Mashadi have shown that the answer is affirmative in a finite dimension *n*-normed space and the space l_p with $1\le p<\infty$ in [4]. Turan and Bilici give affirmative answer for an almost *2*-Banach lattices in [6]. In this study, we show that the second question has affirmative answer. We refer to [5] for definitions and notations not explained here.

2. MAIN RESULTS

To be effortless we will take $n=2$ throughout the study. However, it can be given for an arbitrary $n\epsilon\mathbb{N}$, $2\le n$. Let $(X,\|.\|)$ be a normed space,

$$\|x_1,x_2\|_G = \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2}} \left| det\begin{pmatrix} f_1(x_1) & f_2(x_1)\\ f_1(x_2) & f_2(x_2)\end{pmatrix}\right|$$

be the *2*-norm denoted by Gähler and

$$\|x\|_G^* = \|x,a\|_G + \|x,b\|_G$$

be derived norm with respect to linearly independent set $\{a,b\}$.

Lemma 1. Let $(X,\|.\|)$ be a normed space. For every $x_1,x_2\in X$ the following inequality holds

$$\|x_1,x_2\|_G \le 2\|x_1\|\|x_2\| .$$

Proof For every $x_1,x_2\in X$ we have

$$\|x_1,x_2\|_G = \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2}} \left| det\begin{pmatrix} f_1(x_1) & f_2(x_1)\\ f_1(x_2) & f_2(x_2)\end{pmatrix}\right|$$
$$= \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2}} |f_1(x_1)f_2(x_2)-f_1(x_2)f_2(x_1)|$$
$$\le \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2}} \{|f_1(x_1)||f_2(x_2)|+|f_1(x_2)||f_2(x_1)|\}$$
$$\le \|x_1\|\|x_2\|+\|x_2\|\|x_1\|$$
$$= 2\|x_1\|\|x_2\|.$$

Lemma 2. Let $(X,\|.\|)$ be a normed space. For every $x,y_1,y_2\in X$, we have

$$\|x\|\|y_1,y_2\|_G \le 2\|y_2\|\|x,y_1\|_G + \|y_1\|\|x,y_2\|_G.$$

Proof For every $x,y_1,y_2\in X$ we get

$$\|x\|\|y_1,y_2\|_G = \|x\| \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2}} \left| det\begin{pmatrix} f_1(y_1) & f_2(y_1)\\ f_1(y_2) & f_2(y_2)\end{pmatrix}\right|$$
$$= \|x\| \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2}} |f_1(y_1)f_2(y_2) - f_1(y_2)f_2(y_1)|$$
$$= \sup_{f_3\epsilon X',\|f_3\|\le 1} |f_3(x)| \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2}} |f_1(y_1)f_2(y_2) - f_1(y_2)f_2(y_1)|$$
$$= \sup_{f_3\epsilon X',\|f_3\|\le 1} |f_3(x)f_1(y_1)f_2(y_2) - f_3(x)f_1(y_2)f_2(y_1)|$$
$$= \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2,3}} \left| \begin{matrix} f_3(x)f_1(y_1)f_2(y_2) - f_3(y_1)f_1(x)f_2(y_2) + f_3(y_1)f_1(x)f_2(y_2) \\ -f_3(y_1)f_1(y_2)f_2(x) + f_3(y_1)f_1(y_2)f_2(x) - f_3(x)f_1(y_2)f_2(y_1)\end{matrix}\right|$$
$$= \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2,3}} \left| \begin{matrix} f_2(y_2)[f_3(x)f_1(y_1) - f_3(y_1)f_1(x)] \\ +f_3(y_1)[f_1(x)f_2(y_2) - f_1(y_2)f_2(x)] \\ +f_1(y_2)[f_3(y_1)f_2(x) - f_3(x)f_2(y_1)]\end{matrix}\right|$$
$$\le \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2,3}} |f_2(y_2)[f_3(x)f_1(y_1) - f_3(y_1)f_1(x)]|$$
$$+ \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2,3}} |f_3(y_1)[f_1(x)f_2(y_2) - f_1(y_2)f_2(x)]|$$
$$+ \sup_{\substack{f_i\in X',\|f_i\|\le 1\\ i=1,2,3}} |f_1(y_2)[f_3(y_1)f_2(x) - f_3(x)f_2(y_1)]|$$
$$\le \|y_2\|\|x,y_1\|_G + \|y_1\|\|x,y_2\|_G + \|y_2\|\|x,y_1\|_G$$
$$= 2\|y_2\|\|x,y_1\|_G + \|y_1\|\|x,y_2\|_G.$$

Proposition 1. Let $(X,\|.\|)$ be a normed space, $\|.,.\|_G$ be the *2*-norm obtained from norm and $\|.\|_G^*$ be derived norm with respect to linearly independent set $\{a,b\}$. Then, the following inequality holds

$$\frac{2\|a,b\|_G}{(\|a\|+\|b\|)}\|x\| \le \|x\|_G^* \le 2(\|a\|+\|b\|)\|x\|.$$

Proof By the inequality in Lemma 1, for any $x\in X$ we obtain

$$\|x\|_G^* = \|x,a\|_G + \|x,b\|_G \le 2\|x\|\|a\| + 2\|x\|\|b\| \le 2(\|a\|+\|b\|)\|x\|.$$

From the inequality in Lemma 2, we have

$$\|x\|\|a,b\|_G \le 2\|b\|\|x,a\|_G + \|a\|\|x,b\|_G \le 2(\|a\|+\|b\|)\|x\|_G^*.$$

Theorem 1. Let $(X,\|.\|)$ be a normed space. A sequence in X convergent in the norm $\|.\|$ if and only if it is convergent in the *2*-norm $\|.,.\|_G$.

Proof If (x_k) converges to an $x \in X$ in the norm, then

$$\lim_{k\to\infty} \|x_k - x\| = 0.$$

Hence, for every $y \in X$, we have

$$\lim_{k\to\infty} \|x_k - x, y\|_G \le \lim_{k\to\infty} 2\|x_k - x\|\,\|y\| = 0.$$

It has been proved that (x_k) $\|.,.\|_G$-converges to x. Conversely, let $(x_k) \subseteq X$ $\|.,.\|_G$- converges to $x \in X$, that is,

$$\lim_{k\to\infty} \|x_k - x, y\|_G = 0$$

for every $y \in X$. Let $\{a,b\}$ be a linearly independent set in X. Then, we have

$$\lim_{k\to\infty} \|x_k - x, a\|_G = 0 \; and \; \lim_{k\to\infty} \|x_k - x, b\|_G = 0$$

so

$$\lim_{k\to\infty} \|x_k - x\|^* = 0.$$

This implies, by Proposition 1,

$$\lim_{k\to\infty} \|x_k - x\| = 0$$

as required.

A sequence (x_k) in an n-normed space X is called a Cauchy sequence if

$$\lim_{k,l\to\infty} \|x_k - x_l, y_1, \dots, y_{n-1}\| = 0$$

for all $y_1, \dots, y_{n-1} \in X$ and all k, l. An n-normed space in which every Cauchy sequence is convergent is called an n-Banach space. We can give the following result from the above theorem.

Corollary 1. Let $(X,\|.\|)$ be a normed space. X is a Banach space if and only if X is a 2-Banach space.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Gähler, S., Lineare 2-normierte räume, Math. Nachr., 28, 1-43, (1964).

[2] Gähler, S., Untersuchungen über verallgemeinerte m-metrische Räume. I, Math. Nachr., 40, 165-189, (1969).

[3] Gunawan, H., The space of p-summable sequences and its n-norms, Bull. Austral. Math. Soc., 64, 137-147, (2001).

[4] Gunawan, H. and Mashadi, M., On n-normed spaces, Int. J. Math. Math. Sci., 27, 631-639, (2001).

[5] Raymond, W. and Freese, Yeol Je Cho, Geometry of linear 2-normed spaces, Nova Science publishers, Inc, Newyork, (2001).

[6] B. Turan and F. Bilici, On almost 2-normed vector lattice, Preprint.

[7] White, A., 2-Banach Spaces, Doctoral Dissertation, Saint Louis University, (1968).