Eastern Anatolian Journal of Science
Volume VI, Issue II, 2020, 9-14

Eastern Anatolian Journal of Science

On $(\alpha,\beta)-$ Convex Function

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Abstract

In this paper, we introduce a new class called $(\alpha,\beta)-$convex of F_{β}^{α} and give some basic properties for this class. We proved a variant of Hermite-Hadamard inequality for this class also we give some relations of $(\alpha,\beta)-$ convex functions with known two functionals F and H.

Keywords: Convex functions, Hadamard inequality, s-convex functions.

1. *Introduction*

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as Hadamard's inequality for convex mappings. For particular choice of the function f in (1.1) yields some classical inequalities of means. Both inequalities hold in reversed direction if f is concave. For further information see (S. S. Dragomir, 2002), (M. Avcı, 2011), (M.E. Özdemir M. A., 2011) and (M.E. Özdemir M. A., 2010). In (Orlicz, 1961), Orlicz introduced the definition of s-convexity of real valued functions:

Received: 09.11.2020
Accepted: 13.12.2020
Published:16.12.2020
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Cite this article as: M. E. Özdemir, On (α,β)- Convex Function, *Eastern Anatolian Journal of Science, Vol. 6, Issue 2, 9-14, 2020.*

Definition 1. Let $s \in (0,1]$ be fixed real number. A function $f:(0,\infty] \to \mathbb{R}$ is said to be $s-$ convex (in the first sense), or that f belongs to the class K_s^1, if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

holds for all $x, y \in (0,\infty]$ with $\alpha^s + \beta^s = 1$, $\alpha, \beta \geq 0$.

In (H. Hudzik, 1994), Hudzik and Maligranda give the definiton of a new s-convex class:

Definition 2. Let $s \in (0,1]$ be fixed real number. A function $f:(0,\infty] \to \mathbb{R}$ is said to be $s-$ convex (in the second sense), or that f belongs to the class K_s^2, if

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y)$$

holds for all $x, y \in (0,\infty]$ and $\alpha \in [0,1]$.

It is clear that convexity mean just the convexity when $s=1$. In (S.S. Dragomir, 1999), Dragomir and Fitzpatrick proved the following variant of Hadamard's inequality which hold for convex functions in the second sense:

Theorem 1. Suppose that $f:[0,\infty) \to [0,\infty)$ is a s-convex function in the second sense, where $s \in (0,1)$ and let $a, b \in [0,\infty), a < b$. If $f \in L_1([a,b])$, then the following inequalities hold.

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1} \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

Again in (S.S. Dragomir, 1999), Dragomir and Fitzpatrick also proved the following Hadamard-type inequality for s-convex functions in the first sense:

Theorem 2. Suppose that $f:[0,\infty)\to[0,\infty)$ is a s-convex function in the first sense, where $s\in(0,1)$ and let $a,b\in[0,\infty)$, $a<b$. If $f\in L_1([a,b])$, then the following inequalities hold.

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx \le\frac{f(a)+sf(b)}{s+1} \tag{1.3}$$

The above inequalities are sharp.

2. *Main Results*

First we introduce a new convex function class called (α,β) −convex function:

Definition 3. Let $(\alpha,\beta)\in(0,1]$ be fixed real numbers. A function $f:(0,\infty)\to\mathbb{R}$ is said to be (α,β) −convex, if

$$f(tx+(1-t)y)\le t^{\alpha}f(x)+(1-t)^{\beta}f(y)$$

Holds for all $x,y\in(0,\infty]$ and $t\in[0,1]$. We denote this by $f\in F_{\beta}^{\alpha}$.

It's easy to see that an (α,β) −convex function is s −convex function in the second sense for $\alpha=\beta=s$ and ordinary convex function for $\alpha=\beta=1$.

Proposition 1. If f is (α,β) −convex then f is non-negative on $[0,\infty)$.

Proof. We have for $u\in\mathbb{R}_+$

$$f(u)=f\left(\frac{u}{2}+\frac{u}{2}\right)\le\left(\frac{1}{2^{\alpha}}+\frac{1}{2^{\beta}}\right)f(u).$$

Since $\left(\frac{1}{2^{\alpha}}+\frac{1}{2^{\beta}}-1\right)f(u)\ge 0$ then $f(u)\ge 0$.

Now we give a Hadamard type inequality fort his class of function.

Theorem 3. Suppose that $f:\mathbb{R}_+\to\mathbb{R}_+$ is an (α,β) −convex function, $\alpha,\beta\in(0,1)$ and $a,b\in\mathbb{R}_+$ with $a<b$. If $f\in L_1([a,b])$, then one has the inequalities:

$$\frac{\dot{H}(2^{\alpha},2^{\beta})}{2}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx \le\frac{f(a)}{\alpha+1}+\frac{f(b)}{\beta+1}. \tag{2.1}$$

Where $\dot{H}(x,y)$ is the harmonic mean of x and y.

Proof. As f is (α,β) −convex, we have for all $t\in[0,1]$:

$$f(ta+(1-t)b)\le t^{\alpha}f(a)+(1-t)^{\beta}f(b).$$

Integrating this inequality on $[0,1]$ we get:

$$\int_0^1 f(ta+(1-t)b)dt \le f(a)\int_0^1 t^{\alpha}dt+f(b)\int_0^1 t^{\beta}dt =\frac{f(a)}{\alpha+1}+\frac{f(b)}{\beta+1}$$

As the change of variable $x=ta+(1-t)b$ gives us that:

$$\int_a^b f(ta+(1-t)b)dt=\frac{1}{b-a}\int_a^b f(x)dx,$$

the second inequality in (2.1) is proved.
To prove the first inequality in (2.1), we observe thet for all $x,y\in I$ we have

$$f\left(\frac{x+y}{2}\right)\le\frac{f(x)}{2^{\alpha}}+\frac{f(y)}{2^{\beta}}.$$

Now let $x=ta+(1-t)b$ and $y=tb+(1-t)a$ with $t\in[0,1]$ then we obtain

$$f\left(\frac{a+b}{2}\right)\le\frac{f(ta+(1-t)b)}{2^{\alpha}}+\frac{f(tb+(1-t)a)}{2^{\beta}}$$

For all $t\in[0,1]$. Integrating this inequality on $[0,1]$, we get the first part of (2.1).

Theorem 4. Let f be a nondecreasing (α,β) −convex function and g be a non-negative convex function on $[0,\infty)$. Then the composition $f\circ g$ of f with g is an (α,β) −convex function.

Proof. Let $h = f \circ g$ then we have for $t \in [0,1]$

$$h(tx+(1-t)y) = f\big(g(tx+(1-t)y)\big)$$

Since f is nondecreasing and g is convex we obtain

$$f\big(g(tx+(1-t)y)\big) \le f\big(tg(x)+(1-t)g(y)\big).$$

By using (α,β) −convexity of f and note that g is non-negative, we have

$$\begin{aligned} &f\big(tg(x)+(1-t)g(y)\big) \\ &\quad \le t^{\alpha} f\big(g(x)\big) + (1-t)^{\beta} f\big(g(y)\big) \\ &= t^{\alpha} h(x) + (1-t)^{\beta} h(y). \end{aligned}$$

That completes the proof.

Suppose that f is Lebesgue integrable on $[a,b]$ and consider the mapping $H:[0,1] \to \mathbb{R}$ given by

$$H(t) := \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx.$$

(S. S. Dragomir, 2002) The following Theorem involves some properties of this mapping associated with (α,β) −convexity:

Theorem 5. Let $f: I \subset \mathbb{R}_+ \to \mathbb{R}$ be an $(\alpha,\beta)-$ convex function on I, $\alpha,\beta \in (0,1]$ and Lebesque integrable on $[a,b]$, $a<b$. Then:

i) H is $(\alpha,\beta) -$ convex on $[0,1]$

ii) We have the inequality:

$$H(t) \ge \frac{\dot{H}(\alpha^2,\beta^2)}{2} f\left(\frac{a+b}{2}\right). \tag{2.2}$$

iii)

$$H(t) \le min\{H_1(t), H_2(t)\}. \tag{2.3}$$

where

$$H_1(t) = t^{\alpha}\frac{1}{b-a}\int_a^b f(x)dx + (1-t)^{\beta} f\left(\frac{a+b}{2}\right)$$

and

$$H_2(t) = \frac{f\left[ta+(1-t)\left(\frac{a+b}{2}\right)\right]}{\alpha+1} + \frac{f\left[tb+(1-t)\left(\frac{a+b}{2}\right)\right]}{\beta+1}$$

iv) If $\widehat{H}(t) = max\{H_1(t), H_2(t)\}$ then

$$\widehat{H}(t) \le t^{\alpha}\left(\frac{f(a)}{\alpha+1}+\frac{f(b)}{\beta+1}\right) + (1-t)^{\beta} f\left(\frac{a+b}{2}\right)\left(\frac{1}{\alpha+1}+\frac{1}{\beta+1}\right).$$

Proof. i) $t_1,t_2 \in [0,1]$ and $x,y \in [a,b]$ with $c+d=1$. We have:

$$\begin{aligned} &H(ct_1+dt_2) \\ &= \int_a^b f\left((ct_1+dt_2)x + [1-(ct_1+dt_2)]\frac{a+b}{2}\right)dx \\ &\le \int_a^b f\left(c\left[t_1x+(1-t_1)\frac{a+b}{2}\right] + d\left[t_2x+(1-t_2)\frac{a+b}{2}\right]\right)dx \\ &\le \frac{1}{(b-a)}\int_a^b \left[c^{\alpha} f\left(t_1x+(1-t_1)\frac{a+b}{2}\right) + d^{\beta} f\left(t_2x+(1-t_2)\frac{a+b}{2}\right)\right]dx \\ &= c^{\alpha}H(t_1) + d^{\beta}H(t_2). \end{aligned}$$

That completes proof.

ii) Suppose that $t \in (0,1]$. Then a simple change of variable $u = tx+(1-t)\frac{a+b}{2}$ gives us

$$\begin{aligned} H(t) &= \frac{1}{t(b-a)}\int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} f(u)du \\ &= \frac{1}{p-q}\int_a^b f(u)du \end{aligned}$$

Where $p = tb+(1-t)\frac{a+b}{2}$ and $q = ta+(1-t)\frac{a+b}{2}$. Applying the first Hermite-Hadamard inequality, we get:

$$\begin{aligned} \frac{1}{p-q}\int_a^b f(u)du &\ge \frac{\dot{H}(2^{\alpha},2^{\beta})}{2} f\left(\frac{p+q}{2}\right) \\ &= \frac{\dot{H}(2^{\alpha},2^{\beta})}{2} f\left(\frac{a+b}{2}\right) \end{aligned}$$

and the inequality (2.2) is obtained. For $t=0$ it can easily be seen that (2.2) is true.

iii) This time if we apply the second Hermite-Hadamard inequality, we also have

$$\frac{1}{p-q}\int_a^b f(u)du \le \frac{f(p)}{\alpha+1}+\frac{f(q)}{\beta+1}$$

$$= \frac{f\left(ta+(1-t)\frac{a+b}{2}\right)}{\alpha+1} + \frac{f\left(tb+(1-t)\frac{a+b}{2}\right)}{\beta+1}$$
$$= H_2(t)$$

For all $t \in [0,1]$. Furthermore, we know that

$$f\left(tx+(1-t)\frac{a+b}{2}\right) \leq t^{\alpha} f(x) + (1-t)^{\beta} f\left(\frac{a+b}{2}\right)$$

for $t \in [0,1]$ and $x \in [a,b]$.
Integrating this inequality on $[a,b]$ we get (2.3) for $H_1(t)$ and the Theorem is proved.

iv) We have

$$H_2(t) \leq \frac{t^{\alpha} f(a) + (1-t)^{\beta} f\left(\frac{a+b}{2}\right)}{\alpha+1} + \frac{t^{\alpha} f(b) + (1-t)^{\beta} f\left(\frac{a+b}{2}\right)}{\beta+1}$$
$$= t^{\alpha}\left(\frac{f(a)}{\alpha+1} + \frac{f(b)}{\beta+1}\right) + (1-t)^{\beta} f\left(\frac{a+b}{2}\right)\left(\frac{1}{\alpha+1} + \frac{1}{\beta+1}\right).$$

We know that

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)}{\alpha+1} + \frac{f(b)}{\beta+1}$$

and

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} \geq 1$$

then we have

$$H_1(t) \leq t^{\alpha}\left(\frac{f(a)}{\alpha+1} + \frac{f(b)}{\beta+1}\right) + (1-t)^{\beta} f\left(\frac{a+b}{2}\right)\left(\frac{1}{\alpha+1} + \frac{1}{\beta+1}\right)$$

and that completes the proof.

$$F(t) := \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy$$

where $t \in [0,1]$ (S. S. Dragomir, 2002). The following Theorem contains results about this mapping:

Theorem 6. Let $f: I \subset \mathbb{R}_+ \to \mathbb{R}$ be an $(\alpha,\beta)-$ convex function on I, $\alpha,\beta \in (0,1]$ and Lebesque integrable on $[a,b]$, $a<b$. Then:

i) F is $(\alpha,\beta)-$ convex.

ii) We have the inequality:

$$\left(\frac{2}{\alpha+1} + \frac{2}{\beta+1}\right)F(t) \geq \dot{H}(2^{\alpha},2^{\beta})\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy, t \in [0,1]. \quad (2.4)$$

iii) We have the inequality:

$$f(t) \geq \dot{H}(2^{\alpha},2^{\beta})H(t) \quad (2.5)$$

iv) We have the inequality:

$$F(t) \leq min\left\{\left(t^{\alpha}+(-t)^{\beta}\right)\frac{1}{b-a}\int_a^b f(x)dx, \frac{f(a)+f(b)}{(\alpha+1)^2} + \frac{f(ta+(1-t)b)+f(tb+(1-t)a)}{(\alpha+1)(\beta+1)}\right\}. \quad (2.6)$$

Proof. i) $t_1,t_2 \in [0,1]$ and $x,y \in [a,b]$ with $c+d=1$. We have:

$$F(ct_1+dt_2) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left((ct_1+dt_2)x + \left(1-(ct_1+dt_2)\right)y\right)dxdy$$
$$= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(c(t_1x+(1-t_1)y) + d(t_2x+(1-t_2)y)\right)dxdy$$
$$\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b \left[c^{\alpha}f(t_1x+(1-t_1)y) + d^{\beta}f(t_2x+(1-t_2)y)\right]dxdy$$
$$= c^{\alpha}F(t_1) + d^{\beta}F(t_2).$$

That completes the proof.

ii) Since f, (α,β) −convex on I then we have:

$$\frac{f(tx+(1-t)y)}{\alpha+1}+\frac{f(ty+(1-t)x)}{\beta+1} \geq \dot{H}\left(2^{\alpha},2^{\beta}\right)f\left(\frac{x+y}{2}\right)$$

for all $t\in[0,1]$ and $x,y\in[a,b]^2$. Integrating this inequality on $[a,b]^2$ we obtain:

$$\int_a^b\int_a^b\left(\frac{f(tx+(1-t)y)}{\alpha+1}+\frac{f(ty+(1-t)x)}{\beta+1}\right)dxdy \geq \dot{H}\left(2^{\alpha},2^{\beta}\right)\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy.$$

With the fact that

$$\int_a^b\int_a^b f(tx+(1-t)y)dxdy = \int_a^b\int_a^b f(ty+(1-t)x)dxdy$$

we get the desired result in (2.4).

iii) We know that

$$F(t)=\frac{1}{b-a}\int_a^b\left[\frac{1}{b-a}\int_a^b f(tx+(1-t)y)dx\right]dy.$$

Now if we take for y fixed

$$H_y(t):=\frac{1}{b-a}\int_a^b f(tx+(1-t)y)dx,$$

for $t\in[0,1]$ and $p=tb+(1-t)y, q=ta+(1-t)y$, we have the identity

$$H_y(t):=\frac{1}{p-q}\int_q^p f(u)du.$$

Applying the Hermite-Hadamard inequality for (α,β) −convex functions, we obtain

$$H_y(t):=\frac{1}{p-q}\int_q^p f(u)du \geq H\left(2^{\alpha},2^{\beta}\right)f\left(\frac{p+q}{2}\right) = \dot{H}\left(2^{\alpha},2^{\beta}\right)f\left(t\frac{a+b}{2}+(1-t)y\right)$$

for all $t\in[0,1], y\in[a,b]$. Integrating on $[a,b]$ over y we get

$$F(t)\geq \dot{H}\left(2^{\alpha},2^{\beta}\right)H(1-t)$$

for all $t\in(0,1)$. We can easily see that $F(t)=F(1-t)$ then inequality (2.5) holds for $t\in(0,1)$ and it also holds for $t=0$ or $t=1$. That completes proof.

iv) Since f is (α,β) −convex on $[a,b]$, we have

$$f(tx+(1-t)y)\leq t^{\alpha}f(x)+(1-t)^{\beta}f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. By integrating this inequality on $[a,b]^2$ fot t we deduce

$$F(t)\leq\left(t^{\alpha}+(1-t)^{\beta}\right)\frac{1}{b-a}\int_a^b f(x)dx$$

For the second part of the inequality (2.6) we note by the second part of the Hermite-Hadamard inequality, that

$$H_y(t):=\frac{1}{p-q}\int_q^p f(u)du\leq \frac{f(tb+(1-t)a)}{\alpha+1}+\frac{f(ta+(1-t)b)}{\beta+1}$$

where $p=tb+(1-t)y$ and $q=ta+(1-t)y$, $t\in[0,1]$. Integrating this inequality on $[a,b]$ over y we deduce

$$F(t)\leq\frac{1}{b-a}\left[\frac{1}{\alpha+1}\int_a^b f(tb+(1-t)y)dy + \frac{1}{\beta+1}\int_a^b f(ta+(1-t)y)dy\right].$$

Then with a simple calculation, we have:

$$\frac{1}{b-a}\int_a^b f(tb+(1-t)y)dy = \frac{1}{r-l}\int_l^r f(u)du\leq\frac{f(r)}{\alpha+1}+\frac{f(l)}{\beta+1} = \frac{f(b)}{\alpha+1}+\frac{f(tb+(1-t)a)}{\beta+1}$$

Where $r=b$ and $l=tb+(1-t)a,\ \ t\in(0,1)$; and similarly,

$$\frac{1}{b-a}\int_a^b f(ta+(1-t)y)dy$$
$$\frac{f(a)}{\alpha+1}+\frac{f(ta+(1-t)b)}{\beta+1}.$$

By addition, it gives the second part of (2.6)
Furthermore, for $t=0$ and $t=1$, the inequality also holds.

For additional information about the mappings H and F see [3].

Theorem 7. Let $f: I \subset \mathbb{R}_+ \to \mathbb{R}$ be an $(\alpha,\beta)-$ convex function on I, $\alpha,\beta \in I^\circ$ with $a<b, t\in[0,1]$ then we have the inequality:

$$\left|\frac{1}{(b-a)^2}\int_a^b\int_a^b \left|t^\alpha f(x)+(1-t)^\beta f(y)\right|dxdy - F_f(t)\right|$$
$$\leq \frac{1}{b-a}\int_a^b f(x)dx - F_f(t)$$

where

$$F_f(t)=\int_a^b\int_a^b f(tx+(1-t)y)dxdy.$$

Proof. Since f is $(\alpha,\beta)-$convex on I and with properties of modulus we have:

$$0\leq t^\alpha f(x)+(1-t)^\beta f(y)-f(tx+(1-t)y)$$
$$=\left|t^\alpha f(x)+(1-t)^\beta f(y)-f(tx+(1-t)y)\right|$$
$$\geq\left|t^\alpha f(x)+(1-t)^\beta f(y)-|f(tx+(1-t)y)|\right|$$

for all $t\in[0,1]$ and $x\in[a,b]$.
Now integrating the above inequality over x,y on $[a,b]^2$, we get

$$\left|\frac{1}{(b-a)^2}\int_a^b\int_a^b \left|t^\alpha f(x)+(1-t)^\beta f(y)\right|dxdy - F_f(t)\right|$$
$$\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b \left|t^\alpha f(x)+(1-t)^\beta f(y)\right|dxdy - F_f(t)$$

For all $t\in[0,1]$. On the other hand, we have

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b t^\alpha f(x)+(1-t)^\beta f(y)\,dxdy$$
$$=\frac{t^\alpha+(1-t)^\beta}{b-a}\int_a^b f(x)dx$$

and that completes the proof.

Conclusion

$(\alpha,\beta)-$convex functions is a general form of $s-$convex functions in the second sense so the results in this work is consistent with earlier works for convex and s-convex functions. Because of this class is new, new inequalities, properties, and generalizations can be found involving this class.

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