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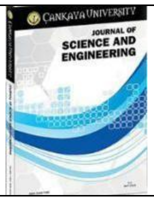

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Solutions of Integro-Differential Difference Equations via Differential Transform Method

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Keywords

Differential transform method, Fredholm and Volterra integrals, Integro-differential difference equations, Semi-analytical and numerical solution, Taylor series expansion.

Abstract

This study presents the application of semi-analytical and numerical solution technique to both Volterra and Fredholm integro-differential difference equations by employing Differential Transform Method depending on Taylor series expansion and introducing the new differential transform theorems with their proofs. To illustrate the computational efficiency and the reliability of the method to other common numerical methods in the open literature, some examples are carried out it is found that the results are highly accurate and reliable.

1. Introduction

The Differential Transform Method (DTM) is a semi-numerical and analytical method based on Taylor series expansion, firstly introduced by Zhou's studies about electrical circuit analysis [18]. In recent years, numerous studies have been realized to extend the application range of this technique to various types of equations such as nonlinear integro-differential equations [1], integro-differential equations [2], difference equations [3], differential-difference equations [4], fractional differential equations [5], integro-differential equation systems [6], fractional integro-differential equations [7], differential equation systems [8], delay differential equations [11], and Volterra integral equation [12] expressing engineering problems, physical phenomena, biological events, or economy. One of these equations is integro-differential difference equations (IDDEs), which are used to model discrete events containing shifted terms in governing integro-differential functions. To solve IDDEs, Polynomial based methods such as the Improved Jacobi matrix method [9], Hybrid Euler-Taylor matrix method [10], the backward substitution method [13], mono-implicit Runge-Kutta method [14], Legendre polynomials [15], Legendre spectral collocation method [16], and Laguerre approach [17] can be employed due to difficulty in solving them analytically, which are reduced to a set of algebraic linear equations. Similarly, DTM can be employed to solve IDDEs. Therefore, the objective of this study is to extend DTM's application border to solve these types of equations by defining the new theorems.

This study is organized as follows. In Section 2, the fundamental theorems and basic definitions used in the transformation operations of various type equations such as the differential, difference, delay, pantograph, and integral, etc. are given, and new theorems for DTM are introduced to solve the Volterra and Fredholm type IDDEs. In Section 3, examples are carried out to demonstrate the application methodology. Also, the comparison

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with other solution techniques is presented to show the accuracy of the DTM. Ultimately, a brief overview and important features of DTM are summarized in Section 4.

2. Differential Transform Method

The differential transform of the k^{th} derivative of a function *f(x)* in a one-dimensional case is defined as follows:

$$F(k)=\frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{1}$$

where *F(k)* is the transformed function of the original function *f(x)* and the inverse differential transform of the *F(k)* function is defined by

$$f(x)=\sum_{k=0}^{\infty}(x-x_0)^k F(k). \tag{2}$$

Any function expressed by differential, difference, integral equations, or combinations of these can be written in the following series expansion form with help of equations (1) and (2)

$$f(x)=\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0}. \tag{3}$$

The following DTM theorems can be deduced from the definitions stated in equations (1) and (2), which are used in the fundamental transform operations for the one-variable equations

Theorem 1. If $f(x)=g(x)\pm h(x)$, then $F(k)=G(k)\pm H(k)$.

Theorem 2. If $f(x)=c\,g(x)$, then $F(k)=cG(k)$, where $c \in IR$.

Theorem 3. If $f(x)=x^n$, then $F(k)=\delta(k-n)=\begin{cases}1, & k=n\\ 0, & k\neq n\end{cases}$.

Theorem 4. If $f(x)=\sin(ax+b)$, then $F(k)=\frac{a^k}{k!}\sin\left(\frac{\pi}{2}k+b\right)$, where $a, b \in IR$.

Theorem 5. If $f(x)=\cos(ax+b)$, then $F(k)=\frac{a^k}{k!}\cos\left(\frac{\pi}{2}k+b\right)$, where $a, b \in IR$.

Theorem 6. If $f(x)=c^{ax+b}$, then $F(k)=\frac{1}{k!}c^b a^k \ln^k(c)$, where a, b, c $\in IR$ are const.

Theorem 7. If $f(x)=\frac{d^n g(x)}{dx^n}$, then $F(k)=\frac{(k+n)!}{k!}G(k)$.

Theorem 8. If $f(x)=g(x)h(x)$, then $F(k)=\sum_{l=0}^{k}G(l)H(k-l)=\sum_{l=0}^{k}G(k-l)H(l)$.

Theorem 9. If $f(x)=\int_{x_0}^{x} g(t)dt$, then $F(k)=\frac{G(k-1)}{k}$ for $k\geq 1$.

Theorem 10. If $f(x)=\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_2}\int_{x_0}^{x_1} g(t)dt dx_1 dx_2 \dots dx_{n-1}$, then $F(k)=\frac{(k-n)!}{k!}G(k-n)$.

Theorem 11. If $f(x)=g(x+a)$, then $F(k)=\sum_{h=k}^{N}\binom{h}{k}a^{h-k}G(h)$ for $N\to\infty$.

Theorem 12. If $f(x)=g^{(n)}(x+a)$, then $F(k)=\frac{(k+n)!}{k!}\sum_{h=k+n}^{N}\binom{h}{k+n}a^{h-k-n}G(h)$.

Theorem 13. If $f(x)=p(x+a)g(x+b)$, then

$$F(k)=\sum_{l=0}^{k}\sum_{h_1=l}^{N}\sum_{h_2=k-l}^{N}\binom{h_1}{l}\binom{h_2}{k-l}a^{h_1-l}b^{h_2-k+l}P(h_1)G(h_2) \text{ for } N\to\infty.$$

Theorem 14. If $f(x)=p(x)g(x+a)$, then $F(k)=\sum_{l=0}^{k}\sum_{h=l}^{N}\binom{h}{l}a^{h-l}P(k-l)G(h)$.

Proof. Let the differential transformation of function *p(x)* be *P(k)* and function $g(x+a)$ be *O(k)* at $x = x_0$. Then, by applying Theorem 8, the transformation of *f(x)* can be obtained as

$$F(k)=\sum_{l=0}^{k}P(k-l)O(l),$$

and from Theorem 11 we have

$$O(l)=\sum_{h=l}^{N}\binom{h}{l}a^{h-l}G(h).$$

By utilizing these terms

$$F(k)=\sum_{l=0}^{k}\sum_{h=l}^{\infty}\binom{h}{l}a^{h-l}P(k-l)G(h) \quad .$$

Theorem 15. If $f(x)=p(x)g^{(n)}(x+a)$, then

$$F(k)=\sum_{l=0}^{k}\sum_{h=k-l+n}^{N}\frac{(k-l+n)!}{(k-l)!}\binom{h}{k-l+n}a^{h-k+l-n}P(l)G(h).$$

Theorem 16. If $f(x)=g(x)\int_{x_0}^{x}h(t)dt$, then $F(k)=\sum_{l=0}^{k-1}G(l)\frac{H(k-l-1)}{k-l}$ for $k\geq 1$.

Theorem 17. If $f(x)=\int_{x_0}^{x}g(t)h(t)dt$, then $F(k)=\sum_{l=0}^{k-1}G(l)\frac{H(k-l-1)}{k}$ for $k\geq 1$.

The proof of Theorems 1-8 are not given due to well-known fundamental properties of DTM referred to in various papers [1-8, 11, 12, 18], and the proofs of Theorems 9-17 are available in the study of Arikoglu and Ozkol [2-7]. Newly introduced theorems with the proofs to solve the IDDEs can be given as

Theorem 18. If $f(x)=\int_{x_0}^{x}g(t+a)dt$, then $F(k)=\frac{1}{k}\sum_{h=k-1}^{\infty}\binom{h}{k-1}a^{h-k+1}G(h)$ for $k\geq 1$.

Proof. By using the definitions of DTM given in equations (1) and (2), we can write

$$
\begin{aligned}
f(x) &= \int_{x_0}^{x} g(t+a)\,dt = \int_{x_0}^{x} \sum_{k=0}^{\infty} (t-x_0+a)^k G(k)\,dt = \sum_{k=0}^{\infty} \int_{x_0}^{x} (t-x_0+a)^k\, G(k)\,dt \\
&= \sum_{k=0}^{\infty} \left[(t-x_0+a)^{k+1} \frac{G(k)}{k+1} \right]_{x_0}^{x} = \sum_{k=0}^{\infty} (x-x_0+a)^{k+1} \frac{G(k)}{k+1} - \sum_{k=0}^{\infty} a^{k+1} \frac{G(k)}{k+1} \\
&= (x-x_0)G(0) + aG(0) - aG(0) \\
&+ \frac{1}{2}(x-x_0)^2 G(1) + \frac{1}{2} 2a(x-x_0)G(1) + \frac{1}{2}a^2G(1) - \frac{1}{2}a^2G(1) \\
&+ \frac{1}{3}(x-x_0)^3 G(2) + \frac{1}{3}3a(x-x_0)^2 G(2) + \frac{1}{3}3a^2(x-x_0)G(2) + \frac{1}{3}a^3G(2) - \frac{1}{3}a^3G(2) + \ldots \\
&= (x-x_0)\left[\frac{1}{1}G(0) + \frac{2}{2}aG(1) + \frac{3}{3}a^2G(2) + \frac{4}{4}a^3G(3) + \frac{5}{5}a^4G(4) + \ldots \right] \\
&+ (x-x_0)^2 \left[\frac{1}{2}G(1) + \frac{3}{3}aG(2) + \frac{6}{4}a^2G(3) + \frac{10}{5}a^3G(4) + \ldots \right] \\
&+ (x-x_0)^3 \left[\frac{1}{3}G(2) + \frac{4}{4}aG(3) + \frac{10}{5}a^2G(4) + \ldots \right] + \ldots \\
&= \sum_{h=0}^{\infty} a^{h-0} \frac{h!}{1!(h-0)!} G(h)\,(x-x_0) + \sum_{h=1}^{\infty} a^{h-1} \frac{h!}{2!(h-1)!} G(h)\,(x-x_0)^2 \\
&+ \sum_{h=2}^{\infty} a^{h-2} \frac{h!}{3!(h-2)!} G(h)\,(x-x_0)^3 + \ldots + \sum_{h=k}^{\infty} a^{h-k} \frac{h!}{(k+1)!(h-k)!} G(h)\,(x-x_0)^{k+1} + \ldots \\
&= \sum_{k=0}^{\infty} \sum_{h=k}^{\infty} \binom{h}{k} \frac{a^{h-k}}{k+1} G(h)(x-x_0)^{k+1}.
\end{aligned}
$$

If the index of k is started from $k = 1$ instead of $k = 0$ in the outer sum, we can rewrite $f(x)$ as

$$
f(x) = \sum_{k=1}^{\infty} \sum_{h=k-1}^{\infty} \binom{h}{k-1} \frac{a^{h-k+1}}{k} G(h)\,(x-x_0)^k .
$$

By using equation (2) we get the following transformed function

$$
F(k) = \frac{1}{k} \sum_{h=k-1}^{\infty} \binom{h}{k-1} a^{h-k+1} G(h) \text{ where } k \geq 1.
$$

Theorem 19. If $f(x) = \int_{x_0}^{x} \int_{x_0}^{x_{n-1}} \ldots \int_{x_0}^{x_3} \int_{x_0}^{x_2} \int_{x_0}^{x_1} g(t+a)\,dt\,dx_1\,dx_2\,dx_3 \ldots dx_{n-1}$, then

$$
F(k) = \frac{(k-n)!}{k!} \sum_{h=k-n}^{N} \binom{h}{k-n} a^{h-k+n} G(h) \quad \text{for} \quad N \to \infty .
$$

Proof. By employing the definition of DTM expressed in equation (2), we can write

$$
f(x) = \int_{x_0}^{x} \int_{x_0}^{x_{n-1}} \ldots \int_{x_0}^{x_3} \int_{x_0}^{x_2} \int_{x_0}^{x_1} \sum_{k=0}^{\infty} (t-x_0+a)^k G(k)\,dt\,dx_1\,dx_2\,dx_3 \ldots dx_{n-1} .
$$

From Theorem 18, the expression above can be rewritten as follows

$$f(x)=\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_3}\int_{x_0}^{x_2}\int_{x_0}^{x_1}\sum_{k=0}^{\infty}\sum_{h=k}^{\infty}\binom{h}{k}a^{h-k}G(h)(t-x_0)^k\,dt\,dx_1\,dx_2\,dx_3\dots dx_{n-1}$$
$$=\sum_{k=0}^{\infty}\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_3}\int_{x_0}^{x_2}\int_{x_0}^{x_1}\sum_{h=k}^{\infty}\binom{h}{k}a^{h-k}G(h)(t-x_0)^k\,dt\,dx_1\,dx_2\,dx_3\dots dx_{n-1}$$
$$=\sum_{k=0}^{\infty}\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_3}\int_{x_0}^{x_2}\sum_{h=k}^{\infty}\binom{h}{k}a^{h-k}G(h)\frac{(x_1-x_0)^{k+1}}{k+1}\,dx_1\,dx_2\,dx_3\dots dx_{n-1}$$
$$=\sum_{k=0}^{\infty}\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_3}\sum_{h=k}^{\infty}\binom{h}{k}a^{h-k}G(h)\frac{(x_2-x_0)^{k+2}}{(k+1)(k+2)}dx_2\,dx_3\dots dx_{n-1}$$
$$=\sum_{k=0}^{\infty}\int_{x_0}^{x}\int_{x_0}^{x_{n-1}}\dots\int_{x_0}^{x_4}\sum_{h=k}^{\infty}\binom{h}{k}a^{h-k}G(h)\frac{(x_3-x_0)^{k+3}}{(k+1)(k+2)(k+3)}dx_3\dots dx_{n-1}$$

$$=\sum_{k=0}^{\infty}\sum_{h=k}^{\infty}\binom{h}{k}a^{h-k}G(h)\frac{(x-x_0)^{k+n}}{(k+1)(k+2)\dots(k+n)}$$
$$=\sum_{k=0}^{\infty}\sum_{h=k}^{\infty}\binom{h}{k}a^{h-k}G(h)\frac{k!}{(k+n)!}(x-x_0)^{k+n}.$$

If the index of k is started from $k = n$ instead of $k = 0$, we can rewrite $f(x)$ as

$$f(x)=\sum_{k=n}^{\infty}\sum_{h=k-n}^{\infty}\binom{h}{k-n}a^{h-k+n}G(h)\frac{(k-n)!}{k!}(x-x_0)^k.$$

By employing the definition in equation (2), we get the following transformed function

$$F(k)=\sum_{h=k-n}^{\infty}\binom{h}{k-n}a^{h-k+n}G(h)\frac{(k-n)!}{k!}(x-x_0)^k \text{ where } k\geq n.$$

Theorem 20. If $f(x)=p(x)\int_{x_0}^{x}g(t+a)dt$, then $F(k)=\sum_{l=1}^{k}\sum_{h=l-1}^{N}a^{h-l+1}\binom{h}{l-1}P(k-l)\frac{G(h)}{l}$.

Proof. Let the differential transformation of function $p(x)$ be $P(k)$ and function $\int_{x_0}^{x}g(t+a)dt$ be $O(k)$ at $x = x_0$.

Then, by applying Theorem 8, the transformation of $f(x)$ can be obtained as

$$F(k)=\sum_{l=0}^{k}P(k-l)O(l),$$

and from Theorem 18 we have

$$O(l)=\sum_{h=l-1}^{\infty}\binom{h}{l-1}a^{h-l+1}\frac{G(h)}{l}.$$

By utilizing these terms

$$F(k)=\sum_{l=1}^{k}P(k-l)\sum_{h=l-1}^{N}\binom{h}{l-1}a^{h-l+1}\frac{G(h)}{l}=\sum_{l=1}^{k}\sum_{h=l-1}^{N}\binom{h}{l-1}a^{h-l+1}P(k-l)\frac{G(h)}{l},\ N\to\infty.$$

Theorem 21. If $f(x)=\int_{x_0}^{x} g_1(t+a_1)\,g_2(t+a_2)\,dt$, the

$$F(k)=\frac{1}{k}\sum_{k_1=0}^{k-1}\sum_{h_1=k_1}^{N}\sum_{h_2=k-k_1-1}^{N}\binom{h_1}{k_1}\binom{h_2}{k-k_1-1}a_1^{h_1-k_1}a_2^{h_2-k+k_1+1}G_1(h_1)G_2(h_2)\quad,\ N\to\infty\,.$$

Proof. Let the product of inner functions be equal to function *g(t)* that differential transform of it with the help of Theorem 13 can be rewritten as

$$G(k)=\sum_{k_1=0}^{k}\sum_{h_1=k_1}^{N}\sum_{h_2=k-k_1}^{N}\binom{h_1}{k_1}\binom{h_2}{k-k_1}a^{h_1-k_1}b^{h_2-k+k_1}G_1(h_1)G_2(h_2)\,.$$

and the inverse transform of G(k) function can be written from equation (2) as follow

$$g(t)=\sum_{k=0}^{\infty}G(k)(t-x_0)^k\,.$$

By substituting *g(t)* function into integral expression, *f(x)* function can be rewritten as

$$f(x)=\int_{x_0}^{x}g(t)\,dt=\int_{x_0}^{x}\sum_{k=0}^{\infty}(t-x_0)^k\,G(k)\,dt=\sum_{k=0}^{\infty}\int_{x_0}^{x}(t-x_0)^k\,G(k)\,dt$$

$$=\sum_{k=0}^{\infty}\left[(t-x_0)^{k+1}\frac{G(k)}{k+1}\right]_{x_0}^{x}=\sum_{k=0}^{\infty}(x-x_0)^{k+1}\frac{G(k)}{k+1}$$

$$=\sum_{k=0}^{\infty}\sum_{k_1=0}^{k}\sum_{h_1=k_1}^{N}\sum_{h_2=k-k_1}^{N}(x-x_0)^{k+1}\binom{h_1}{k_1}\binom{h_2}{k-k_1}a^{h_1-k_1}b^{h_2-k+k_1}\,\frac{G_1(h_1)G_2(h_2)}{k+1}\,.$$

If the index of *k* is started from *k = 1* instead of *k = 0*, we can obtain *f(x)* as

$$f(x)=\sum_{k=1}^{\infty}\sum_{k_1=0}^{k-1}\sum_{h_1=k_1}^{N}\sum_{h_2=k-k_1-1}^{N}(x-x_0)^k\binom{h_1}{k_1}\binom{h_2}{k-k_1-1}a^{h_1-k_1}b^{h_2-k+k_1+1}\,\frac{G_1(h_1)G_2(h_2)}{k}\,.$$

Again, applying equation (2), we get the following transformed function

$$F(k)=\sum_{k_1=0}^{k-1}\sum_{h_1=k_1}^{N}\sum_{h_2=k-k_1-1}^{N}\binom{h_1}{k_1}\binom{h_2}{k-k_1-1}a^{h_1-k_1}b^{h_2-k+k_1+1}\,\frac{G_1(h_1)G_2(h_2)}{k}\ \text{ where }\ k\ge 1.$$

Theorem 22. If $f(x)=\int_{x_0}^{x} g_1(t+a_1)\,g_2(t+a_2)\ldots g_{n-1}(t+a_{n-1})\,g_n(t+a_n)\,dt$, then

$$\begin{aligned}F(k)=\frac{1}{k}&\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\sum_{h_1=k_1}^{N}\sum_{h_2=k_2-k_1}^{N}\cdots\sum_{h_{n-1}=k_{n-1}-k_{n-2}}^{N}\sum_{h_n=k-k_{n-1}-1}^{N}\binom{h_1}{k_1}\\&\times\binom{h_2}{k_2-k_1}\cdots\binom{h_{n-1}}{k_{n-1}-k_{n-2}}\binom{h_n}{k-k_{n-1}-1}\\&\times a_1^{h_1-k_1}a_2^{h_2-k_2+k_1}\ldots a_{n-1}^{h_{n-1}-k_{n-1}+k_{n-2}}a_n^{h_n-k+k_{n-1}+1}\\&\times G_1(h_1)G_2(h_2)\ldots G_{n-1}(h_{n-1})G_n(h_n)\,.\end{aligned}$$

Proof. The proof of Theorem 22 is not given here, and it can be obtained by generalizing the proof of Theorem 21.

Theorem 23. If $f(x)=\int_{x_0}^{x} p(t)g(t+a)dt$, then $F(k)=\frac{1}{k}\sum_{l=0}^{k-1}\sum_{h=l}^{N}\binom{h}{l}a^{h-l}P(k-l-1)G(h)$ for $N\to\infty$.

Proof. Let the product of inner functions be equal to *u(t)* function that differential transform of it with the help of Theorem 14 can be written as

$$U(k)=\sum_{l=0}^{k}\sum_{h=l}^{N}\binom{h}{l}a^{h-l}P(k-l)G(h),$$

and the inverse transform of *U(k)* function can be written as follow

$$u(t)=\sum_{k=0}^{\infty}(t-x_0)^k U(k).$$

By substituting *g(t)* function into integral expression, *f(x)* function can be written as

$$\begin{aligned} f(x)&=\int_{x_0}^{x} u(t)dt=\int_{x_0}^{x}\sum_{k=0}^{\infty}(t-x_0)^k U(k)dt=\sum_{k=0}^{\infty}\int_{x_0}^{x}(t-x_0)^k U(k)dt \\ &=\sum_{k=0}^{\infty}\left[(t-x_0)^{k+1}\frac{U(k)}{k+1}\right]\Bigg|_{x_0}^{x}=\sum_{k=0}^{\infty}(x-x_0)^{k+1}\frac{U(k)}{k+1} \\ &=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{h=l}^{N}(x-x_0)^{k+1}\binom{h}{l}a^{h-l}\frac{P(k-l)G(h)}{k+1}. \end{aligned}$$

If the index of k is started from $k = 1$ instead of $k = 0$, function *f(x)* can be written as

$$f(x)=\sum_{k=1}^{\infty}\sum_{l=0}^{k-1}\sum_{h=l}^{N}(x-x_0)^k\binom{h}{l}a^{h-l}\frac{P(k-l-1)G(h)}{k}.$$

By using equation (2), we get the following transformed function

$$F(k)=\sum_{l=0}^{k-1}\sum_{h=l}^{N}\binom{h}{l}a^{h-l}\frac{P(k-l-1)G(h)}{k} \quad \text{where } k\geq 1.$$

Remark 1. In the solution of the Fredholm integrals, the integral sign is definite and $\alpha, \beta \in IR$ are constant values, then *f(x)* function can be written by utilizing DTM definitions in equations (1) and (2) as follow

$$f(x)=\int_{x_0}^{x} g(t)dt=\sum_{k=1}^{N}\left[(\alpha-x_0)^k-(\beta-x_0)^k\right]\frac{G(k-1)}{k}\ ,\ N\to\infty.$$

3. Applications and Numerical Results

Example 1. Let's first consider the following the pantograph type Volterra integro-differential equation that was considered in the studies of Reutskiy [13], Rihan et al. [14], and Yuzbasi [17]

$$u'(x)=u(x-1)+\int_{x-1}^{x} u(t)dt \tag{4}$$

with the initial condition $u(0)=1$. At $x_0 = 0$, equation (4) gives out another initial condition

$$u'(0)=u(-1)+\alpha, \tag{5}$$

where $\alpha \in IR$ is a constant value

$$\alpha = \int_{-1}^{0} u(t)dt = \int_{0}^{1} u(t-1)dt = \sum_{k=1}^{N} (1-0)^k \frac{1}{k} \sum_{h=k-1}^{N} \binom{h}{k-1} (-1)^{h-k+1} U(h).$$

The exact solution to this problem is $u(x) = e^x$. Before applying concerned Theorems, equation (4) must be rearranged as

$$\begin{aligned} u'(x) &= u(x-1) + \int_0^x u(t)dt - \int_0^{x-1} u(t)dt = u(x-1) + \int_0^x u(t)dt - \int_1^x u(t-1)dt \\ &= u(x-1) + \int_0^x u(t)dt - \int_0^x u(t-1)dt + \int_0^1 u(t-1)dt. \end{aligned} \tag{6}$$

Then, the problem can be reduced into a set of linear algebraic equations, and the following recurrence relation can be obtained for $k \geq 1$

$$(k+1)U(k+1) = \sum_{h=k}^{N} \binom{h}{k} (-1)^{h-k} U(h) + \frac{U(k-1)}{k} - \frac{1}{k} \sum_{h=k-1}^{\infty} \binom{h}{k-1} (-1)^{h-k+1} U(h) + \alpha\, \delta(k), \tag{7}$$

and the transformed initial conditions at $x_0 = 0$ can be obtained

$$U(0) = 1, \text{ and } U(1) = \alpha + \sum_{k=0}^{N} (-1)^k U(k). \tag{8}$$

Using the inverse transform definition in equation (2), we get the following series solution for the sufficiently larger chosen number of terms $N = 8$,

$$\begin{aligned} u(x) &= 1 + 0.999980\,x + 0.499972\,x^2 + 0.166659\,x^3 + 0.041699\,x^4 + 0.008315\,x^5 \\ &+ 0.001369\,x^6 + 0.000223\,x^7 + 0.000013\,x^8 + O(x^9). \end{aligned} \tag{9}$$

Numerical results obtained with DTM for $N = 8$ can be seen in Table 1.

Table 1. Comparison of the absolute errors and numerical results for $N = 8$

x	Exact	DTM	Error	Ref.[13]	Error	Ref. [17]	Error
0.0	1.000000000	1.000000000	0.0e+00		0.0e+0	1.000000000	3.1e-14
0.2	1.221402758	1.221397739	5.0e-06	-	3.9e-8	1.221404341	1.5e-6
0.4	1.491824697	1.491812590	1.2e-05	-	3.8e-8	1.491826867	2.1e-6
0.6	1.822118800	1.822097946	2.1e-05	-	1.4e-7	1.822120801	2.0e-6
0.8	2.225540928	2.225509327	3.2e-05	-	2.1e-7	2.225542667	1.7e-6
1.0	2.718281828	2.718234072	4.8e-05	-	2.4e-7	2.718283674	1.8e-6

Example 2. Now, we introduce the following the Fredholm IDDE with variable coefficients, which is also solved by using the Hybrid Euler-Taylor matrix method in the study of Balci and Sezer [10]

$$y''(x) - e^{x+2} y'(x+1) = e^{-x} + \int_0^1 y(t-1)dt, \tag{10}$$

with the initial conditions $y(0)=2$ and $y'(0)=-1$. The exact solution of equation (10) is given $y(x)=1+e^{-x}$. Applying concerning Theorems to equation (10), the problem is expressed by the following recurrence relation for $k \geq 1$, which represents a set of linear algebraic equations

$$(k+2)(k+1)Y(k+2)-e^{2}\sum_{l=0}^{k}\sum_{h=k-l+1}^{N}\frac{(k-l+1)!}{(k-l)!}\binom{h}{k-l+1}\frac{1}{l!}Y(h)=\frac{(-1)^{k}}{k!}+\alpha.\delta(k), \tag{11}$$

where $\alpha=\int_{0}^{1}y(t-1)dt$ and equation (12) can be rewritten as follows

$$Y(k+2)=\frac{k!}{(k+2)!}\left(e^{2}\sum_{l=0}^{k}\sum_{h=k-l+1}^{N}\binom{h}{k-l+1}\frac{(k-l+1)Y(h)}{l!}+\frac{(-1)^{k}}{k!}+\alpha.\delta(k)\right), \tag{12}$$

and the transformed initial conditions at $x_0 = 0$ can be obtained

$$Y(0)=2 \quad \text{and} \quad Y(1)=-1. \tag{13}$$

By using the inverse transform property in equation (12), the series solution of equation (10) is obtained in the terms of α and x. To obtain the approximate solution by using DTM, the value of α is firstly obtained by utilizing Remark 1 given for the solution of Fredholm integrals as follow

$$\alpha=\sum_{k=1}^{N}\frac{1}{k}\sum_{h=k-1}^{N}\binom{h}{k-1}(-1)^{h-k+1}Y(h), \tag{14}$$

and then algebraic equations in equations (13) and (14) are solved together. Ultimately, the following series solutions are obtained for $N = 2$ *and* $N = 5$, respectively.

$$\begin{aligned} y(x)&=2-x+0.304352x^{2}+O(x^{3})\\ y(x)&=2-x+0.512537x^{2}-0.174662x^{3}+0.038322x^{4}-0.003914x^{5}+O(x^{6}). \end{aligned} \tag{15}$$

In Table 2, numerical results are presented with a comparison to exact results and Ref. [10].

Table 2. Error analysis and numerical results for the different values of N

N	x	Exact	DTM	Error	Ref. [10]	Error
2	0.0	2.000000000	2.000000000	0.00000e+00	1.999995000	0.5000e−5
	0.2	1.818730753	1.812174117	6.55664e-03	1.811862600	0.6868153e−2
	0.4	1.670320046	1.648696471	2.16236e-02	1.647457400	0.22862646e−1
	0.6	1.548811636	1.509567060	3.92446e-02	1.506779400	0.42032236e−1
	0.8	1.449328964	1.394785885	5.45431e-02	1.389828600	0.59500364e−1
	1.0	1.367879441	1.304352946	6.35265e-02	1.296605000	0.71274441e−1
5	0.0	2.000000000	2.000000000	0.00000e+00	1.999550000	0.450000e−3
	0.2	1.818730753	1.819164272	4.33520e-04	1.818747472	0.16719e−4
	0.4	1.670320046	1.671768634	1.44859e-03	1.671286512	0.16719e−4
	0.6	1.548811636	1.551448833	2.63720e-03	1.550846672	0.2035036e−2
	0.8	1.449328964	1.453011618	3.68265e-03	1.452280432	0.2951468e−2
	1.0	1.367879441	1.372284441	4.40500e-03	1.371450000	0.3570559e−2

Example 3. Next, the following third-order linear Fredholm integro-differential difference equation with a variable coefficient [9, 15, 16] is introduced

$$y'''(x)+y''(x-1)-xy'(x)-xy(x-1)=\alpha+g(x)+\int_{-1}^{1}y(t-1)dt\,, \tag{17}$$

where $\alpha=1-\cos(2)$, $g(x)=-\left(\sin(x-1)+\cos(x)\right)(x+1)$, and the initial conditions are given as follows

$$y(0)=0\,,\ \ y'(0)=1\,,\text{ and }\ \ y''(0)=0\,. \tag{18}$$

In this problem, the exact solution is given $y(x)=\sin(x)$. Using concerned Theorems, equation (17) can be transformed the following recurrence relation

$$\begin{aligned}&\frac{(k+3)!}{k!}Y(k+3)+(k+1)\sum_{h=k+1}^{N}\binom{h}{k+1}(-1)^{h-k-1}Y(h)-\sum_{l=0}^{k}\sum_{h=k-l+1}^{N}(k-l+1)\binom{h}{k-l+1}\times\\&(-1)^{h-k+l-1}\delta(l-1)Y(h)-\sum_{l=0}^{k}\sum_{h=k-l}^{N}\binom{h}{k-l}(-1)^{h-k+l}\delta(l-1)Y(h)=G(k)+(\alpha+\beta)\delta(k),\end{aligned} \tag{19}$$

where $\beta=\int_{-1}^{1}y(t-1)dt$, and

$$G(k)=-\frac{1}{k!}\left(\cos\left(\frac{\pi}{2}k\right)+\sin\left(\frac{\pi}{2}k-1\right)\right)-\sum_{l=0}^{k}\frac{\delta(l-1)}{(k-l)!}\left(\cos\left(\frac{\pi}{2}(k-l)\right)+\sin\left(\frac{\pi}{2}(k-l)-1\right)\right),$$

and the initial conditions at $x_0 = 0$ are transformed as follows

$$Y(0)=0\,,\ Y(1)=1\ \text{, and}\ \ Y(2)=0\,. \tag{20}$$

Fredholm integral term β in equation (19) can be written by using Remark 1 as follow

$$\beta=\sum_{k=1}^{N}\frac{(1)^k-(-1)^k}{k}\sum_{h=k-1}^{N}\binom{h}{k-1}(-1)^{h-k+1}Y(h)\,. \tag{21}$$

Using the recurrence relation in equation (19), transformed initial conditions in equation (20), and Fredholm Integral term in equation (21), *Y(k)* for $k\geq 1$ is obtained and then employing inverse transform rule, the following series solution of equation (17) is obtained for $N = 6$

$$y(x)=x-0.178773x^3+0.053303x^4+0.0089682x^5+0.004715x^6+O(x^7)\,. \tag{22}$$

Table 3. Comparison of the absolute errors for $N = 6$

x	DTM	Ref. [9]	Ref. [15]	Ref. [16]
-1.0	3.73e-02	3.84e-02	2.53e-02	2.88e-02
-0.8	1.46e-02	1.82e-02	1.19e-02	1.36e-02
-0.6	4.13e-03	7.00e-03	4.57e-03	5.22e-02
-0.4	5.77e-04	1.86e-03	1.21e-03	1.38e-03
-0.2	1.17e-05	2.04e-03	1.33e-04	1.52e-04
0.0	0.00e+00	0	0.00e+00	0
0.2	1.82e-04	1.48e-03	1.00e-04	1.10e-04
0.4	2.11e-03	9.67e-03	6.89e-04	7.20e-04
0.6	9.25e-03	2.55e-03	2.06e-03	1.90e-03
0.8	2.65e-02	4.44e-03	4.56e-03	3.34e-03
1.0	5.99e-02	5.76e-03	8.97e-03	4.36e-03

Example 4. In this problem, we will consider the second-order linear Volterra IDDE

$$u''(x)=u(2x)-e^{x-1}\int_0^x u(t+1)\,dt\ , \tag{23}$$

and the initial conditions at $x_0 = 0$ are defined as follows

$$u(0)=1\ \text{and}\ u'(0)=1. \tag{24}$$

Here, the exact solution of equation (23) is $y=e^x$. Employing concerned DTM Theorems, equation (23), can be transformed for $k\geq 1$ as follows

$$\frac{(k+2)!}{k!}U(k+2)=2^k\,U(k)-\frac{1}{e}\sum_{l=1}^{k}\sum_{h=l-1}^{N}\binom{h}{l-1}\frac{1}{(k-l)!}\frac{U(h)}{l}\,, \tag{25}$$

and the initial conditions in equation (24) transforms to

$$U(0)=1\ \ \text{and}\ U(1)=1\,. \tag{26}$$

Taking *N = 9* and applying the inverse transform formula in equation (2) after finding *Y(k)* values from equations (25) and (26), we get the following series of solution

$$\begin{aligned} u(x) = & 1 + x + 0.500029x^2 + 0.166664x^3 + 0.041674x^4 + 0.0089682x^5 \\ & + 0.008330x^6 + 0.001392x^7 + 0.000196x^8 + 0.000029x^9 + O\left(x^{10}\right). \end{aligned} \tag{27}$$

Ultimately, a comparison of exact analytical results with the numerical results is presented in Table 4 for different values of *N*.

Table 4. Numerical results vs. the exact result for different values of *N*

x	Exact	*N = 5*	*N = 6*	*N = 7*	*N = 8*	*N = 9*
0.0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.2	1.22140276	1.22623384	1.22084977	1.22153392	1.22139161	1.22140395
0.4	1.49182470	1.51141446	1.48958429	1.49235706	1.49177958	1.49182953
0.6	1.82211880	1.86737584	1.81691200	1.82335660	1.82201434	1.82213005
0.8	2.22554093	2.30877243	2.21574257	2.22785101	2.22534589	2.22556213
1.0	2.71828183	2.85307913	2.70155424	2.72210580	2.71795014	2.71831767

Example 5. Ultimately, let us solve the following third-order linear Volterra IDDE

$$u'''(x) = -\frac{3}{4}u'(x) - \frac{x}{4}\cos(2) + \int_{1}^{0.5x+1} \cos(t)u(t-2)\,dt \ , \tag{28}$$

with the initial conditions

$$u(0) = 1,\ u'(0) = 0, \text{ and } u''(0) = -1. \tag{29}$$

The exact solution of equation (28) is $y = \cos(x)$. Before applying DTM Theorems, equation (28) needs to be rearranged by changing the definite integral boundaries as follow

$$u'''(x) = -\frac{3}{4}u'(x) - \frac{x}{4}\cos(2) + \int_{0}^{0.5x} \cos(t+1)u(t-1)\,dt \ . \tag{30}$$

Using concerned DTM Theorems, equation (30) is transformed to

$$\begin{aligned} \frac{(k+3)!}{k!}U(k+3) = & -\frac{3}{4}(k+1)U(k+1) - \frac{\cos(2)}{4}\delta(k-1) + \frac{0.5^k}{k}\sum_{l=0}^{k-1}\sum_{h=l}^{N}\binom{h}{l}(-1)^{h-l} \\ & \times U(h)\cos\left(\frac{\pi}{2}(k-l-1)+1\right)\frac{1}{(k-l-1)!}\ , \end{aligned} \tag{31}$$

and the transformed initial conditions at $x_0 = 0$ are $U(0) = 1$, $U(1) = 0$, and $U(2) = -1/2$

By using equations (31), (32), and (2), the following series solution of equation (28) is obtained for $N = 5$, $N = 7$ respectively.

$$\begin{aligned} u(x) &= 1 - 0.5x^2 - 0.000360x^3 - 0.041686x^4 + O(x^6) \\ u(x) &= 1 - 0.5x^2 - 0.000124x^3 + 0.041674x^4 + 0.000004x^5 - 0.001388x^6 + O(x^8). \end{aligned} \tag{33}$$

For different values of series term N, a comparison of exact analytical results to the approximate numerical results and absolute errors are presented in Table 5.

Table 5. Comparison of numerical results to the exact results and absolute errors

		$N = 5$		$N = 7$	
x	Exact	Approx. Result	Abs. Error	Approx. Result	Abs. Error
-1.0	0.540302306	0.542046967	1.744661e-03	0.540398861	9.655491e-05
-0.8	0.696706709	0.697259370	5.526602e-04	0.696765270	5.856050e-05
-0.6	0.825335615	0.825480448	1.448332e-04	0.825361866	2.625078e-05
-0.4	0.921060994	0.921090252	2.925778e-05	0.921068921	7.926565e-06
-0.2	0.980066578	0.980069583	3.005543e-06	0.980067573	9.952127e-07
0.0	1.000000000	1.000000000	0.000000e+00	1.000000000	0.000000e+00
0.2	0.980066578	0.980063813	2.765049e-06	0.980065586	9.918687e-07
0.4	0.921060994	0.921044087	1.690696e-05	0.921053090	7.903815e-06
0.6	0.825335615	0.825324642	1.097276e-05	0.825308811	2.680398e-05
0.8	0.696706709	0.696890052	1.833423e-04	0.696640747	6.596188e-05
1.0	0.540302306	0.541325643	1.023337e-03	0.540158759	1.435465e-04

4. Conclusion

The solution of integral equations is a challenging job, and it is most difficult to obtain the analytical solution. Improved Jacobi matrix method [9], Hybrid Euler–Taylor matrix method [10], the backward substitution method [13], mono-implicit Runge-Kutta method [14], Legendre polynomials [15], Legendre spectral collocation method [16], and Laguerre approach [17] are some of the employed numerical methods, until now. In this study, DTM is extended to solve Volterra and Fredholm type integro-differential difference equations with variable coefficients by introducing new theorems. Concerned Theorems can be deduced from the basic definitions in equations (1) and (2) and fundamental DTM Theorems, easily. Ultimately, some examples are carried out to demonstrate the reliability and accuracy of DTM, and their approximate results in the case of employing DTM are compared to approximate results obtained by using other common numerical solution techniques and exact analytical results of these examples. It is shown that DTM is one of the reliable semi-numerical and analytical solver tools based on Taylor series expansion for the solution of IDDEs.

Declaration of Competing Interest

The author declares that there is no competing financial interests or personal relationships that influence the work in this paper.

Authorship Contribution Statement

İbrahim Özkol: Reviewing and editing, Methodology.

Mustafa Tolga Yavuz: Writing-Original draft preparation, Software, Visualization.

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