e-ISSN: 2564-7954 CUSJE 18(1): 009-023 (2021) Research Article

Çankaya University

Journal of Science and Engineering

CUJSE

https://dergipark.org.tr/cankujse

A New Extension of Modified Gamma and Beta Functions

Umar Muhammad Abubakar[1*], Salim Rabı'u Kabara[2], Muhammad Auwal Lawan[3], Faisal Adamu Idris[4]

[1,2,3] Department of Mathematics, Faculty of Computing and Mathematical Science, Kano University of Science and Technology, Nigeria
[4] Mathematical Science Department, Sa'adatu Rimi College of Education, Nigeria

Keywords

Gamma Function, Beta Function, Mittag-Leffler Function, Modified Gamma Function, Modified Beta Function, Beta Distribution.

Abstract

In this research paper, a new extension of modified Gamma and Beta functions is presented and various functional, symmetric, first and second summation relations, Mellin transforms and integral representations are obtained. Furthermore, mean, variance and moment generating function for the beta distribution of the new extension of the modified beta function are also obtained.

1. Introduction

The Classical Euler gamma and beta functions [1] are given by:

$$B(\lambda_1,\lambda_2)=\int_0^1 t^{\lambda_1-1}(1-t)^{\lambda_2-1}\,dt=\frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)}. \tag{1}$$

Where,

$$\Gamma(\lambda_1)=\int_0^\infty t^{\lambda_1-1}e^{-t}dt\,,\mathrm{Re}(\lambda_1)>0\,,\mathrm{Re}(\lambda_2)>0. \tag{2}$$

Classical Euler gamma and beta functions with their connection with Macdonald, error and Whittaker functions [1, 2] was extended as follows:

$$\Gamma_\varpi(x)=\int_0^\infty t^{\lambda_1-1}\exp\left(-t-\frac{\varpi}{t}\right)dt. \tag{3}$$

Where,

$$\mathrm{Re}(\lambda_1)>0\,,\ \varpi\geq 0\,,$$

* Corresponding Author: umabubakar347@gmail.com
Received: August 5, 2020, Accepted: November 21, 2020

and

$$B\left(\lambda_1,\lambda_2,\varpi\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}\exp\left(-\frac{\varpi}{t\left(1-t\right)}\right)dt\,. \tag{4}$$

Where

$$\operatorname{Re}\left(\varpi\right)>0,\operatorname{Re}(\lambda_1)>0,\operatorname{Re}(\lambda_2)>0$$

In 2014, Lee et al [3] generalized beta function given Chaudhry et al [2] as follows:

$$B\left(\lambda_1,\lambda_2,\varpi;\omega\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}\exp\left(\frac{\varpi}{t^{\omega}\left(1-t\right)^{\omega}}\right)dt \tag{5}$$

Where

$$\operatorname{Re}\left(\varpi\right)>0,\operatorname{Re}(\lambda_1)>0,\operatorname{Re}(\lambda_2)>0,\operatorname{Re}(\omega)>0$$

In 2014, Choi et al [4] extended the beta function given by Chaudhry et al [2] as follows:

$$B\left(\lambda_1,\lambda_2,\varpi_1,\varpi_2\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}\exp\left(-\frac{\varpi_1}{t}-\frac{\varpi_2}{1-t}\right)dt$$

Where

$$\operatorname{Re}\left(\varpi_1\right)>0,\operatorname{Re}\left(\varpi_2\right)>0,\operatorname{Re}(\lambda_1)>0,\operatorname{Re}(\lambda_2)>0$$

In 2011, Ozergin et al [5] presented the following generalizations of gamma and beta functions:

$$\Gamma_{\varpi}^{(\kappa_1,\kappa_2)}\left(\lambda_1\right)=\int_0^1 t^{\lambda_1-1}\,{}_1F_1\left(\kappa_1;\kappa_2;-t-\frac{\varpi}{t}\right)dt \tag{6}$$

Where

$$\operatorname{Re}\left(\lambda_1\right)>0,\operatorname{Re}\left(\kappa_1\right)>0,\operatorname{Re}\left(\kappa_2\right)>0,\operatorname{Re}\left(\varpi\right)>0,\operatorname{Re}\left(x\right)>0$$

$$B_{\varpi}^{(\kappa_1,\kappa_2)}\left(\lambda_1,\lambda_1\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}\,{}_1F_1\left(\kappa_1;\kappa_2;-\frac{\varpi}{t\left(1-t\right)}\right)dt \tag{7}$$

Where

$$\operatorname{Re}\left(\kappa_1\right)>0,\operatorname{Re}\left(\kappa_2\right)>0,\operatorname{Re}\left(\varpi\right)>0,\operatorname{Re}\left(\lambda_1\right)>0,\operatorname{Re}\left(\lambda_1\right)>0$$

In 2013, Parmar [6] generalized the result obtained by Ozergin et al in [5] as follows:

$$\Gamma_{\sigma}^{(\kappa_1,\kappa_2;\omega)}\left(\lambda_1\right)=\int_0^1 t^{\lambda_1-1}\,{}_1F_1\left(\kappa_1;\kappa_2;-t^{\omega}-\frac{\varpi}{t^{\omega}}\right)dt \tag{8}$$

Where

$$\operatorname{Re}\left(\kappa_1\right)>0,\operatorname{Re}\left(\kappa_2\right)>0,\operatorname{Re}\left(\varpi\right)>0,\operatorname{Re}\left(x\right)>0,\operatorname{Re}(\omega)>0$$

and

$$B_{\sigma}^{(\kappa_1,\kappa_2;\omega)}\left(\lambda_1,\lambda_1\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}\,{}_1F_1\left(\kappa_1;\kappa_2;-\frac{\varpi}{t^{\omega}\left(1-t\right)^{\omega}}\right)dt \tag{9}$$

Where

$$\operatorname{Re}\left(\lambda_1\right)>0,\operatorname{Re}\left(\lambda_2\right)>0,\operatorname{Re}\left(\varpi\right)>0,\operatorname{Re}\left(\kappa_1\right)>0,\operatorname{Re}\left(\kappa_2\right)>0,\operatorname{Re}(\omega)>0$$

In 2015, Agarwal et al [7] used beta function introduced by Parmar [6] to develop two and three variables Hypergeometric function as follows:

$$F_{1,\varpi}^{(\kappa_1,\kappa_2;\omega)}\left(\lambda_1,\lambda_2,\lambda_3;\lambda_4;x,y\right)=\sum_{r,s=0}^{\infty}\left(\lambda_2\right)_r\left(\lambda_3\right)_s\frac{B_{\varpi}^{(\kappa_1,\kappa_2;\omega)}\left(\lambda_1+r+s,\lambda_4-\lambda_1\right)}{B\left(\lambda_1,\lambda_4-\lambda_1\right)}\frac{x^r}{r!}\frac{y^s}{s!}$$

$$\left(\max\left\{|x|,|y|\right\}<1;\operatorname{Re}\left(\varpi\right)\geq 0;\min\left\{\operatorname{Re}\left(\kappa_1\right)\geq 0,\operatorname{Re}\left(\kappa_2\right)\geq 0,\operatorname{Re}\left(\omega\right)\geq 0\right\}\right)$$

$$F_{2,\varpi}^{\left(\kappa_1,\kappa_2,\kappa_1^1,\kappa_2^1;\omega\right)}\left(\lambda_1,\lambda_2,\lambda_3;\lambda_4,\lambda_5;x,y\right)=\sum_{r,s=0}^{\infty}\left(\lambda_1\right)_{r+s}\frac{B_{\varpi}^{(\kappa_1,\kappa_1;\omega)}\left(\lambda_2+r,\lambda_4-\lambda_2\right)}{B\left(\lambda_2,\lambda_4-\lambda_2\right)}$$

$$\times\frac{B_{\varpi}^{\left(\kappa_1^1,\kappa_2^1;\omega\right)}\left(\lambda_3+s,\lambda_5-\lambda_3\right)}{B\left(\lambda_3,\lambda_5-\lambda_3\right)}\frac{x^r}{r!}\frac{y^s}{s!}$$

$$\left(\max\left\{|x|,|y|\right\}<1;\operatorname{Re}\left(\varpi\right)\geq 0;\min\left\{\operatorname{Re}\left(\kappa_1\right)\geq 0,\operatorname{Re}\left(\kappa_2\right)\geq 0,\operatorname{Re}\left(\kappa_1^1\right)\geq 0,\operatorname{Re}\left(\kappa_2^1\right)\geq 0,\operatorname{Re}\left(\omega\right)\geq 0\right\}\right)$$

$$F_{D,\varpi}^{(3;\kappa_1,\kappa_2;\omega)}\left(\lambda_1,\lambda_2,\lambda_3;\lambda_4,\lambda_5;x,y,z\right)=\sum_{r,s,t=0}^{\infty}\frac{B_{\varpi}^{(\kappa_1,\kappa_2;\omega)}\left(\lambda_1+r+s+t,\lambda_5-\lambda_1\right)}{B\left(\lambda_1,\lambda_5-\lambda_1\right)}$$

$$\times\left(\lambda_2\right)_r\left(\lambda_3\right)_s\left(\lambda_4\right)_t\frac{x^r}{r!}\frac{y^s}{s!}\frac{z^t}{t!}$$

$$\left(\max\left\{|x|,|y|,|z|\right\}<1;\operatorname{Re}\left(\varpi\right)\geq 0;\min\left\{\operatorname{Re}\left(\kappa_1\right)\geq 0,\operatorname{Re}\left(\kappa_2\right)\geq 0,\operatorname{Re}\left(\omega\right)\geq 0\right\}\right)$$

In 2017, Pucheta [8] introduced an extended beta and gamma functions using one parameter Mittag-Leffler function below:

$$\Gamma^{\kappa_1}\left(\lambda_1\right)=\int_0^1 t^{\lambda_1-1}E_{\kappa_1}\left(-t\right)dt \tag{10}$$

and

$$B_{\varpi}^{\kappa_1}\left(\lambda_1,\lambda_2\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}E_{\kappa_1}\left(-\varpi t\left(1-t\right)\right)dt \tag{11}$$

Recently, many generalizations, modifications, extensions and variants of gamma and beta functions [9 – 28] have been proposed.

In this paper we generalized the result obtained by Pucheta [8] given in equations (10) and (11) by using two parameters Mittag – Leffler function. The paper is organized as follows: Section one comprises introduction and related literature. Section two covers Mellin transform functional, symmetry and summation relations. Section 3 discusses integral representations. Section 4 contains some statistical applications

2. Main Result

In this part, we introduce a new extension of the modified gamma and beta function with their properties such as functional relation, summation relations and Mellin transform.

Definition 2.1. Let $\kappa_1,\kappa_2\in\mathbb{R}^+$, $\lambda_1\epsilon\,\mathbb{C}$ such that $\mathrm{Re}\left(\lambda_1\right)>0$. Then, the extended gamma function is given by:

$$\Gamma^{\kappa_1,\kappa_2}\left(\lambda_1\right)=\int_0^\infty t^{\lambda_1-1}E_{\kappa_1,\kappa_2}\left(-t\right)\,dt \tag{12}$$

where $E_{\kappa_1,\kappa_2}\left(,\right)$ is two parameters Mittag – Leffler function denoted by

$$E_{\kappa_1,\kappa_2}\left(-t\right)=\sum_{r=0}^{\infty}\frac{\left(-1\right)^r t^r}{\Gamma\left(\kappa_1 r+\kappa_2\right)} \tag{13}$$

Other verities and generalizations of Mittag – Leffler function can be found in [29 – 33].

Remarks 2.1.

1. If $\kappa_2=1$, then $\Gamma^{\kappa_1,\kappa_2}\left(\lambda_1\right)=\Gamma^{\kappa_1}\left(\lambda_1\right)$ given in equation (10)
2. If $\kappa_1=\kappa_2=1$, then $\Gamma^{\kappa_1,\kappa_2}\left(\lambda_1\right)=\Gamma\left(\lambda_1\right)$ given in equation (2)

Lemma 2.1. Let $\lambda_1\in\mathbb{C}$, $\mathrm{Re}\left(\lambda_1\right)>0$ and $\kappa_1,\kappa_2\in\mathbb{R}^+$ then

$$\Gamma^{\kappa_1,\kappa_2}\left(\lambda_1\right)=\frac{\Gamma\left(\lambda_1+1\right)\Gamma\left(1-\left(\lambda_1+1\right)\right)}{\Gamma\left(\kappa_2-\kappa_1\left(1+\lambda_1\right)\right)} \tag{14}$$

Proof

Let $\vartheta=\lambda_1+1$

$$\Gamma^{\kappa_1,\kappa_2}\left(\lambda_1+1\right)=\Gamma^{\kappa_1,\kappa_2}\left(\vartheta\right)=\int_0^\infty t^{\vartheta-1}E_{\kappa_1,\kappa_2}\left(-t\right)dt$$

$$=\mathrm{M}\left\{E_{\kappa_1,\kappa_2}\left(-t\right)\right\}\left(\vartheta\right)=\frac{\Gamma\left(\vartheta\right)\Gamma\left(1-\vartheta\right)}{\Gamma\left(\kappa_2-\kappa_1\vartheta\right)} \tag{15}$$

Where $\mathrm{M}\left\{E_{\kappa_1,\kappa_2}\left(-t\right)\right\}\left(\vartheta\right)$ is Mellin transforms. Replacing $\vartheta=\lambda_1+1$ we obtain the required result.

Definition 2.2. Let $\varpi>0$, $\kappa_1,\kappa_2\in\mathbb{R}^+$ and $\lambda_1,\lambda_2\in\mathbb{C}$ such that $\mathrm{Re}\left(\lambda_1\right)>0$, $\mathrm{Re}\left(\lambda_2\right)>0$. Then, the new extended beta function is given by

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_{12}-1}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt \tag{16}$$

Remarks 2.2.

1. If $\kappa_2=1$, then $B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=B_{\varpi}^{\kappa_1}\left(\lambda_1,\lambda_2\right)$ given in equation (11)
2. If $\kappa_1=\kappa_2=1$, then $\varpi=0$, then $B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=B\left(\lambda_1,\lambda_2\right)$ given in equation (1)

Theorem 2.1. Let $\varpi\geq 0$, $\kappa_1,\kappa_2\in\mathbb{R}^+$, $\lambda_1,\lambda_2\in\mathbb{C}$ such that $\mathrm{Re}\left(\lambda_1\right)>0$ and $\mathrm{Re}\left(\lambda_2\right)>0$. Then

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\sum_{r=o}^{\infty}\frac{\left(-1\right)^r\varpi^r}{\Gamma\left(\kappa_1 r+\kappa_2\right)}B\left(\lambda_1+r,\lambda_2+r\right) \tag{17}$$

Proof

Putting the definition of two parameters Mittag – Leffler function equation (16), we obtain

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}\sum_{r=o}^{\infty}\frac{\left(-1\right)^r\varpi^r t^r\left(1-t\right)^r}{\Gamma\left(\kappa_1 r+\kappa_2\right)}dt \tag{18}$$

On interchanging the order of integration and summation we have:

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\sum_{r=o}^{\infty}\frac{\left(-1\right)^r\varpi^r}{\Gamma\left(\kappa_1 r+\kappa_2\right)}\int_0^1 t^{\lambda_1+r-1}\left(1-t\right)^{\lambda_2+r-1}dt \tag{19}$$

Using the definition of classical beta function given in equation (1), we obtain the result.

Theorem 2.2. (Functional Relation) Let $\varpi\geq 0$, $\kappa_1,\kappa_2\in\mathbb{R}^+$, $\lambda_1,\lambda_2\in\mathbb{C}$ such that $\mathrm{Re}\left(\lambda_1+1\right)>0$ and $\mathrm{Re}\left(\lambda_2+1\right)>0$. Then, the new extended functional relation is given by:

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2+1\right)+B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1+1,\lambda_2\right)=B\left(\lambda_1,\lambda_2\right) \tag{20}$$

Proof

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2+1\right)+B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda+1,\lambda_2\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt$$

$$+\int_0^1 t^{\lambda_1}\left(1-t\right)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt$$

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2+1\right)+B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1+1,\lambda_2\right)=\int_0^1\left[t^{-1}+\left(1-t\right)^{-1}\right]t^{\lambda_1}\left(1-t\right)^{\lambda_2}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt$$

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2+1\right)+B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1+1,\lambda_2\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt$$

$$B_{\sigma}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2+1\right)+B_{\sigma}^{\kappa_1,\kappa_2}\left(\lambda_1+1,\lambda_2\right)=B_{\sigma}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)$$

Theorem 2.3. (Symmetry Relation) Let $\varpi \geq 0$, $\mathrm{Re}\left(\lambda_1\right)>0$ and $\mathrm{Re}\left(\lambda_2\right)>0$. Then, the new extended beta symmetry relation is given by:

$$B_{\sigma}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=B_{\sigma}^{\kappa_1,\kappa_2}\left(\lambda_2,\lambda_1\right) \tag{21}$$

Proof

On substituting $t=1-u$ and interchanging the variables, we obtain the required result.

Theorem 2.4. (First Summation Relation) Let $\varpi \geq 0$, $\kappa_1,\kappa_2 \in \mathbb{R}^+$, $\lambda_1,\lambda_2 \in \mathbb{C}$ such that $\mathrm{Re}\left(\lambda_1\right)>0$ and $\mathrm{Re}\left(1-\lambda_2\right)>0$. Then, the new extended beta first summation relation is given by:

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,1-\lambda_2\right)=\sum_{r=0}^{\infty}\frac{\left(\lambda_2\right)_r}{r!}B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1+r,1\right) \tag{22}$$

Proof

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,1-\lambda_2\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{-\lambda_2}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt$$

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,1-\lambda_2\right)=\int_0^1 x^{\lambda_1-1}\sum_{r=0}^{\infty}\frac{\left(\lambda_2\right)_r x^r}{r!}E_{\kappa_1,\kappa_2}\left(-\varpi x\left(1-x\right)\right)dx \tag{23}$$

On interchanging the order of integration and summation, equation (23) reduces to

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,1-\lambda_2\right)=\sum_{r=0}^{\infty}\frac{\left(\lambda_2\right)_r}{r!}\int_0^1 t^{\lambda_1+r-1}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt \tag{24}$$

On using extended beta representation, we obtain the required result from equation (24)

Theorem 2.5. (Second Summation Relation) Let $\sigma \geq 0$, $\kappa_1,\kappa_2 \in \mathbb{R}^+$, $\lambda_1,\lambda_2 \in \mathbb{C}$ such that $\mathrm{Re}\left(\lambda_1\right)>0$ $\mathrm{Re}\left(\lambda_2\right)>0$. Then the new extended second summation relation is given by:

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\sum_{r=0}^{\infty}B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1+r,\lambda_2+1\right) \tag{25}$$

Proof

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt$$

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2}\left(1-t\right)^{-1}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt$$

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2}\sum_{r=0}^{\infty}x^r E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt\ \left(\left(1-t\right)^{-1}=\sum_{r=0}^{\infty}t^r,\left|t\right|<1\right)$$

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\sum_{r=0}^{\infty}\int_0^1 t^{\lambda_1+r-1}\left(1-t\right)^{\lambda_2}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt$$

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_1\right)=\sum_{r=0}^{\infty}B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1+r,\lambda_2+1\right)$$

Theorem 2.6. (Mellin Transform). Let $\varpi\geq 0$, $\kappa_1,\kappa_2\in\mathbb{R}^+$ $\vartheta\in\mathbb{C}$, such that $\mathrm{Re}\left(\vartheta\right)>0$, $\mathrm{Re}\left(\lambda_1-\vartheta\right)>0$ and $\mathrm{Re}\left(\lambda_2-\vartheta\right)>0$. Then, the new extended Mellin transformation is given by:

$$\mathrm{M}\left\{B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)\right\}\left(\vartheta\right)=B\left(\lambda_1-\vartheta,\lambda_2-\vartheta\right)\Gamma^{\kappa_1,\kappa_2}\left(\vartheta\right)\tag{26}$$

Proof

$$\mathrm{M}\left\{B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)\right\}\left(\vartheta\right)=\int_0^{\infty}\varpi^{\vartheta-1}\left(\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\varpi t\left(1-t\right)\right)dt\right)d\varpi\tag{27}$$

Using uniform convergence of integral we can interchange the order of integration, equation (27) yield

$$\mathrm{M}\left\{B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)\right\}\left(\vartheta\right)=\int_0^1 t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}\int_0^{\infty}\varpi^{\vartheta-1}E_{a,b}\left(-\varpi t\left(1-t\right)\right)dt\,d\varpi\tag{28}$$

Letting $u=\varpi t\left(1-t\right)$ and $w=t$ then $d\varpi=t^{-1}\left(1-t\right)^{-1}du$ and $dw=dt$, equation (28) gives

$$\mathrm{M}\left\{B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)\right\}\left(\vartheta\right)=\int_0^1 w^{\lambda_1-\vartheta-1}\left(1-w\right)^{\lambda_2-\vartheta-1}dw\int_0^{\infty}u^{\vartheta-1}E_{\kappa_1,\kappa_2}\left(-u\right)du$$

$$\mathrm{M}\left\{B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)\right\}\left(\vartheta\right)=\int_0^1 w^{\lambda_1-\vartheta-1}\left(1-w\right)^{\lambda_2-\vartheta-1}dw\,\Gamma^{a,b}\left(\vartheta\right)$$

$$\mathrm{M}\left\{B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)\right\}\left(\vartheta\right)=B\left(\lambda_1-\vartheta,\lambda_2-\vartheta\right)\Gamma^{\kappa_1,\kappa_2}\left(\vartheta\right)\tag{29}$$

Remarks 2.3.

1. Putting $\kappa_2=1$, in (26), we get $\mathrm{M}\left\{B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)\right\}\left(\vartheta\right)=B\left(\lambda_1-\vartheta,\lambda_2-\vartheta\right)\Gamma^{\kappa_1}\left(\vartheta\right)$ given in Pucheta [8].

2. Putting $\kappa_1=\kappa_2=1$, and $\vartheta=1$, in (26), we obtain $\int_0^{\infty} B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=B(\lambda_1-1,\lambda_2-1)$ given in Chaudhury [1].

3. Integral Representations

Theorem 3.1. The following integral transforms holds true:

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=2\int_0^{\frac{\pi}{2}}\cos^{2\lambda_1-1}\phi\sin^{2\lambda_2-1}\phi E_{\kappa_1,\kappa_2}\left(-\varpi\cos^2\phi\sin^2\phi\right)d\phi \tag{30}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\int_0^1 t^{n\lambda_1-1}\left(1-t^n\right)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\varpi t^n\left(1-t^n\right)\right)dt \tag{31}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\frac{1}{\alpha^{\lambda_1+\lambda_2-1}}\int_0^{\alpha} t^{\lambda_1-1}(\alpha-t)^{\lambda_2-1}E_{\lambda_1}\left(\frac{-\varpi t(\alpha-t)}{\alpha^2}\right)dt \tag{32}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=(1+\alpha)^{\lambda_1-1}\alpha^{\lambda_2-1}\int_0^1\frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{(t+\alpha)^{\lambda_1+\lambda_2}}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi\alpha(1+\alpha)t(1-t)}{(t+\alpha)^2}\right)dt \tag{33}$$

Proof

In equation (16), putting $t=\cos^2\phi$ then $dt=-2\cos\phi\sin\varphi d\phi$ when $t=0:\phi=\frac{\pi}{2}$ and $t=1:\phi=0$. Therefore

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\int_{\frac{\pi}{2}}^{0}\cos^{2\lambda-2}\theta\sin^{2\lambda_2-2}\theta E_{\kappa_1,\kappa_2}\left(-\varpi\cos^2\theta\sin^2\theta\right)\left(-2\cos\theta\sin\theta d\theta\right) \tag{34}$$

On simplifying, we get the required result. In equation (16), putting $t=u^n$ then $dt=nu^{n-1}$ when $t=0:u=0$ and $t=1:u=1$. Therefore

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\int_0^1 u^{n(\lambda_1-1)}(1-u)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\varpi u^n\left(1-u^n\right)\right)nu^{n-1}du \tag{35}$$

On simplifying, we obtain the desired result. In equation (16), putting $t=\frac{u}{\alpha}$ then $dt=\frac{du}{\alpha}$ when $t=0:u=0$ and $t=1:u=\alpha$. Therefore

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\int_0^{\alpha}\left(\frac{u}{\alpha}\right)^{\lambda_1-1}\left(\frac{\alpha-u}{\alpha}\right)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\varpi\frac{u}{\alpha}\left(\frac{a-u}{\alpha}\right)\right)\frac{du}{\alpha} \tag{36}$$

On simplifying we obtain the required result. In equation (16), putting $t=\frac{(1+\alpha)u}{(u+\alpha)}$ then $dt=\frac{a(1+\alpha)}{(u+\alpha)^2}du$ when $t=0:u=0$ and $t=1:u=1$. Therefore

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\int_0^1\left(\frac{(1+\alpha)u}{u+\alpha}\right)^{\lambda_1-1}\left(\frac{a(1-u)}{u+\alpha}\right)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\frac{\sigma\alpha(1+\alpha)u(1-u)}{(u+\alpha)^2}\right)$$
$$\times\frac{\alpha(1+\alpha)}{(u+\alpha)^2}du \tag{37}$$

On simplifying, we obtain the desired result.

Theorem 3.2. The following integral transformations holds true:

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\int_0^{\infty}\frac{t^{\lambda_1-1}}{(1+t)^{\lambda_1+\lambda_2}}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi t}{(1+t)^2}\right)dt \tag{38}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\frac{1}{2}\int_0^{\infty}\frac{t^{\lambda_1-1}+t^{\lambda_1-1}}{(1+t)^{\lambda_1+\lambda_1}}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi t}{(1+t)^2}\right)dt \tag{39}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\int_0^{1}\frac{t^{\lambda_1-1}+t^{\lambda_2-1}}{(1+t)^{\lambda_1+\lambda_2}}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi t}{(1+t)^2}\right)dt \tag{40}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\alpha^{\lambda_1}\beta^{\lambda_2}\int_0^{\infty}\frac{t^{\lambda_1-1}}{(\beta+\alpha t)^2}E_{a,b}\left(-\frac{\varpi abt}{(\beta+\alpha t)^2}\right)dt \tag{41}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=2\alpha^{\lambda_1}\beta^{\lambda_2}\int_0^{\frac{\pi}{2}}\frac{\sin^{2\lambda_1-1}\phi\cos^{2\lambda_1-1}\phi}{(\cos^2\phi+a\sin^2\phi)^{\lambda_1+\lambda_2}}E_{\kappa_1,\kappa_2}\left(-\frac{\sigma\alpha\beta\tan^2\phi}{(\beta+\alpha\tan^2\phi)^2}\right)d\phi \tag{42}$$

Proof

In equation (16), putting $t=\frac{u}{1+u}$, $dt=\frac{du}{(1+u)^2}$, when $t=0:u=0$ and $t=1:u\to\infty$.

Therefore

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\int_0^{\infty}\frac{u^{\lambda_1-1}}{(1+u)^{\lambda_1-1}}\frac{1}{(1+u)^{\lambda_2-1}}E_{\kappa_1,\kappa_2}\left(\frac{-\varpi u}{(1+u)^2}\right)\frac{du}{(1+u)^2}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\int_0^{\infty}\frac{u^{\lambda_1-1}}{(1+u)^{\lambda_1+\lambda_2}}E_{a,b}\left(-\frac{\varpi u}{(1+u)^2}\right)du \tag{43}$$

On interchanging the variables, we obtain the required result. Using symmetric property in (43) we obtain:

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\int_0^{\infty}\frac{u^{\lambda_2-1}}{(1+u)^{\lambda_1+\lambda_2}}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi u}{(1+u)^2}\right)du \tag{44}$$

On adding equation (43) and (44) we obtain the required result. Using equation (3.15) we get:

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\int_0^1 \frac{u^{\lambda_1-1}}{\left(1+u\right)^{\lambda_1+\lambda_2}} E_{a,b}\left(-\frac{\sigma u}{\left(1+u\right)^2}\right)du$$
$$+\int_1^{\infty} \frac{u^{p-1}}{\left(1+u\right)^{\lambda_1+\lambda_2}} E_{\kappa_1,\kappa_2}\left(-\frac{\varpi u}{\left(1+u\right)^2}\right)du \qquad (45)$$

Setting $u=t^{-1},\ du=-t^{-2}dt$ when $u=1:t=1$ and $u\to\infty:t=0$. On the second integral of the right hand side of equation (45) give the desired result.

In equation (38), using $t=\frac{\alpha}{\beta}u$ then $dt=\frac{\alpha}{\beta}du$ when $x=0:u=0$ and $t\to\infty:u\to\infty$. Therefore,

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\int_0^{\infty} \frac{\left(\frac{\alpha}{\beta}u\right)^{\lambda_1-1}}{\left(1+\frac{\alpha}{\beta}u\right)^{\lambda_1+\lambda_2}} E_{\kappa_1,\kappa_2}\left(-\frac{\varpi\left(\frac{\alpha}{\beta}u\right)}{\left(1+\frac{\alpha}{\beta}u\right)^2}\right)\frac{\alpha}{\beta}du \qquad (46)$$

On simplifying, we obtained the desired result. In equation (41), putting $x=\tan^2\phi,\ dx=2\tan\phi\sec^2\phi$ when $x=0:\phi=0$ and $x\to\infty:\phi=\frac{\pi}{2}$. Therefore,

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\alpha^{\lambda_1}\beta^{\lambda_2}\int_0^{\frac{\pi}{2}} \frac{\left(\tan^2\phi\right)^{\lambda_1-1}}{\left(1+\alpha\tan^2\phi\right)^{\lambda_1+\lambda_2}} E_{\kappa_1,\kappa_2}\left(-\frac{\varpi\alpha\beta\tan^2\phi}{\left(\beta+\alpha\tan^2\phi\right)^2}\right)2\tan\phi\sec\phi d\phi \qquad (47)$$

On simplifying we get the desired result.

Theorem 3.3. The following integral representation holds true:

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\alpha^{\lambda_1}\beta^{\lambda_2}\int_0^1 \frac{t^{\lambda_1-1}\left(1-t\right)^{\lambda_2-1}}{\left\{\alpha+\left(\beta-\alpha\right)t\right\}^{\lambda_1+\lambda_2}} E_{\kappa_1,\kappa_2}\left(-\frac{\varpi\alpha\beta t\left(1-t\right)}{\left\{\beta+\left(\alpha-\beta\right)t\right\}^2}\right)dt \qquad (48)$$

$$B_{\varpi}^{\kappa_1,\kappa_2}\left(\lambda_1,\lambda_2\right)=\beta^{\lambda_2}\left(\beta+\gamma\right)^{\lambda_1}\int_0^1 \frac{x^{\lambda_1-1}\left(1-x\right)^{\lambda_2-1}}{\left(\beta+\gamma x\right)^{\lambda_1+\lambda_2}} E_{\kappa_1,\kappa_2}\left(-\frac{\varpi\alpha\beta x\left(1-x\right)}{\left(\beta+\gamma x\right)^2}\right)dx \qquad (49)$$

Proof

In equation (16), putting $\frac{\alpha}{u}-\frac{\beta}{t}=\alpha-\beta$ then $dt=\frac{\alpha\beta}{\left\{\alpha+\left(\beta-\alpha\right)u\right\}^{\lambda_1+\lambda_2}}du$ when $t=0:u=0$ and $t=1:u=1$ give the desired result. Lastly, interchanging α and β in equation (48) and substitute $\alpha-\beta=\gamma$ give equation (49).

Theorem 3.4. The following integral representations holds true:

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=(\beta-\alpha)^{1-\lambda_1-\lambda_2}\int_{\alpha}^{\beta}(t-\alpha)^{\lambda_1-1}(\beta-t)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi(t-\alpha)(\beta-t)}{(\beta-\alpha)^2}\right)dt \tag{50}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=2^{1-\lambda_1-\lambda_2}\int_{-1}^{1}(t+1)^{\lambda_1-1}(1-t)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi(t+1)(1-t)}{4}\right)dt \tag{51}$$

Proof

Firstly, in (16) putting $t=\dfrac{u-\alpha}{\beta-\alpha}$, then $dt=\dfrac{du}{\beta-\alpha}$, when $t=0:u=\alpha$ and $t=1:u=\beta$. Therefore

$$B_{\sigma}^{a,b}(p,q)=\int_{\alpha}^{\beta}\frac{(u-\alpha)^{p-1}}{(\beta-\alpha)^{p-1}}\frac{(\beta-u)^{q-1}}{(\beta-\alpha)^{q-1}}E_{a,b}\left(-\frac{\sigma(u-\alpha)(\beta-u)}{(\beta-\alpha)^2}\right)\frac{du}{(\beta-\alpha)}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=(\beta-\alpha)^{1-\lambda_1-\lambda_2}\int_{\alpha}^{\beta}(u-\alpha)^{\lambda_1-1}(\beta-u)^{\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi(u-\alpha)(\beta-u)}{(\beta-\alpha)^2}\right)du \tag{50}$$

On interchanging the variables, we get the required result. Lastly, in (50) putting $\alpha=-1$ and $\beta=1$, we obtain the required result

Theorem 3.5. The following formulas hold.

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\alpha^{\lambda_2}\beta^{\lambda_1}\int_{0}^{1}\frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{\{\alpha+(\beta-\alpha)t\}^{\lambda_1+\lambda_2}}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi\alpha\beta t(1-t)}{\{\alpha+(\beta-\alpha)t\}^2}\right)dt \tag{51}$$

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=\beta^{\lambda_2}(\beta+\gamma)^{\lambda_1}\int_{0}^{1}\frac{t^{\lambda_1-1}(1-t)^{\lambda_2-1}}{(\beta+\gamma t)^{\lambda_1+\lambda_2}}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi\alpha\beta t(1-t)}{(\beta+\gamma t)^2}\right)dt \tag{52}$$

Proof

Firstly, in equation (16), putting $\dfrac{\alpha}{u}-\dfrac{\beta}{t}=\alpha-\beta$ then $dt=\dfrac{\alpha\beta}{\{\alpha+(\beta-\alpha)u\}}du$ when $t=0:u=0$ and $t=1:u=1$, therefore

$$B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)=a^{\lambda_2-1}b^{\lambda_1-1}\int_{0}^{1}\frac{u^{\lambda_1-1}(1-u)^{\lambda_2-1}}{\{\alpha+(\beta-\alpha)u\}^{\lambda_1+\lambda_2}}E_{\kappa_1,\kappa_2}\left(-\frac{\varpi\alpha\beta u(1-u)}{\{\alpha+(\beta-\alpha)u\}^2}\right) \times \frac{\alpha\beta}{\{\alpha+(\beta-\alpha)u\}^2}du \tag{53}$$

On simplifying, we obtain the desired result in equation (53). Lastly, in equation (53) interchanging α and β and later replacing $\alpha-\beta=\gamma$ give the required result.

Theorem 3.6. The following integral representation formula hold true:

$$\Gamma^{\kappa_1,\kappa_2}(\lambda_1)\Gamma^{\kappa_1,\kappa_2}(\lambda_2)=4\int_0^{\frac{\pi}{2}}\int_0^{\infty} r^{2(\lambda_1+\lambda_2-1)} r\cos^{2\lambda_1-1}\theta\sin^{2\lambda_2-1}\theta E_{\kappa_1,\kappa_2}\left(-\varpi\cos^2\theta\right)E_{\kappa_1,\kappa_2}\left(-\varpi\sin^2\theta\right)dr\,d\theta$$

Proof

In equation (12), putting $\lambda_1=n^2$ and $\lambda_2=m^2$ yield:

$$\Gamma^{\kappa_1,\kappa_2}(\lambda_1)=2\int_0^{\infty} n^{2\lambda_1-1}E_{\kappa_1,\kappa_2}\left(-\varpi n^2\right)dn \quad (54)$$

$$\Gamma^{\kappa_1,\kappa_2}(\lambda_2)=2\int_0^{\infty} m^{2\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\varpi m^2\right)dm \quad (55)$$

Multiplying (54) and (55), yield:

$$\Gamma^{\kappa_1,\kappa_2}(\lambda_1)\Gamma^{\kappa_1,\kappa_2}(\lambda_2)=4\int_0^{\infty}\int_0^{\infty} n^{2\lambda_1-1}m^{2\lambda_2-1}E_{\kappa_1,\kappa_2}\left(-\varpi n^2\right)E_{\kappa_1,\kappa_2}\left(-\varpi m^2\right)dn\,dm \quad (56)$$

Putting $n=r\cos\theta$ and $m=r\sin\theta$ give the result

Remarks 3.1.

1. If $b=1$, then $\Gamma^{\kappa_1,\kappa_2}(\lambda_1)\Gamma^{\kappa_1,\kappa_2}(\lambda_2)=\Gamma^{\kappa_1}(\lambda_1)\Gamma^{\kappa_2}(\lambda_2)$ given by Pucheta [8]
2. If $\kappa_1=\kappa_2=1$ and $x=r^2$, then $\Gamma^{\kappa_1,\kappa_2}(\lambda_1)\Gamma^{\kappa_1,\kappa_2}(\lambda_2)=\Gamma(\lambda_1)\Gamma(\lambda_2)=B(\lambda_1,\lambda_2)\Gamma(\lambda_1+\lambda_2)$ given in [1]

Other integral formulas for related generalized gamma and beta functions are given by Abubakar and Kabara in [34, 35].

4. The Beta Distribution of $B_{\varpi}^{\kappa_1,\kappa_2}(\lambda_1,\lambda_2)$

The extended modified beta distribution $B_{\sigma}^{a,b}(h,g)$, where h and g satisfy the condition $-\infty<h<\infty$, $-\infty<g<\infty$ and $\sigma>0$ as

$$f(x)=\begin{cases}\dfrac{1}{B_{\varpi}^{\kappa_1,\kappa_2}(h,g)}t^{h-1}(1-t)^{g-1}E_{a,b}\left(-\varpi t(1-t)\right), & 0<t<1\\ 0, & \text{otherwise}\end{cases} \quad (57)$$

$$(h,g\in\mathbb{R},\bar{\varpi},\kappa_1,\kappa_2\in\mathbb{R}^+)$$

The r^{th} moment of X for any real number, r is given by:

$$E\left(X^r\right)=\frac{B_{\varpi}^{\kappa_1,\kappa_2}(h+r,g)}{B_{\varpi}^{\kappa_1,\kappa_2}(h,g)},\ (h,g\in\mathbb{R},\bar{\varpi},\kappa_1,\kappa_2\in\mathbb{R}^+) \quad (58)$$

For $r=1$, the mean is given by:

$$\mu = E(X) = \frac{B_{\varpi}^{\kappa_1,\kappa_2}(h+1,g)}{B_{\varpi}^{\kappa_1,\kappa_2}(h,g)}, \ (h, g \in \mathbb{R}, \sigma, \kappa_1, \kappa_2 \in \mathbb{R}^+ \tag{59}$$

The variance of the distribution is given by:

$$\delta^2 = E(X^2) - \{E(X)\}^2$$

$$\delta^2 = \frac{B_{\varpi}^{\kappa_1,\kappa_2}(h,g) B_{\varpi}^{\kappa_1,\kappa_2}(h+2,g) - \{B_{\varpi}^{\kappa_1,\kappa_2}(h+1,g)\}^2}{\{B_{\varpi}^{\kappa_1,\kappa_2}(h,g)\}^2} \tag{60}$$

The moment generating function of the distribution is given by:

$$M(t) = \frac{1}{B_{\varpi}^{\kappa_1,\kappa_2}(h,g)} \sum_{n=0}^{\infty} B_{\varpi}^{\kappa_1,\kappa_2}(h+n,g) \frac{t^n}{n!} \tag{61}$$

The cumulative distribution is defined as

$$F(t) = \frac{B_{\varpi,t}^{\kappa_1,\kappa_2}(h,g)}{B_{\varpi}^{\kappa_1,\kappa_2}(h,g)} \tag{62}$$

Where

$$B_{\varpi,t}^{\kappa_1,\kappa_2}(h,g) = \int_0^t t^{h-1}(1-t)^{g-1} E_{\kappa_1,\kappa_2}(-\varpi t(1-t))\,dt, \ (h, g \in \mathbb{R}, \varpi, \kappa_1, \kappa_2 \in \mathbb{R}^+)$$

is the extended modified incomplete beta function.

5. Conclusion

By using two parameters Mittag – Leffler function, we have defined a new modified gamma $\Gamma^{\kappa_1,\kappa_2}$ and beta functions $B_{\varpi}^{\kappa_1,\kappa_2}$. In their special cases, these generalizations include the extension of gamma and beta functions which were presented in [8]. We have investigated some properties of these generalized functions, most of which are analogous with the classical and other related generalized gamma and beta functions.

It is expected that the results obtained in this study will be prove significant in area of statistic, physics, engineering and applied mathematics.

Acknowledgements

The authors are thankful to the editor and the reviewers for their useful comments and suggestions which helped in improving the presentation of this paper.

Declaration of Competing Interest

The authors declare that there is no competing interest or personal relationships that influence the work in this research paper.

Authorship Contribution Statement

Umar Muhammad Abubakar: Conceptualization, Writing-Original Draft.
Salim Rabiu Kabara: Formal Analysis, Writing-Review and Editing.
Dr. Muhammad Auwal Lawan: Supervision.
Faisal Adamu Idris: Methodology, Resources.

References

[1] M.A. Chaudhry, S.M. Zubair, On a class of incomplete gamma functions with application, Chapman & Hall / CRC, 2002.

[2] M.A. Chaudhry, A. Qadir, M. Rafique, S.M. Zubair, "Extension of Euler beta function," Journal of Computational and Applied Mathematics, vol. 78, pp. 19-32, 1997.

[3] D.M. Lee, A.K. Rathie, R.K. Parmar, Y.S Kim, "Generalized extended beta function, hypergeometric and confluent hypergeometrc functions," Honam Mathematical Journal, Vol. 33 no. 2, pp. 187 – 206, 2011.

[4] J. Choi, A.K. Pallie, R.K. Parmar, "Extension of extended beta, hypergeometric and confluent hypergeometric functions," Honam Mathematical Journal., vol. 36, no. 2, pp. 357 – 385, 2014.

[5] E. Ozergin, M.A. Ozarslan, A. Altin, "Extension of gamma, beta and hypergeometric function," Journal of Computational and Applied Mathematics, vol. 235, pp. 4601 – 4610, 2011.

[6] R.K. Parmar, "A new generalization of gamma, beta, hypergeometric and confluent hypergeometric functions," Le Mathematiche, vol. LXVIII, no. II, pp. 33 – 54, 2013.

[7] P. Agarwal, J. Choi, S. Jain, "Extended hypergeometric functions of two and three variable," Communication of the Korean Mathematical Society, vol. 30, no. 4, pp. 403 – 414, 2015.

[8] P.I. Pucheta, "An new extended beta function," International Journal of Mathematics and Its Applications, vol. 5, no. 3-C, pp. 255 – 260, 2017.

[9] M. Chand, H. Hachimi, R. Rani, "New extension of beta function and its applications," International Journal of Mathematical Sciences, Article ID: 6451592, 25 pages, 2018.

[10] K.S. Gehlot and K.S. Nisar, "Extension of two parameter gamma, beta functions and its properties," Applications and Applied Mathematics: An International Journal, Special Issue 6, pp. 37 – 55, 2020.

[11] E.C. Emmanuel, "On analytical review of the gamma functions," Asian Research Journal of Current Science, vol. 2 no. 1, pp. 28 – 33, Article noARJOCS.163, 2020.

[12] M. Ghayasuddin, M. Ali and R.B. Paris, "Certain new extension of beta and related functions,", 2020.

[13] M.A.H. Kulip, F.F. Mohsen and S.S. Barahmah, "Further extended gamma and beta functions in term of generalized Wright functions," Electronic Journal of University of Aden for Basic and Applied Sciences, vol. 1, no. 2, pp. 78 – 83, 2020.

[14] A. Ata, I.O. Kiyma, "A study on certain properties of generalized special function defined by Fox – Wright function," Applied Mathematics and Nonlinear Sciences, vol. 5, no. 1, pp. 147 – 162, 2020.

[15] F. He, A. Bakhet, M. Abdullah, M. Hidan, "On the extended hypergeometric functions and their applications for the derivatives of the extended Jacobi matrix polynomials," Mathematical Problems in Engineering, Article ID: 4268361, 8 pages, 2020.

[16] R. Sahin, O. Yagci, "Fractional calculus of the extended hypergeometric function," Applied Mathematics and Nonlinear Sciences, vol. 5 no. 1, pp. 369 – 384, 2020.

[17] K.S. Nisar, D.l. Suthar, A. Agarwal, S.D. Purohit, "Fractional calculus operators with Appell function kernels applied to Srivastava polynomials and extended Mittag – Leffler function," Advance in Difference Equations, vol. 148, 14 pages, 2020.

[18] T. Kim, D.S. Kim, "Note on the degenerate gamma function," Russian Journal of Mathematical Physics, vol. 27, no. 3, pp. 352 – 358, 2020.

[19] A. Tassaddiq, “An application of theory of distributions to the family of λ – generalized gamma function,” Mathematics, vol. 5, no. 6, pp. 5839 – 5858, 2020.

[20] N. Khan, T. Usman, M. Aman, “Extended beta, hypergeometric and confluent hypergeometric function,” Transactions of National Academy of Science of Azerbaijan. Series of Physical – Technical and Mathematical Sciences, Issue Mathematics, vol. 39, no. 1, pp. 83 – 97, 2019.

[21] M. Gayasuddin, N. Khan, M. Ali, “A study of the extended beta, Gauss and confluent hypergeometric functions,” International Journal of Applied Mathematics, vol. 33, no. 10, pp. 1 – 13, 2020.

[22] D. Baleanu, P. Agarwal, R.K. Parmar, M.M. Alqurashi, S. Salahshour, “Extension of the fractional derivative operator of the Riemann – Liouville,” Journal of Nonlinear Sciences and Applicatons, vol. 10, pp. 2914 – 2924, 2017.

[23] P. Agarwal, F. Qi, m. Chand, G. Singh, “Some fractional differential equations involving generalized hypergeometric functions,” Journal of Applied Analysis, vol. 25, no. 1, pp. 37 – 44, 2019.

[24] R.K. Parmar, T.K. Pogany, “On the Mathieu – type series for the unified Gauss hypergeometric functions,” Applicable Analysis and Discrete Mathematics, vol. 14, pp. 138 – 149, 2020.

[25] A. Koraby, M. Ahmed, M. Khaled, E. Ahmed, M. Magdy, “Generalization of beta functions in term of Mittag – Leffler function,” Frontiers in Scientific Research and Tehnology, vol. 1, pp. 81 – 88, 2020.

[26] K.Tilahun, H. Tadessee, D.L. Suthar, “The extended Bessel – Maitland function and integral operators associated with fractional calculus,” Journal of Mathematics, Article ID: 7582063, 8 pages, 2020.

[27] D.L. Suthar, D. Baleanu, S.D. Purohit, E. Ucar, “Certain k – fractional operators and images forms of k – struve function,” Mathematics, vol. 5, no. 3, pp. 1706 – 1719, 2020.

[28] D.L. Suthar, A.M. Khan, A. Alaria, S.D. Puhohit, J. Singh, “Extended Bessel – Maitland function and its properties pertaining of integral transforms and fractional calculus,” Mathematics, vol. 5, no. 2, pp. 1400 – 1414, 2020.

[29] S. Joshi, E. Mittal and R.M. Pandey, “On Euler types interval Involving Mittag – Leffler functions,” Boletim da Sociedate Paranaense de Matematica, vol. 38, no. 2, pp. 125 – 134, 2020.

[30] N.U. Khan and S.W. Khan, “A new extension of the Mittag – Leffler Function,” Plastine Journal Mathematics, vol. 9, no. 2, pp. 977 – 983, 2020.

[31] M. Saif, A.H. Khan and K.S. Nisar, “Integral transform of extended Mittag – Leffler in term of Fox – Wright function,” Palestine Journal of Mathematics, vol. 9, no. 1, pp. 456 – 463, 2020.

[32] K.S. Ghelot, “Differential equation of p – k Mittag – Leffler function,” Palestine Journal of Mathematics, vol. 9, no. 2, pp. 940 – 944, 2020.

[33] P.I. Pucheta, “An extended p – k Mittag – Leffler function,” Palestine Journal of Mathematics, vol. 9, no. 2, pp. 785 – 791, 2020.

[34] U.M. Abubakar and S.R. Kabara, “A note on a new extended gamma and beta functions and their properties,” IOSR Journal of Mathematics, vol. 15, no. 5, pp. 1 – 6, 2019.

[35] U.M. Abubakar and S.R. Kabara, “Some results on the extension of the extended beta function,” IOSR Journal of Mathematics, vol. 15, no. 5, pp. 7 – 12, 2019.